Lecture 8. Sets

Oleg Viro

February 29, 2016

Sets and maps

Language rather than theory.

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Compositions

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Theorem. $f : X \to Y$ is injection $\iff$ $\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

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Compositions, image, preimage, identity, injection

Theorem. $f : X \to Y$ is injection $\Longleftrightarrow$
$\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Left inverse; invertible from left.

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Theorem. $f : X \to Y$ is surjection $\iff$ $\exists g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

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Right inverse; invertible from right.

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Theorem. $f : X \to Y$ is bijection $\Longleftrightarrow$
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Inverse; invertible. The inverse is unique, because left inverse = right inverse.

The number of elements

Sets A and B contain the same number of elements

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$$\iff \exists \text{ bijection } A \to B.$$

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This is a definition.

Cantor, 1878.

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$\iff \exists$ bijection $A \to B$.

Properties of injections, surjections, bijections.

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Transitivity, reflexivity, symmetry.

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Pre-orders

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Pre-orders, Total strict orders

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Equivalence classes.

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Definition for inequality between cardinal numbers.

Cantor-Bernstein-Schroeder theorem. If there exist injections $X \to Y$ and $Y \to X$, then $\exists$ a bijection $X \to Y$.

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Cantor-Bernstein-Schroeder theorem. If there exist injections $X \to Y$ and $Y \to X$, then $\exists$ a bijection $X \to Y$.

$a \leq b, \; b \leq a \implies a = b$.

Grand Hotel Hilbert

Hotel with infinite number of rooms, numerated by natural numbers.

(1924)

Grand Hotel Hilbert

Hotel full. One more guest came.

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Are all infinite sets equipotent?

The sets of subsets

Theorem. $2^a \neq a$ for any cardinal number a.

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Why 2?

The sets of subsets

Theorem. $2^a \neq a$ for any cardinal number a.

2^A is the set of all subsets of the set A.

Why 2? What is 3^A?

The sets of subsets

Theorem. $2^a \neq a$ for any cardinal number a.

Definition. A^B is the set of all maps $B \to A$. Justify!

The sets of subsets

Theorem. $2^a \neq a$ for any cardinal number a.

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Proof of Theorem. Let $f : A \to \{0, 1\}^A$ be a bijection.
$f(x)$ is a map $A \to \{0, 1\}$ for each $x \in A$.

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Define $\phi : A \to \{0,1\}$ by formula $\phi(x) = 1 - f(x)(x)$.

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Then $\phi(x) \neq f(x)(x)$.

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Then $\phi(x) \neq f(x)(x)$. Hence $\phi \neq f(x)$ for any $x \in A$.

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Then $\phi(x) \neq f(x)(x)$. Hence $\phi \neq f(x)$ for any $x \in A$.

Hence $f : A \to \{0,1\}^A$ is not even a surjection.

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Define $\phi : A \to \{0,1\}$ by formula $\phi(x) = 1 - f(x)(x)$.

Then $\phi(x) \neq f(x)(x)$. Hence $\phi \neq f(x)$ for any $x \in A$.

Hence $f : A \to \{0,1\}^A$ is not even a surjection.

So, it's not a bijection. This contradicts to the assumption! $\square$

Continuum

Theorem. $\operatorname{card}(\mathbb{R}) > \aleph_0$

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There is an injection $2^{\mathbb{N}} \to [0,1) : (x_n)_{n=1,\ldots} \mapsto \sum_{n=1}^{\infty} \frac{x_n}{10^n}$

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Hence $\aleph_0 = \operatorname{card}\mathbb{N} < \operatorname{card} 2^{\mathbb{N}} \leq \operatorname{card}[0,1) \leq \operatorname{card}\mathbb{R}$. $\square$

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In particular, transcendental numbers exist.

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Hence $\aleph_0 = \operatorname{card}\mathbb{N} < \operatorname{card} 2^{\mathbb{N}} \leq \operatorname{card}[0,1) \leq \operatorname{card}\mathbb{R}$. $\square$

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Continuum hypothesis. There is no intermediate cardinal number between $\aleph_0$ and continuum$= \operatorname{card}\mathbb{R}$

Counter-intuitive equalities

$$\operatorname{card}(a,b) = \operatorname{card}(0,1) = \operatorname{card}\mathbb{R}\,.$$

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Counter-intuitive equalities

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$\operatorname{card}(\textsf{square}) = \operatorname{card}(\textsf{segment})$.

$\operatorname{card}(\textsf{cube}) = \operatorname{card}(\textsf{segment})$.

$\operatorname{card}(\mathbb{R}^n) = \operatorname{card}(\mathbb{R})$ for any n .

Counter-intuitive equalities

$\mathrm{card}(a, b) = \mathrm{card}(0, 1) = \mathrm{card}\,\mathbb{R}$.

$\mathrm{card}(\text{square}) = \mathrm{card}(\text{segment})$.

$\mathrm{card}(\text{cube}) = \mathrm{card}(\text{segment})$.

$\mathrm{card}(\mathbb{R}^n) = \mathrm{card}(\mathbb{R})$ for any n.

$\mathrm{card}(\mathbb{R}^\infty) = \mathrm{card}(\mathbb{R})$ for any n.