Lecture 3. Pythagorean Triples

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Warm Up Exercise

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Problem: Find all Pythagorean triples.

Number theory approach Diophantine equation

Primitive solutions

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Lemma 1. A Pythagorean triple (a, b, c) is primitive if and only if a, b and c have no common divisors.

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Lemma 1. A Pythagorean triple (a, b, c) is primitive if and only if
a, b and c have no common divisors.

Lemma 2. Presence of odds. In a primitive Pythagorean triple,
at least one of the numbers is odd.

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$even + even = even\,$,

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The answer: c must be odd,
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Introduced in 1798 by Carl Friedrich Gauss (Gauβ) (1777-1855),
when he was 21, in Disquisitiones Arithmeticae.

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Exercise. Prove that moreover $n \equiv 1 \mod 2 \implies n^2 \equiv 1 \mod 8$.

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Rewrite $a^2 + b^2 = c^2$ as $b^2 = c^2 - a^2$ and further as $b^2 = (c - a)(c + a)$.

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Denote $c + a = 2P$, $c - a = 2Q$ and $b = 2R$.

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For a primitive Pythagorean triple, P and Q have no common divisor.
Recall $b^2 = c^2 - a^2 = (c+a)(c-a) = 4PQ = 4R^2$.

Solution

Summary: a primitive Pythagorean triple (a, b, c) can be obtained from relatively prime integers P and Q that are related to a, b, c by formulas $c = P + Q$, $a = P - Q$ and $b^2 = 4PQ$.

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Verify: for any integers u, v (no matter odd, relatively prime, or not)

$$\begin{aligned} a^2 + b^2 &= (u^2 - v^2)^2 + (2uv)^2 \\ &= u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 = c^2 \end{aligned}$$

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Pythagorean triples come from Geometry.

Where is Geometry?

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If (a, b) is a Pythagorean point and $c = \sqrt{a^2 + b^2}$,
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Conversely, if (p, q) is a rational point on $x^2 + y^2 = 1$, then on the line $y = \frac{q}{p}x$ there are Pythagorean points.

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The rest is a point (a, b) on the Cartesian plane. It has integer coordinates (a, b) and integer distance $c = \sqrt{a^2 + b^2}$ from the origin.

This is a geometric incarnation of Pythagorian triple (a, b, c), a Pythagorian point.

If (a, b) is a Pythagorean point and $c = \sqrt{a^2 + b^2}$, then $(\frac{a}{c}, \frac{b}{c})$ is a rational point on the circle $x^2 + y^2 = 1$.

Conversly, if (p, q) is a rational point on $x^2 + y^2 = 1$, then on the line $y = \frac{q}{p}x$ there are Pythagorean points.

The problem that we solved was find all Pythagorain points.

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Reformulation: find all rational points on the unit circle $x^2 + y^2 = 1$.

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Riddle:

Draw on a picture all the heroes: α, $\cos\alpha$, $\sin\alpha$, β, and $t = \tan\beta$.