3.5. Residue Theory

On the contents of the lecture. At last, the reader learns something, which Euler did not know, and which he would highly appreciate. Residue theory allows one to evaluate a lot of integrals which were not accessible by the Newton-Leibniz formula.

Monotone curve. A *monotone curve* Γ is defined as a subset of the complex plane which is the image of a monotone path. Nonempty intersections of vertical and horizontal lines with a monotone curve are either points or closed intervals.

The points of the monotone curve which have an extremal sum of real and imaginary parts are called its *endpoints*, the other points of the curve are called its *interior* points.

A continuous injective monotone path p whose image coincides with Γ is called a *parametrization* of Γ.

LEMMA 3.5.1. *Let $p_1\colon [a,b] \to \mathbb{C}$ and $p_2\colon [c,d] \to \mathbb{C}$ be two parametrizations of the same monotone curve Γ. Then $p_1^{-1}p_2\colon [c,d] \to [a,b]$ is a continuous monotone bijection.*

PROOF. Set $P_i(t) = \operatorname{Re} p_i(t) + \operatorname{Im} p_i(t)$. Then P_1 and P_2 are continuous and strictly monotone. And $p_1(t) = p_2(\tau)$ if and only if $P_1(t) = P_2(\tau)$. Hence $p_1^{-1}p_2 = P_1^{-1}P_2$. Since P_1^{-1} and P_2 are monotone continuous, the composition $P_1^{-1}P_2$ is monotone continuous. $\square$

Orientation. One says that two parametrizations p_1 and p_2 of a monotone curve Γ have the *same orientation*, if $p_1^{-1}p_2$ is increasing, and one says that they have *opposite orientations*, if $p_1^{-1}p_2$ is decreasing.

Orientation divides all parametrizations of a curve into two classes. All elements of one orientation class have the same orientation and all elements of the other class have the opposite orientation.

An oriented curve is a curve with fixed *circulation direction*. A choice of orientation means distinguishing one of the orientation classes as positive, corresponding to the oriented curve. For a monotone curve, to specify its orientation, it is sufficient to indicate which of its endpoints is its beginning and which is the end. Then all positively oriented parametrizations start with its beginning and finish at its end, and negatively oriented parametrizations do the opposite.

If an oriented curve is denoted by Γ, then its *body*, the curve without orientation, is denoted $|\Gamma|$ and the curve with the same body but with opposite orientation is denoted $-\Gamma$.

If Γ' is a monotone curve which is contained in an oriented curve Γ, then one defines the *induced orientation* on Γ' by Γ as the orientation of a parametrization of Γ' which extends to a positive parametrization of Γ.

Line integral. One defines the integral $\int_\Gamma f(z)\,dg(z)$ along a oriented monotone curve Γ as the integral $\int_p f(z)\,dg(z)$, where p is a positively oriented parametrization of Γ. This definition does not depend on the choice of p, because different parametrizations are obtained from each other by an increasing change of variable (Lemma 3.5.1).

One defines a *partition of a curve* Γ by a point x as a pair of monotone curves Γ_1, Γ_2, such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = x$. And we write in this case $\Gamma = \Gamma_1 + \Gamma_2$.

The Partition Rule for the line integral is

$$\int_{\Gamma_1+\Gamma_2} f(z)\, dg(z) = \int_{\Gamma_1} f(z)\, dg(z) + \int_{\Gamma_2} f(z)\, dg(z), \tag{3.5.1}$$

where the orientations on Γ_i are induced by an orientation of Γ. To prove the Partition Rule consider a positive parametrization $p\colon [a,b] \to \Gamma$. Then the restrictions of p over $[a, p^{-1}(x)]$ and $[p^{-1}(x), b]$ give positive parametrizations of Γ_1 and Γ_2. Hence the equality (3.5.1) follows from $\int_a^{p^{-1}(x)} f(z)\, dg(z) + \int_{p^{-1}(x)}^b f(z)\, dg(z) = \int_a^b f(z)\, dg(z)$.

A sequence of oriented monotone curves $\{\Gamma_k\}_{k=1}^n$ with disjoint interiors is called a *chain* of monotone curves and denoted by $\sum_{k=1}^n \Gamma_k$. The body of a chain $C = \sum_{k=1}^n \Gamma_k$ is defined as $\bigcup_{k=1}^n |\Gamma_k|$ and denoted by $|C|$. The interior of the chain is defined as the union of interiors of its elements.

The integral of a form $f\,dg$ along the chain is defined as $\int_{\sum_{k=1}^n \Gamma_k} f\,dg = \sum_{k=1}^n \int_{\Gamma_k} f\,dg$.

One says that two chains $\sum_{k=1}^n \Gamma_k$ and $\sum_{k=1}^m \Gamma'_k$ have the same orientation, if the orientations induced by Γ_k and Γ'_j on $\Gamma_k \cap \Gamma'_j$ coincide in the case when $\Gamma_k \cap \Gamma'_j$ has a nonempty interior. Two chains with the same body and orientation are called *equivalent.*

LEMMA 3.5.2. *If two chains $C = \sum_{k=1}^n \Gamma_k$ and $C' = \sum_{k=1}^m \Gamma'_k$ are equivalent then the integrals along these chains coincide for any form $f dg$.*

PROOF. For any interior point x of the chain C, one defines the subdivision of C by x as $\Gamma_j^+ + \Gamma_j^- + \sum_{k=1}^n \Gamma_k[k \neq j]$, where Γ_j is the curve containing x and $\Gamma_j^+ + \Gamma_j^-$ is the partition of Γ by x. The subdivision does not change the integral along the chain due to the Partition Rule.

Hence we can subdivide C step by step by endpoints of C' to construct a chain Q whose endpoints include all endpoints of P'. And the integral along Q is the same as along P. Another possibility to construct Q is to subdivide C' by endpoints of C. This construction shows that the integral along Q coincides with the integral along C'. Hence the integrals along C and C' coincide. □

Due to this lemma, one can introduce the integral of a differential form along any oriented piecewise monotone curve Γ. To do this one considers a *monotone partition* of Γ into a sequence $\{\Gamma_k\}_{k=1}^n$ of monotone curves with disjoint interiors and equip all Γ_k with the induced orientation. One gets a chain and the integral along this chain does not depend on the partition.

Contour integral. A *domain* D is defined as a connected bounded part of the plane with piecewise monotone boundary. The boundary of D denoted ∂D is the union of finitely many monotone curves. And we suppose that $\partial D \subset D$, that is we consider a closed domain.

For a monotone curve Γ, which is contained in the boundary of a domain D, one defines the *induced orientation* of Γ by D as the orientation of a parametrization which leaves D on the left during the movement along Γ around D.

One introduces the integral $\oint_{\partial D} f(z)dg(z)$ as the integral along any chain whose body coincides with ∂D and whose orientations of curves are induced by D.

Due to Lemma 3.5.2 the choice of chain does not affect the integral.

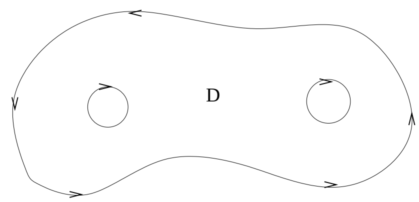

FIGURE 3.5.1. Contour integral

LEMMA 3.5.3. *Let D be a domain and l be either a vertical or a horizontal line, which bisects D into two parts: D' and D'' lying on the different sides of l. Then* $\oint_{\partial D} f(z)dz = \oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz$.

PROOF. The line l intersects the boundary of D in a finite sequence of points and intervals $\{J_k\}_{k=1}^m$.

Set $\partial' D = \partial D \cap \partial D'$ and $\partial'' D = \partial D \cap \partial D''$. The intersection $\partial' D \cap \partial'' D$ consists of finitely many points. Indeed, the interior points of J_k do not belong to this intersection, because their small neighborhoods have points of D only from one side of l. Hence

$$\int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

The boundary of D' consists of $\partial' D$ and some number of intervals. And the boundary of D'' consists of $\partial'' D$ and the same intervals, but with opposite orientation. Therefore

$$L = \int_{l\cap\partial D'} f(z)\,dz = -\int_{l\cap\partial D''} f(z)\,dz.$$

On the other hand

$$\oint_{\partial D'} f(z)dz = \int_{\partial' D} f(z)\,dz + L \text{ and } \oint_{\partial D''} f(z)dz = \int_{\partial'' D} f(z)\,dz - L,$$

hence

$$\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

□

LEMMA 3.5.4 (Estimation). *If $|f(z)| \leq B$ for any z from a body of a chain $C = \sum_{k=1}^n \Gamma_k$, then* $\left|\int_C f(z)\,dz\right| \leq 4Bn \operatorname{diam}|C|$.

PROOF. By Lemma 3.3.6 for any k one has $\left|\int_{\Gamma_k} f(z)\,dz\right| \leq 4B|A_k - B_k| \leq 4B \operatorname{diam}|C|$ where A_k and B_k are endpoints of Γ_k. The summation of these inequalities proves the lemma. □

THEOREM 3.5.5 (Cauchy). *If a function f is complex differentiable in a domain D then* $\oint_{\partial D} f(z)dz = 0$.

PROOF. Fix a rectangle R with sides parallel to the coordinate axis which contains D and denote by $|R|$ its area and by P its perimeter.

The proof is by contradiction. Suppose $\left|\oint_{\partial D} f(z)\,dz\right| \neq 0$. Denote by c the ratio of $\left|\oint_{\partial D} f(z)\,dz\right|/|R|$. We will construct a nested sequence of rectangles $\{R_k\}_{k=0}^{\infty}$ such that

- $R_0 = R$, $R_{k+1} \subset R_k$;
- R_{2k} is similar to R;
- $|\oint_{\partial(R_k\cap D)} f(z)\,dz| \geq c|R_k|$, where $|R_k|$ is the area of R_k.

The induction step: Suppose R_k is already constructed. Divide R_k in two equal rectanges R_k' and R_k'' by drawing either a vertical, if k is even, or a horizontal, if k is odd, interval joining the middles of the opposite sides of R_k. Set $D_k = D \cap R_k$, $D' = D \cap R_k'$, $D'' = D \cap R_k''$. We state that at least one of the following inequalities holds:

$$\left|\oint_{\partial D'} f(z)dz\right| \geq c|R_k'|, \qquad \left|\oint_{\partial D''} f(z)dz\right| \geq c|R_k''|. \tag{3.5.2}$$

Indeed, in the opposite case one gets

$$\left|\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz\right| < c|R_k'| + c|R_k'| = c|R_k|.$$

Since $\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \oint_{\partial D_k} f(z)dz$ by Lemma 3.5.3 we get a contradiction with the hypothesis $|\int_{p_k} f(z)\,dz| \geq c|R_k|$. Hence, one of the inequalities (3.5.2) holds. If the first inequality holds we set $R_{k+1} = R_k'$ else we set $R_{k+1} = R_k''$.

After constructing the sequence $\{R_k\}$, consider a point z_0 belonging to $\bigcap_{k=1}^{\infty} R_k$. This point belongs to D, because all its neighborhoods contain points of D. Consider the linearization $f(z) = f(z_0) + f'(z_0)(z-z_0) + o(z)(z-z_0)$. Since $\oint_{\partial D_k}(f(z_0) + f'(z_0)(z-z_0))dz = 0$ one gets

$$\left|\oint_{\partial D_k} o(z)(z-z_0)dz\right| = \left|\oint_{\partial D_k} f(z)dz\right| \geq c|R_k|. \tag{3.5.3}$$

The boundary of D_k is contained in the union $\partial R_k \cup R_k \cap \partial D$. Consider a monotone partition $\partial D = \sum_{k=1}^{n} \Gamma_k$. Since the intersection of R_k with a monotone curve is a monotone curve, one concludes that $\partial D \cap R_k$ is a union of at most n monotone curves. As ∂R_k consists of 4 monotone curves we get that ∂D_k is as a body of a chain with at most $4+n$ monotone curves.

Denote by P_k the perimeter of R_k. And suppose that $o(x)$ is bounded in R_k by a constant o_k. Then $|o(x)(z-z_0)| \leq P_k o_k$ for all $z \in R_k$.

Since $\operatorname{diam} \partial D_k \leq \frac{P_k}{2}$ by the Estimation Lemma 3.5.4, we get the following inequality:

$$\left|\oint_{\partial D_k} o(z)(z-z_0)dz\right| \leq 4(4+n)P_k o_k \frac{P_k}{2} = 2(4+n)o_k P_k^2. \tag{3.5.4}$$

The ratio $P_k^2/|R_k|$ is constant for even k. Therefore the inequalities (3.5.3) and (3.5.4) contradict each other for $o_k < \frac{c|R_k|}{2(4+n)P_k^2} = \frac{c|R|}{2(4+n)P^2}$. However the inequality $|o(x)| < \frac{c|R|}{2(4+n)P^2}$ holds for some neighborhood V of z_0 as $o(x)$ is infinitesimally small at z_0. This is a contradiction, because V contains some R_{2k}. $\square$

Residues. By $\oint_{z_0}^{r} f(z)\,dz$ we denote the integral along the boundary of the disk $\{|z - z_0| \le r\}$.

LEMMA 3.5.6. *Suppose a function $f(z)$ is complex differentiable in the domain D with the exception of a finite set of points $\{z_k\}_{k=1}^{n}$. Then*

$$\oint_{\partial D} f(z)dz = \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz,$$

where r is so small that all disks $|z - z_k| < r$ are contained in D and disjoint.

PROOF. Denote by D' the complement of the union of the disks in D. Then $\partial D'$ is the union of ∂D and the boundary circles of the disks. By the Cauchy Theorem 3.5.5, $\oint_{\partial D'} f(z)dz = 0$. On the other hand this integral is equal to the sum $\oint_{\partial D} f(z)dz$ and the sum of integrals along boundaries of the circles. The orientation induced by D' onto the boundaries of these circles is opposite to the orientation induced from the circles. Hence

$$0 = \oint_{\partial D'} f(z)dz = \oint_{\partial D} f(z)dz - \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz.$$

□

A *singular point* of a complex function is defined as a point where either the function or its derivative are not defined. A singular point is called isolated, if it has a neighborhood, where it is the only singular point. A point is called a *regular point* if it not a singular point.

One defines the *residue* of f at a point z_0 and denotes it as $\operatorname{res}_{z_0} f$ as the limit $\lim_{r\to 0} \frac{1}{2\pi i} \oint_{z_0}^{r} f(z)dz$. The above lemma shows that this limit exists for any isolated singular point and moreover, that all integrals along sufficiently small circumferences in this case are the same.

Since in all regular points the residues are 0 the conclusion of Lemma 3.5.6 for a function with finitely many singular points can be presented in the form:

$$\boxed{\oint_{\partial D} f(z)dz = 2\pi i \sum_{z\in D} \operatorname{res}_z f.} \tag{3.5.5}$$

An isolated singular point z_0 is called a *simple pole* of a function $f(z)$ if there exists a nonzero limit $\lim_{z\to z_0} f(z)(z - z_0)$.

LEMMA 3.5.7. *If z_0 is a simple pole of $f(z)$ then* $\operatorname{res}_{z_0} f = \lim_{z\to z_0}(z - z_0)f(z)$.

PROOF. Set $L = \lim_{z\to z_0}(z - z_0)f(z)$. Then $f(z) = L + \frac{o(z)}{(z-z_0)}$, where $o(z)$ is infinitesimally small at z_0. Hence

$$\oint_{z_0}^{r} \frac{o(z)\,dz}{z - z_0} = \oint_{z_0}^{r} f(z)\,dz - \oint_{z_0}^{r} \frac{L}{z - z_0}\,dz. \tag{3.5.6}$$

Since the second integral from the right-hand side of (3.5.6) is equal to $2L\pi i$ and the other one is equal to $2\pi i \operatorname{res}_{z_0} f$ for sufficiently small r, we conclude that the integral from the left-hand side also is constant for sufficiently small r. To prove that $L = \operatorname{res}_{z_0} f$ we have to prove that this constant $c = \lim_{r\to 0} \oint_{z_0}^{r} \frac{o(z)}{z-z_0}\,dz$ is 0. Indeed, assume that $|c| > 0$. Then there is a neighborhood U of z_0 such that $|o(z)| \le \frac{|c|}{32}$

for all $z \in U$. Then one gets a contradiction by estimation of $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right|$ (which is equal to $|c|$ for sufficiently small r) from above by $\frac{|c|}{\sqrt{2}}$ for r less than the radius of U. Indeed, the integrand is bounded by $\frac{|c|}{32r}$ and the path of integration (the circle) can be divided into four monotone curves of diameter $r\sqrt{2}$: quarters of the circle. Hence by the Estimation Lemma 3.5.4 one gets $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right| \leq 16\sqrt{2}\frac{|c|}{32} = \frac{|c|}{\sqrt{2}}$. □

REMARK 3.5.8. *Denote by $\Gamma(r,\phi,z_0)$ an arc of the circle $|z-z_0| = r$, whose angle measure is ϕ. Under the hypothesis of Lemma 3.5.7 the same arguments prove the following*

$$\lim_{r\to 0}\int_{\Gamma(\phi,r,0z)} f(z)\,dz = i\phi \lim_{z\to z_0} f(z)(z-z_0).$$

Problems.

1. Evaluate $\oint_1^1 \frac{dz}{1+z^4}$.
2. Evaluate $\oint_0^1 \frac{dz}{\sin z}$.
3. Evaluate $\oint_0^1 \frac{dz}{e^z-1}$.
4. Evaluate $\oint_0^1 \frac{dz}{z^2}$.
5. Evaluate $\oint_0^1 \sin\frac{1}{z}\,dz$.
6. Evaluate $\oint_0^1 ze^{\frac{1}{z}}\,dz$.
7. Evaluate $\oint_0^{5/2} z^2\cot\pi z\,dz$.
8. Evaluate $\oint_2^{\frac{1}{2}} \frac{z\,dz}{(z-1)(z-2)^2}$.
9. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{5+3\cos\phi}$.
10. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{(1+\cos^2\phi)^2}$.
11. Evaluate $\int_0^{2\pi} \frac{d\phi}{(1+\cos\phi)^2}$.
12. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$.
13. Evaluate $\int_0^{+\infty} \frac{dx}{(1+x^2)(4+x^2)}$.
14. Evaluate $\int_{-\infty}^{+\infty} \frac{1+x^2}{1+x^4}$.
15. Evaluate $\int_{-\infty}^{+\infty} \frac{x^3}{1+x^6}\,dx$.