4.2. Bernoulli Numbers

On the contents of the lecture. In this lecture we give explicit formulas for telescoping powers. These formulas involve a remarkable sequence of numbers, which were discovered by Jacob Bernoulli. They will appear in formulas for sums of series of reciprocal powers. In particular, we will see that $\frac{\pi^2}{6}$, the sum of Euler series, contains the second Bernoulli number $\frac{1}{6}$.

Summation Polynomials. Jacob Bernoulli found a general formula for the sum $\sum_{k=1}^{n} k^q$. To be precise he discovered that there is a sequence of numbers $B_0, B_1, B_2, \dots, B_n, \dots$ such that

$$\sum_{k=1}^{n} k^q = \sum_{k=0}^{q+1} B_k \frac{q^{\underline{k-1}} n^{q+1-k}}{k!}. \tag{4.2.1}$$

The first 11 of the *Bernoulli numbers* are $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}$. The right-hand side of (4.2.1) is a polynomial of degree $q+1$ in n. Let us denote this polynomial by $\psi_{q+1}(n)$. It has the following remarkable property: $\delta\psi_{q+1}(x) = (1+x)^q$. Indeed the latter equality holds for any natural value n of the variable, hence it holds for all x, because two polynomials coinciding in infinitely many points coincide. Replacing in (4.2.1) $q+1$ by m, n by x and reversing the order of summation, one gets the following:

$$\begin{aligned}
\psi_m(x) &= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{m-k-1}}}{(m-k)!} x^k \\
&= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)!}{k!(m-k)!} x^k \\
&= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{k-1}}}{k!} x^k.
\end{aligned}$$

Today's lecture is devoted to the proof of this Bernoulli theorem.

Telescoping powers. Newton's Formula represents x^m as a factorial polynomial $\sum_{k=0}^{n} \frac{\delta^k 0^m}{k!} x^{\underline{k}}$, where $\Delta^k 0^m$ denotes the value of $\delta^k x^m$ at $x = 0$. Since $\delta x^{\underline{k}} = k x^{\underline{k-1}}$, one immediately gets a formula for a polynomial $\phi_{m+1}(x)$ which telescopes x^m in the form

$$\phi_{m+1}(x) = \sum_{k=0}^{\infty} \frac{\Delta^k 0^m}{(k+1)!} x^{\underline{k+1}}$$

This polynomial has the property $\phi_{m+1}(n) = \sum_{k=0}^{n-1} k^m$ for all n.

The polynomials $\phi_m(x)$, as follows from Lemma 4.1.2, are characterized by two conditions:

$$\Delta\phi_m(x) = x^{m-1}, \qquad \phi_m(1) = 0.$$

LEMMA 4.2.1 (on differentiation). $\phi'_{m+1}(x) = \phi'_{m+1}(0) + m\phi_m(x)$.

PROOF. Differentiation of $\Delta\phi_{m+1}(x) = x^m$ gives $\Delta\phi'_{m+1}(x) = mx^{m-1}$. The polynomial $m\phi_m$ has the same differences, hence $\Delta(\phi'_{m+1}(x) - m\phi_m(x)) = 0$. By Lemma 4.1.2 this implies that $\phi'_{m+1}(x) - m\phi_m(x)$ is constant. Therefore, $\phi'_{m+1}(x) -$

$m\phi_m(x) = \phi'_{m+1}(0) - m\phi_m(0)$. But $\phi_m(1) = 0$ and $\phi_m(0) = \phi_m(1) - \delta\phi_m(0) = 0 - 0^{m-1} = 0$. □

Bernoulli polynomials. Let us introduce the m-th *Bernoulli number* B_m as $\phi'_{m+1}(0)$, and define the *Bernoulli polynomial* of degree $m > 0$ as $B_m(x) = m\phi_m(x) + B_m$. The Bernoulli polynomial $B_0(x)$ of degree 0 is defined as identically equal to 1. Consequently $B_m(0) = B_m$ and $B'_{m+1}(0) = (m+1)B_m$.

The Bernoulli polynomials satisfy the following condition:

$$\Delta B_m(x) = mx^{m-1} \quad (m > 0).$$

In particular, $\Delta B_m(0) = 0$ for $m > 1$, and therefore we get the following *boundary conditions* for Bernoulli polynomials:

$$\begin{aligned} B_m(0) = B_m(1) = B_m & \quad \text{for } m > 1, \text{ and} \\ B_1(0) = -B_1(1) = B_1. & \end{aligned}$$

The Bernoulli polynomials, in contrast to $\phi_m(x)$, are *normed*: their leading coefficient is equal to 1 and they have a simpler rule for differentiation:

$$B'_m(x) = mB_{m-1}(x)$$

Indeed, $B'_m(x) = m\phi'_m(x) = m((m-1)\phi_{m-1}(x) + \phi'_m(0)) = mB_{m-1}(x)$, by Lemma 4.2.1.

Differentiating $B_m(x)$ at 0, k times, we get $B_m^{(k)}(0) = m^{\underline{k-1}}B'_{m-k+1}(0) = m^{\underline{k-1}}(m-k+1)B_{m-k} = m^{\underline{k}}B_{m-k}$. Hence the Taylor formula gives the following representation of the Bernoulli polynomial:

$$B_m(x) = \sum_{k=0}^{m} \frac{m^{\underline{k}}B_{m-k}}{k!}x^k.$$

Characterization theorem. The following important property of Bernoulli polynomials will be called the *Balance property*:

$$\int_0^1 B_m(x)\,dx = 0 \quad (m > 0). \tag{4.2.2}$$

Indeed, $\int_0^1 B_m(x)\,dx = \int_0^1 (m+1)B'_{m+1}(x)\,dx = \Delta B_{m+1}(0) = 0$.

The Balance property and the Differentiation rule allow us to evaluate Bernoulli polynomials recursively. Thus, $B_1(x)$ has 1 as leading coefficient and zero integral on $[0,1]$; this allows us to identify $B_1(x)$ with $x - 1/2$. Integration of $B_1(x)$ gives $B_2(x) = x^2 - x + C$, where C is defined by (4.2.2) as $-\int_0^1 x^{\underline{2}}dx = \frac{1}{6}$. Integrating $B_2(x)$ we get $B_3(x)$ modulo a constant which we find by (4.2.2) and so on. Thus we obtain the following theorem:

THEOREM 4.2.2 (characterization). *If a sequence of polynomials $\{P_n(x)\}$ satisfies the following conditions:*

- $P_0(x) = 1$,
- $\int_0^1 P_n(x)\,dx = 0$ *for* $n > 0$,
- $P'_n(x) = nP_{n-1}(x)$ *for* $n > 0$,

then $P_n(x) = B_n(x)$ *for all* n.

Analytic properties.

LEMMA 4.2.3 (on reflection). *$B_n(x) = (-1)^n B_n(1-x)$ for any n.*

PROOF. We prove that the sequence $T_n(x) = (-1)^n B_n(1-x)$ satisfies all the conditions of Theorem 4.2.2. Indeed, $T_0 = B_0 = 1$,

$$\int_0^1 T_n(x)\,dx = (-1)^n \int_1^0 B_n(x)\,dx = 0$$

and

$$\begin{aligned} T_n(x)' &= (-1)^n B_n'(1-x) \\ &= (-1)^n n B_{n-1}(1-x)(1-x)' \\ &= (-1)^{n+1} n B_{n-1}(x) \\ &= n T_{n-1}(x). \end{aligned}$$

□

LEMMA 4.2.4 (on roots). *For any odd $n > 1$ the polynomial $B_n(x)$ has on $[0,1]$ just three roots: $0, \frac{1}{2}, 1$.*

PROOF. For odd n, the reflection Lemma 4.2.3 implies that $B_n(\frac{1}{2}) = -B_n(\frac{1}{2})$, that is $B_n(\frac{1}{2}) = 0$. Furthermore, for $n > 1$ one has $B_n(1) - B_n(0) = n0^{n-1} = 0$. Hence $B_n(1) = B_n(0)$ for any Bernoulli polynomial of degree $n > 1$. By the reflection formula for an odd n one obtains $B_n(0) = -B_n(1)$. Thus any Bernoulli polynomial of odd degree greater than 1 has roots $0, \frac{1}{2}, 1$.

The proof that there are no more roots is by contradiction. In the opposite case consider $B_n(x)$, of the least odd degree > 1 which has a root α different from the above mentioned numbers. Say $\alpha < \frac{1}{2}$. By Rolle's Theorem 4.1.7 $B_n'(x)$ has at least three roots $\beta_1 < \beta_2 < \beta_3$ in $(0,1)$. To be precise, $\beta_1 \in (0,\alpha)$, $\beta_2 \in (\alpha, \frac{1}{2})$, $\beta_3 \in (\frac{1}{2}, 1)$. Then $B_{n-1}(x)$ has the same roots. By Rolle's Theorem $B_{n-1}'(x)$ has at least two roots in $(0,1)$. Then at least one of them differs from $\frac{1}{2}$ and is a root of $B_{n-2}(x)$. By the minimality of n one concludes $n-2 = 1$. However, $B_1(x)$ has the only root $\frac{1}{2}$. This is a contradiction. □

THEOREM 4.2.5. *$B_n = 0$ for any odd $n > 1$. For $n = 2k$, the sign of B_n is $(-1)^{k+1}$. For any even n one has either $B_n = \max_{x\in[0,1]} B_n(x)$ or $B_n = \min_{x\in[0,1]} B_n(x)$. The first occurs for positive B_n, the second for negative.*

PROOF. $B_{2k+1} = B_{2k+1}(0) = 0$ for $k > 0$ by Lemma 4.2.4. Above we have found that $B_2 = \frac{1}{6}$. Suppose we have established that $B_{2k} > 0$ and that this is the maximal value for $B_{2k}(x)$ on $[0,1]$. Let us prove that $B_{2k+2} < 0$ and it is the minimal value for $B_{2k+2}(x)$ on $[0,1]$. The derivative of B_{2k+1} in this case is positive at the ends of $[0,1]$, hence $B_{2k+1}(x)$ is positive for $0 < x < \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$, by Lemma 4.2.4 on roots and the Theorem on Intermediate Values. Hence, $B_{2k+2}'(x) > 0$ for $x < \frac{1}{2}$ and $B_{2k+2}'(x) < 0$ for $x > \frac{1}{2}$. Therefore, $B_{2k+2}(x)$ takes the maximal value in the middle of $[0,1]$ and takes the minimal values at the ends of $[0,1]$. Since the integral of the polynomial along $[0,1]$ is zero and the polynomial is not constant, its minimal value has to be negative. The same arguments prove that if B_{2k} is negative and minimal, then B_{2k+2} is positive and maximal. □

LEMMA 4.2.6 (Lagrange Formula). *If f is a differentiable function on $[a,b]$, then there is a $\xi \in (a,b)$, such that*

$$f(b) = f(a) + f'(\xi)\frac{f(b)-f(a)}{b-a}. \tag{4.2.3}$$

PROOF. The function $g(x) = f(x) - (x-a)\frac{f(b)-f(a)}{b-a}$ is differentiable on $[a,b]$ and $g(b) = g(a) = 0$. By Rolle's Theorem $g'(\xi) = 0$ for some $\xi \in [a,b]$. Hence $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. Substitution of this value of $f'(\xi)$ in (4.2.3) gives the equality. □

Generating function. The following function of two variables is called the *generating function of Bernoulli polynomials.*

$$B(x,t) = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \tag{4.2.4}$$

Since $B_k \leq \frac{k!}{2^k}$, the series on the right-hand side converges for $t < 2$ for any x. Let us differentiate it termwise as a function of x, for a fixed t. We get $\sum_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x,t)$. Consequently $(\ln B(x,t))'_x = \frac{B'_x(x,t)}{B(x,t)} = t$ and $\ln B(x,t) = xt + c(t)$, where the constant $c(t)$ depends on t. It follows that $B(x,t) = \exp(xt)k(t)$, where $k(t) = \exp(c(t))$. For $x = 0$ we get $B(0,t) = k(t) = \sum_{k=0}^{\infty} B_k\frac{t^k}{k!}$. To find $k(t)$ consider the difference $B(x+1,t) - B(x,t)$. It is equal to $\exp(xt+t)k(t) - \exp(xt)$. On the other hand the difference is $\sum_{k=0}^{\infty} \Delta B_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x,t)$. Comparing these expressions we get explicit formulas for the generating functions of Bernoulli numbers:

$$k(t) = \frac{t}{\exp t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}t^k,$$

and Bernoulli polynomials:

$$B(x,t) = \sum_{k=+}^{0-1} B_k(x)\frac{t^k}{k!} = \frac{t\exp(tx)}{\exp t - 1}.$$

From (4.2.4) one gets $t = (\exp t - 1)\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}$. Substituting $\exp t - 1 = \sum_{k=1}^{\infty}\frac{t^k}{k!}$ in this equality, by the Uniqueness Theorem 3.6.9, one gets the equalities for the coefficients of the power series

$$\sum_{k=1}^{n} \frac{B_{n-k}}{(n-k)!k!} = 0 \quad \text{for } n > 1.$$

Add $\frac{B_n}{n!}$ to both sides of this equality and multiply both sides by $n!$ to get

$$B_n = \sum_{k=0}^{n} \frac{B_k n^{\underline{k}}}{k!} \quad \text{for } n > 1. \tag{4.2.5}$$

The latter equality one memorizes via the formula $B^n = (B+1)^n$, where after expansion of the right hand side, one should move down all the exponents at B turning the powers of B into Bernoulli numbers.

Now we are ready to prove that

$$\phi_m(1+x) = \frac{B_m(x+1)}{m} - \frac{B_m}{m} = \sum_{k=0}^{m} B_{m-k}\frac{(m-1)^{\underline{k-1}}}{k!}x^k = \psi_m(x). \tag{4.2.6}$$

Putting $x = 0$ in the right hand side one gets $\psi_m(0) = B_m(m-1)^{\underline{-1}} = \frac{B_m}{m}$. The left-hand side takes the same value at $x = 0$, because $B_m(1) = B_m(0) = B_m$. It remains to prove the equality of the coefficients in (4.2.6) for positive degrees.

$$\begin{aligned}\frac{B_m(x+1)}{m} &= \frac{1}{m}\sum_{k=0}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}(1+x)^k\\ &= \frac{1}{m}\sum_{k=0}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}\sum_{j=0}^{k}\frac{k^{\underline{j}}x^j}{j!}\end{aligned}$$

Now let us change the summation order and change the summation index of the interior sum by $i = m - k$.

$$\begin{aligned}&= \frac{1}{m}\sum_{j=0}^{m}\frac{x^j}{j!}\sum_{k=j}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}k^{\underline{j}}\\ &= \frac{1}{m}\sum_{j=0}^{m}\frac{x^j}{j!}\sum_{i=0}^{m-j}\frac{m^{\underline{m-i}}B_i}{(m-i)!}(m-i)^{\underline{j}}\end{aligned}$$

Now we change $\frac{m^{\underline{m-i}}(m-i)^{\underline{j}}}{(m-i)!}$ by $\frac{(m-j)^{\underline{i}}m^{\underline{j}}}{i!}$ and apply the identity (4.2.5).

$$\begin{aligned}&= \sum_{j=0}^{m}\frac{x^j m^{\underline{j}}}{mj!}\sum_{i=0}^{m-j}\frac{B_i(m-j)^{\underline{i}}}{i!}\\ &= \sum_{j=0}^{m}\frac{(m-1)^{\underline{j-1}}x^j}{j!}B_{m-j}.\end{aligned}$$

Problems.

1. Evaluate $\int_0^1 B_n(x)\sin 2\pi x\,dx$.
2. Expand $x^4 - 3x^2 + 2x - 1$ as a polynomial in $(x-2)$.
3. Calculate the first 20 Bernoulli numbers.
4. Prove the inequality $|B_n(x)| \le |B_n|$ for even n.
5. Prove the inequality $|B_n(x)| \le \frac{n}{4}|B_{n-1}|$ for odd n.
6. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \int_0^1 f'(x)B_1(x)\,dx$.
7. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \frac{\Delta f'(0)}{2} - \int_0^1 f''(x)B_2(x)\,dx$.
8. Deduce $\Delta B_n(x) = nx^{n-1}$ from the balance property and the differentiation rule.
9. Prove that $B_n(x) = B_n(1-x)$, using the generating function.
10. Prove that $B_{2n+1} = 0$, using the generating function.
11. Prove that $B_m(nx) = n^{m-1}\sum_{k=0}^{n-1} B_m\left(x + \frac{k}{n}\right)$.
12. Evaluate $B_n(\frac{1}{2})$.
13. Prove that $B_{2k}(x) = P(B_2(x))$, where $P(x)$ is a polynomial with positive coefficient (Jacobi Theorem).
14. Prove that $B_n = \sum_{k=0}^{\infty}(-1)^k\frac{\Delta^k 0^n}{k+1}$.
***15.** Prove that $B_m + \sum \frac{1}{k+1}[k+1$ is prime and k is divisor of $m]$ is an integer (Staudt Theorem).