4.6. Divergent Series

On the contents of the lecture. "Divergent series is a pure handiwork of Diable. It is a full nonsense to say that $1^{2n} - 2^{2n} + 3^{2n} - \cdots = 0$. Do you keep to die laughing about this?" (N.H. Abel letter to ...). The twist of fate: now one says that that the above mentioned equality holds in *Abel's sense*.

The earliest analysts thought that any series, convergent or divergent, has a sum given by God and the only problem is to find it correctly. Sometimes they disagreed what is the correct answer. In the nineteenth century divergent series were expelled from mathematics as a "handiwork of Diable" (N.H. Abel). Later they were rehabilitated (see G.H. Hardy's book *Divergent Series*[1]). Euler remains the unsurpassed master of divergent series. For example, with the help of divergent series he discovered Riemann's functional equation of the ζ-function a hundred years before Riemann.

Evaluations with divergent series. Euler wrote: "My pen is clever than myself." Before we develop a theory let us simply follow to Euler's pen. The fundamental equality is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \tag{4.6.1}$$

Now we, following Euler, suppose that this equality holds for all $x \neq 1$. In the second lecture we were confused by some unexpected properties of divergent series. But now in contrast with the second lecture we do not hurry up to land. Let us look around.

Substituting $x = -e^y$ in (4.6.1) one gets

$$1 - e^y + e^{2y} - e^{3y} + \cdots = \frac{1}{1+e^y}.$$

On the other hand

$$\frac{1}{1+e^y} = \frac{1}{e^y - 1} - \frac{2}{e^{2y} - 1}. \tag{4.6.2}$$

Since

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k. \tag{4.6.3}$$

One derives from (4.6.2) via (4.6.3)

$$\frac{1}{e^y + 1} = \sum_{k=1}^{\infty} \frac{B_k(1-2^k)}{k!} y^{k-1}. \tag{4.6.4}$$

Let us differentiate repeatedly n-times the equality (4.6) by y. The left-hand side gives $\sum_{k=0}^{\infty}(-1)^k k^n e^{ky}$. In particular for $y = 0$ we get $\sum_{k=0}^{\infty}(-1)^k k^n$. We get on the right-hand side by virtue of (4.6.4) the following

$$\left(\frac{d}{dy}\right)^n \frac{1}{1+e^y} = \frac{B_{n+1}(1-2^{n+1})}{n+1}.$$

Combining these results we get the following equality

$$1^n - 2^n + 3^n - 4^n + \cdots = \frac{B_{n+1}(2^{n+1}-1)}{n+1}. \tag{4.6.5}$$

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[1] G.H. Hardy, *Divergent Series*, Oxford University Press, 1949.

Since odd Bernoulli numbers vanish, we get

$$1^{2n} - 2^{2n} + 3^{2n} - 4^{2n} + \cdots = 0.$$

Consider an even analytic function $f(x)$, such that $f(0) = 0$. In this case $f(x)$ is presented by a power series $a_1x^2 + a_2x^4 + a_3x^6 + \dots$, then

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{f(kx)}{k^2} &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\sum_{n=1}^{\infty}a_nx^{2n}k^{2n}\\ &= \sum_{n=1}^{\infty}a_nx^{2n}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-2}\\ &= a_1x^2(1-1+1-1+\dots)\\ &= \frac{a_1x^2}{2}.\end{aligned}$$

In particular, for $f(x) = 1 - \cos x$ this equality turns into

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1-\cos kx}{k^2} = \frac{x^2}{4}. \tag{4.6.6}$$

For $x = \pi$ the equality (4.6.6) gives

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Since

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \left(1-\frac{1}{4}\right)\sum_{k=1}^{\infty}\frac{1}{k^2}$$

one derives the sum of the Euler series:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

We see that calculations with divergent series sometimes give brilliant results. But sometimes they give the wrong result. Indeed the equality (4.6.6) generally is untrue, because on the left-hand side we have a periodic function and on the right-hand side a non-periodic one. But it is true for $x \in [-\pi, \pi]$. Termwise differentiation of (4.6.6) gives the true equality (3.4.2), which we know from Lecture 3.4.

Euler's sum of a divergent series. Now we develop a theory justifying the above evaluations. Euler writes that the value of an infinite expression (in particular the sum of a divergent series) is equal to the value of a finite expression whose expansion gives this infinite expression. Hence, numerical equalities arise by substituting a numerical value for a variable in a generating functional identity. To evaluate the sum of a series $\sum_{k=0}^{\infty} a_k$ Euler usually considers its *power generating function* $g(z)$ represented by the power series $\sum_{k=0}^{\infty} a_kz^k$, and supposes that the sum of the series is equal to $g(1)$.

To be precise suppose that the power series $\sum_{k=0}^{\infty} a_kz^k$ converges in a neighborhood of 0 and there is an analytic function $g(z)$ defined in a domain U containing a path p from 0 to 1 and such that $g(z) = \sum_{k=0}^{\infty} a_kz^k$ for z sufficiently close to 0 and 1 is a regular point of g. Then the series $\sum_{k=0}^{\infty} a_k$ is called *Euler summable* and the value $g(1)$ is called its *analytic Euler sum* with respect to p. And we will use a special sign $\simeq$ to denote the analytical sum.

By the Uniqueness Theorem 3.6.9 the value of analytic sum of a series is uniquely defined for a fixed p. But this value generally speaking depends on the path. For example, let us consider the function $\sqrt{1+x}$. Its binomial series for $x=-2$ turns into

$$-1+1-\frac{1}{2!}-\frac{1\cdot 3}{3!}-\frac{1\cdot 3\cdot 5}{4!}-\dots-\frac{(2k-1)!!}{(k+1)!}-\dots.$$

For $p(t)=e^{i\pi t}$ one sums up this series to i, because it is generated by the function $\exp\frac{\ln(1+z)}{2}$ defined in the upper half-plane. And along $p(t)=e^{-i\pi t}$ this series is summable to $-i$ by $\exp\frac{-\ln(1+z)}{2}$ defined in the lower half-plane.

For a fixed path the analytic Euler sum evidently satisfies the Shift, Multiplication and Addition Formulas of the first lecture. But we see that the analytic sum of a real series may be purely imaginary. Hence the rule $\operatorname{Im}\sum_{k=0}^{\infty}a_k\simeq\sum_{k=0}^{\infty}\operatorname{Im}a_k$ fails for the analytic sum. The Euler sum along $[0,1]$ coincides with the Abel sum of the series in the case when both of them exist.

In above evaluations we apply termwise differentiation to functional series. If the Euler sum $\sum_{k=1}^{\infty}f_k(z)$ is equal to $F(z)$ for all z in a domain this does not guarantee the possibility of termwise differentiation. To guarantee it we suppose that the function generating the equality $\sum_{k=1}^{\infty}f_k(z)\simeq F(z)$ analytically depends on z. To formalize the last condition we have to introduce analytic functions of two variables.

Power series of two variables. A power series of two variables z,w is defined as a formal unordered sum $\sum_{k,m}a_{km}z^kw^m$, over $\mathbb{N}\times\mathbb{N}$ — the set of all pairs of nonnegative integers.

For a function of two variables $f(z,w)$ one defines its *partial derivative* $\frac{\partial f(z_0,w_0)}{\partial z}$ with respect to z at the point (z_0,w_0) as the limit of $\frac{f(z_0+\Delta z,w_0)-f(z_0,w_0)}{\Delta z}$ as Δz tends to 0.

LEMMA 4.6.1. *If $\sum a_{km}z_1^kw_1^m$ absolutely converges, then both $\sum a_{km}z^kw^m$ and $\sum ma_{km}z^kw^{m-1}$ absolutely converge provided $|z|<|z_1|$, $|w|<|w_1|$. And for any fixed z, such that $|z|<|z_1|$ the function $\sum ma_kz^kw^{m-1}$ is the partial derivative of $\sum a_{km}z^kw^m$ with respect to w.*

PROOF. Since $\sum|a_{km}||z_1|^k|w_1|^m<\infty$ the same is true for $\sum|a_{km}||z|^k|w|^m$ for $|z|<|z_1|$, $|w|<|w_1|$. By the Sum Partition Theorem we get the equality

$$\sum a_{km}z^kw^m=\sum_{m=0}^{\infty}w^m\sum_{k=0}^{\infty}a_{km}z^k.$$

For any fixed z the right-hand side of this equality is a power series with respect to w as the variable. By Theorem 3.3.9 its derivative by w, which coincides with the partial derivative of the left-hand side, is equal to

$$\sum_{m=0}^{\infty}mw^{m-1}\sum_{k=0}^{\infty}a_{km}z^k=\sum ma_{km}w^{m-1}z^k.$$

□

Analytic functions of two variables. A function of two variables $F(z,w)$ is called analytic at the point (z_0,w_0) if for (z,w) sufficiently close to (z_0,w_0) it can be presented as a sum of a power series of two variables.

THEOREM 4.6.2.

(1) *If $f(z,w)$ and $g(z,w)$ are analytic functions, then $f+g$ and fg are analytic functions.*
(2) *If $f_1(z), f_2(z)$ and $g(z,w)$ are analytic functions, then $g(f_1(z), f_2(w))$ and $f_1(g(z,w))$ are analytic functions.*
(3) *The partial derivative of any analytic function is an analytic function.*

PROOF. The third statement follows from Lemma 4.6.1. The proofs of the first and the second statements are straightforward and we leave them to the reader. □

Functional analytical sum. Let us say that a series $\sum_{k=1}^{\infty} f_k(z)$ of analytic functions is *analytically summable* to a function $F(z)$ in a domain $U \subset \mathbb{C}$ along a path p in $\mathbb{C} \times \mathbb{C}$, such that $p(0) \in U \times 0$ and $p(1) \in U \times 1$, if there exists an analytic function of two variables $F(z,w)$, defined on a domain W containing p, $U \times 0$, $U \times 1$, such that for any $z_0 \in U$ the following two conditions are satisfied:

(1) $F(z_0, 1) = F(z_0)$.
(2) $F(z,w) = \sum \frac{f_m^{(k)}(z_0)}{k!}(z-z_0)^k w^m$ for sufficiently small $|w|$ and $|z-z_0|$.

Let us remark that the analytic sum does not change if we change p keeping it inside W. That is why one says that the sum is evaluated along the domain W.

To denote the functional analytical sum we use the sign $\cong$. And we will write also $\cong_W$ and $\cong_p$ to specify the domain or the path of summation.

The function $F(z,w)$ will be called the *generating function* for the analytical equality $\sum_{k=1}^{\infty} f_k(z) \cong F(z)$.

LEMMA 4.6.3. *If $f(z)$ is an analytic function in a domain U containing 0, such that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for sufficiently small $|z|$, then $f(z) \cong_W \sum_{k=0}^{\infty} a_k z^k$ in U for $W = \{(z,w) \mid wz \in U\}$.*

PROOF. The generating function of this analytical equality is $f((z-z_0)w)$. □

LEMMA 4.6.4 (on substitution). *If $F(z) \cong_p \sum_{k=0}^{\infty} f_k(z)$ in U and $g(z)$ is an analytic function, then $F(g(z)) \cong_{g(p)} \sum_{k=0}^{\infty} f_k(g(z))$ in $g^{-1}(U)$.*

PROOF. Indeed, if $F(z,w)$ generates $F(z) \cong_p \sum_{k=0}^{\infty} f_k(z)$, then $F(g(z),w))$ generates $F(g(z)) \cong_{g(p)} \sum_{k=0}^{\infty} f_k(g(z))$. □

N. H. Abel was the first to have some doubts about the legality of termwise differentiation of functional series. The following theorem justifies this operation for analytic functions.

THEOREM 4.6.5. *If $\sum_{k=1}^{\infty} f_k(z) \cong_p F(z)$ in U then $\sum_{k=1}^{\infty} f_k'(z) \cong_p F'(z)$ in U.*

PROOF. Let $F(z,w)$ be a generating function for $\sum_{k=1}^{\infty} f_k(z) \cong_p F(z)$. We demonstrate that its partial derivative by z (denoted $F'(z,w)$) is the generating function for $\sum_{k=1}^{\infty} f_k'(z) \cong_p F'(z)$. Indeed, locally in a neighborhood of $(z_0, 0)$ one has $F(z,w) = \sum \frac{f_m^{(k)}(z_0)}{k!} w^m (z-z_0)^k$. By virtue of Lemma 4.6.1 its derivative by z is $F'(z,w) = \sum \frac{f_m^{(k)}(z_0)}{(k-1)!} w^m (z-z_0)^{k-1} = \sum \frac{f_m'^{(k)}(z_0)}{k!} w^m (z-z_0)^k$. □

The dual theorem on termwise integration is the following one.

THEOREM 4.6.6. *Let $\sum_{k=1}^{\infty} f_k \cong F$ be generated by $F(z,w)$ defined on $W = U \times V$. Then for any path q in U one has $\int_q F(z)\,dz \simeq \sum_{k=1}^{\infty} \int_q f_k(z)\,dz$.*

PROOF. The generating function for integrals is defined as $\int_q F(z,w)\,dz$. □

The proof of the following theorem is left to the reader.

THEOREM 4.6.7. *If* $\sum_{k=0}^{\infty} f_k \cong_p F$ *and* $\sum_{k=0}^{\infty} g_k \cong_p G$ *then* $\sum_{k=0}^{\infty}(f_k+g_k) \cong_p F+G$, $\sum_{k=1}^{\infty} f_k \cong_p F - f_0$, $\sum_{k=0}^{\infty} cf_k \cong_p cF$

Revision of evaluations. Now we are ready to revise the above evaluation equipped with the theory of analytic sums. Since all considered generating functions in this paragraph are single valued, the results do not depend on the choice of the path of summation. That is why we drop the indications of path below.

The equality (4.6.1) is the analytical equivalence generated by $\frac{1}{1-tx}$. The next equality (4.6.7) is the analytical equivalence by Lemma 4.6.4. The equality (4.6.3) is analytical equivalence due to Lemma 4.6.3. Termwise differentiation of (4.6.7) is correct by virtue of Theorem 4.6.5. Therefore the equality (4.6.5) is obtained by the restriction of an analytical equivalence. Hence the Euler sum of $\sum_{k=1}^{\infty}(-1)^k k^{2n}$ is equal to 0. Since the series $\sum_{k=1}^{\infty}(-1)^k k^{2n} z^k$ converges for $|z|<1$ its value coincides with the value of the generating function. And the limit $\lim\limits_{z\to 1-0}\sum_{k=1}^{\infty}(-1)^k k^{2n} z^k$ gives the Euler sum, which is zero. Hence as a result of our calculations we have found *Abel's sum* $\sum_{k=1}^{\infty}(-1)^k k^{2n} = 0$.

Now we choose another way to evaluate the Euler series. Substituting $x = e^{\pm i\theta}$ in (4.6.1) for $0<\theta<2\pi$ one gets

$$\begin{aligned} 1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+\ldots &\cong \frac{1}{1-e^{i\theta}}, \\ 1+e^{-i\theta}+e^{-2i\theta}+e^{-3i\theta}+\ldots &\cong \frac{1}{1-e^{-i\theta}}. \end{aligned} \tag{4.6.7}$$

Termwise addition of the above lines gives for $\theta\in(0,2\pi)$ the following equality

$$\cos\theta+\cos 2\theta+\cos 3\theta+\cdots \cong -\frac{1}{2}. \tag{4.6.8}$$

Integration of (4.6.8) from π to x with subsequent replacement of x by θ gives by Theorem 4.6.6:

$$\sum_{k=1}^{\infty}\frac{\sin k\theta}{k} \cong \frac{\pi-\theta}{2} \quad (0<\theta<2\pi).$$

A second integration of the same type gives

$$\sum_{k=1}^{\infty}\frac{\cos k\theta-(-1)^k}{k^2} \cong \frac{(\pi-\theta)^2}{4}.$$

Putting $\theta=\frac{\pi}{2}$ we get

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} \simeq \frac{\pi^2}{16}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} + 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$

one gets

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6}.$$

Problems.

1. Prove that the analytic sum of convolution of two series is equal to the product of analytic sums of the series.

2. Suppose that for all $n \in \mathbb{N}$ one has $A_n \simeq \sum_{k=0}^{\infty} a_{n,k}$ and $B_n \simeq \sum_{k=0}^{\infty} a_{k,n}$. Prove that the equality $\sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} B_k$ holds provided there is an analytic function $F(z,w)$ coinciding with $\sum a_{k,n} z^k w^n$ for sufficiently small $|w|$,$|z|$ which is defined on a domain containing a path joining $(0,0)$ with $(1,1)$ analytically extended to $(1,1)$ (i.e., $(1,1)$ is a regular point of $F(z,w)$).