3.2. Exponential Functions

On the contents of the lecture. We solve the principal differential equation $y' = y$. Its solution, the exponential function, is expanded into a power series. We become acquainted with hyperbolic functions. And, finally, we prove the irrationality of e.

Debeaune's problem. In 1638 F. Debeaune posed Descartes the following geometrical problem: find a curve $y(x)$ such that for each point P the distances between V and T, the points where the vertical and the tangent lines cut the x-axis, are always equal to a given constant a. Despite the efforts of Descartes and Fermat, this problem remained unsolved for nearly 50 years. In 1684 Leibniz solved the problem via infinitesimal analysis of this curve: let x, y be a given point P (see the picture). Then increase x by a small increment of b, so that y increases almost by yb/a. Indeed, in small the curve is considered as the line. Hence the point P' of the curve with vertical projection V', one considers as lying on the line TP. Hence the triangle $TP'V'$ is similar to TPV. As $TV = a$, $TV' = b+a$ this similarity gives the equality $\frac{a+b}{y+\delta y} = \frac{a}{y}$ which gives $\delta y = yb/a$.

Repeating we obtain a sequence of values

$$y,\ y(1+\tfrac{b}{a}),\ y(1+\tfrac{b}{a})^2,\ y(1+\tfrac{b}{a})^3, \dots.$$

We see that "in small" $y(x)$ transforms an arithmetic progression into a geometric one. This is the inverse to what the logarithm does. And the solution is a function which is the inverse to a logarithmic function. Such functions are called *exponential*.

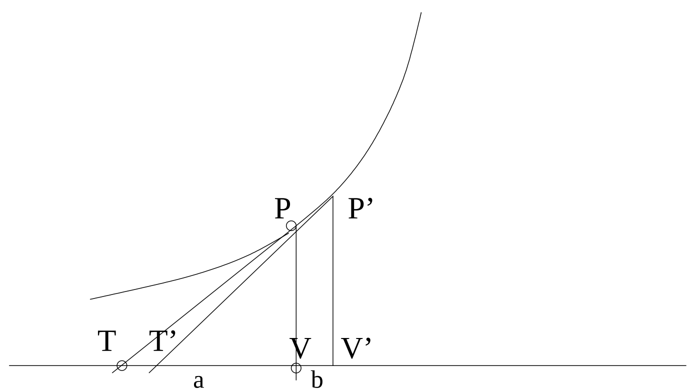

FIGURE 3.2.1. Debeaune's problem

Tangent line and derivative. A tangent line to a smooth convex curve at a point x is defined as a straight line such that the line intersects the curve just at x and the whole curve lies on one side of the line.

We state that the equation of the tangent line to the graph of function f at a point x_0 is just the principal part of linearization of $f(x)$ at x_0. In other words, the equation is $y = f(x_0) + (x - x_0)f'(x_0)$.

First, consider the case of a horizontal tangent line. In this case $f(x_0)$ is either maximal or minimal value of $f(x)$.

LEMMA 3.2.1. *If a function $f(x)$ is differentiable at an extremal point x_0, then $f'(x_0) = 0$.*

PROOF. Consider the linearization $f(x) = f(x_0)+f'(x_0)(x-x_0)+o(x))(x-x_0)$. Denote $x-x_0$ by δx, and $f(x)-f(x_0)$ by $\delta f(x)$. If we suppose that $f'(x_0) \neq 0$, then, for sufficiently small δx, we get $|o(x\pm\delta x)| < |f'(x)|$, hence $\operatorname{sgn}(f'(x_0)+o(x_0+\delta x)) = \operatorname{sgn}(f'(x_0)+o(x_0-\delta x))$, and $\operatorname{sgn}\delta f(x) = \operatorname{sgn}\delta x$. Therefore the sign of $\delta f(x)$ changes whenever the sign of δx changes. The sign of $\delta f(x)$ cannot be positive if $f(x_0)$ is the maximal value of $f(x)$, and it cannot be negative if $f(x_0)$ is the minimal value. This is the contradiction. □

THEOREM 3.2.2. *If a function $f(x)$ is differentiable at x_0 and its graph is convex, then the tangent line to the graph of $f(x)$ at x_0 is $y = f(x_0)+f'(x_0)(x-x_0)$.*

PROOF. Let $y = ax+b$ be the equation of a tangent line to the graph $y = f(x)$ at the point x_0. Since $ax+b$ passes through x_0, one has $ax_0+b = f(x_0)$, therefore $b = f(x_0)-ax_0$, and it remains to prove that $a = f'(x_0)$. If the tangent line $ax+b$ is not horizontal, consider the function $g(x) = f(x)-ax$. At x_0 it takes either a maximal or a minimal value and $g'(x_0) = 0$ by Lemma 3.2.1. On the other hand, $g'(x_0) = f'(x_0)-a$. □

Differential equation. The Debeaune problem leads to a so-called differential equation on $y(x)$. To be precise, the equation of the tangent line to $y(x)$ at x_0 is $y = y(x_0)+y'(x_0)(x-x_0)$. So the x-coordinate of the point T can be found from the equation $0 = y(x_0)+y'(x_0)(x-x_0)$. The solution is $x = x_0 - \frac{y(x_0)}{y'(x_0)}$. The x-coordinate of V is just x_0. Hence TV is equal to $\frac{y(x_0)}{y'(x_0)}$. And Debeaune's requirement is $\frac{y(x_0)}{y'(x_0)} = a$. Or $ay' = y$. Equations that include derivatives of functions are called *differential equations*. The equation above is the simplest differential equation. Its solution takes one line. Indeed passing to differentials one gets $ay'\,dx = y\,dx$, further $ady = y\,dx$, then $a\frac{dy}{y} = dx$ and $a\,d\ln y = dx$. Hence $a\ln y = x+c$ and finally $y(x) = \exp(c+\frac{x}{a})$, where $\exp x$ denotes the function inverse to the natural logarithm and c is an arbitrary constant.

Exponenta. The function inverse to the natural logarithm is called the *exponential function*. We shall call it the *exponenta* to distinguish it from other exponential functions.

THEOREM 3.2.3. *The exponenta is the unique solution of the differential equation $y' = y$ such that $y'(0) = 1$.*

PROOF. Differentiation of the equality $\ln\exp x = x$ gives $\frac{\exp' x}{\exp x} = 1$. Hence $\exp x$ satisfies the differential equation $y' = y$. For $x = 0$ this equation gives $\exp'(0) = \exp 0$. But $\exp 0 = 1$ as $\ln 1 = 0$.

For the converse, let $y(x)$ be a solution of $y' = y$. The derivative of $\ln y$ is $\frac{y'}{y} = 1$. Hence the derivative of $\ln y(x) - x$ is zero. By Theorem 3.1.16 from the previous lecture, this implies $\ln y(x) - x = c$ for some constant c. If $y'(0) = 1$, then $y(0) = 1$ and $c = \ln 1 - 0 = 0$. Therefore $\ln y(x) = x$ and $y(x) = \exp\ln y(x) = \exp x$. □

Exponential series. Our next goal is to prove that

$$\exp x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{3.2.1}$$

where $0! = 1$. This series is absolutely convergent for any x. Indeed, the ratio of its subsequent terms is $\frac{x}{n}$ and tends to 0, hence it is eventually majorized by any geometric series.

Hyperbolic functions. To prove that the function presented by series (3.2.1) is virtually monotone, consider its odd and even parts. These parts represent the so-called *hyperbolic functions*: hyperbolic sine $\operatorname{sh} x$, and hyperbolic cosine $\operatorname{ch} x$.

$$\operatorname{sh}(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \operatorname{ch}(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

The hyperbolic sine is an increasing function, as all odd powers are increasing over the whole line. The hyperbolic cosine is increasing for positive x and decreasing for negative. Hence both are virtually monotone; and so is their sum.

Consider the integral $\int_0^x \operatorname{sh} t\, dt$. As all terms of the series representing sh are increasing, we can integrate the series termwise. This integration gives $\operatorname{ch} x$. As $\operatorname{sh} x$ is locally bounded, $\operatorname{ch} x$ is continuous by Theorem 3.1.13. Consider the integral $\int_0^x \operatorname{ch} t\, dt$; here we also can integrate the series representing ch termwise, because for positive x all the terms are increasing, and for negative x, decreasing. Integration gives $\operatorname{sh} x$, since the continuity of $\operatorname{ch} x$ was already proved. Further, by Theorem 3.1.13 we get that $\operatorname{sh} x$ is differentiable and $\operatorname{sh}' x = \operatorname{ch} x$. Now returning to the equality $\operatorname{ch} x = \int_0^x \operatorname{sh} t\, dt$ we get $\operatorname{ch}' x = \operatorname{sh} x$, as $\operatorname{sh} x$ is continuous.

Therefore $(\operatorname{sh} x + \operatorname{ch} x)' = \operatorname{ch} x + \operatorname{sh} x$. And $\operatorname{sh} 0 + \operatorname{ch} 0 = 0 + 1 = 1$. Now by the above Theorem 3.2.3 one gets $\exp x = \operatorname{ch} x + \operatorname{sh} x$.

Other exponential functions. The exponenta as a function inverse to the logarithm transforms sums into products. That is, for all x and y one has

$$\exp(x+y) = \exp x \exp y.$$

A function which has this property (i.e., transform sums into products) is called *exponential*.

THEOREM 3.2.4. *For any positive a there is a unique differentiable function denoted by a^x called the exponential function to base a, such that $a^1 = a$ and $a^{x+y} = a^x a^y$ for any x, y. This function is defined by the formula* $\exp a \ln x$.

PROOF. Consider $l(x) = \ln a^x$. This function has the property $l(x+y) = l(x) + l(y)$. Therefore its derivative at any point is the same: it is equal to $k = \lim_{x\to 0} \frac{l(x)}{x}$. Hence the function $l(x) - kx$ is constant, because its derivative is 0. This constant is equal to $l(0)$, which is 0. Indeed $l(0) = l(0+0) = l(0) + l(0)$. Thus $\ln a^x = kx$. Substituting $x = 1$ one gets $k = \ln a$. Hence $a^x = \exp(x \ln a)$. So if a differentiable exponential function with base a exists, it coincides with $\exp(x \ln a)$. On the other hand it is easy to see that $\exp(x \ln a)$ satisfies all the requirements for an exponential function to base a, that is $\exp(1 \ln a) = a$, $\exp((x+y)\ln a) = \exp(x \ln a)\exp(y \ln a)$; and it is differentiable as composition of differentiable functions. □

Powers. Hence for any positive a and any real b, one defines the number a^b as

$$a^b = \exp(b \ln a)$$

a is called the base, and b is called the exponent. For rational b this definition agrees with the old definition. Indeed if $b = \frac{p}{q}$ then the properties of the exponenta and the logarithm imply $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Earlier, we have defined logarithms to base b as the number c, and called the *logarithm of b to base a*, if $a^c = b$ and denoted $c = \log_a b$.

The basic properties of powers are collected here.

THEOREM 3.2.5.

$$\left(a^b\right)^c = a^{(bc)}, \quad a^{b+c} = a^b a^c, \quad (ab)^c = a^c b^c, \quad \log_a b = \frac{\log b}{\log a}.$$

Power functions. The power operation allows us to define the power function x^α for any real degree α. Now we can prove the equality $(x^\alpha)' = \alpha x^{\alpha-1}$ in its full value. Indeed, $(x^\alpha)' = (\exp(\alpha \ln x))' = \exp'(\alpha \ln x)(\alpha \ln x)' = \exp(\alpha \ln x)\frac{\alpha}{x} = \alpha x^{\alpha-1}$.

Infinite products via the Logarithm.

LEMMA 3.2.6. *Let $f(x)$ be a function continuous at x_0. Then for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ one has $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

PROOF. For any given $\varepsilon > 0$ there is a neighborhood U of x_0 such that $|f(x) - f(x_0)| \leq \varepsilon$ for $x \in U$. As $\lim_{n\to\infty} x_n = x_0$, eventually $x_n \in U$. Hence eventually $|f(x_n) - f(x_0)| < \varepsilon$. □

As we already have remarked, infinite sums and infinite products are limits of partial products.

THEOREM 3.2.7. $\ln \prod_{k=1}^\infty p_k = \sum_{k=1}^\infty \ln p_k$.

PROOF.

$$\begin{aligned}\exp(\textstyle\sum_{k=1}^\infty \ln p_k) &= \exp(\lim_{n\to\infty} \textstyle\sum_{k=1}^n \ln p_k) \\ &= \lim_{n\to\infty} \exp(\textstyle\sum_{k=1}^n \ln p_k) \\ &= \lim_{n\to\infty} \textstyle\prod_{k=1}^n p_k \\ &= \textstyle\prod_{k=1}^\infty p_k.\end{aligned}$$

Now take logarithms of both sides of the equation. □

Symmetric arguments prove the following: $\exp \sum_{k=1}^\infty a_k = \prod_{k=1}^\infty \exp a_k$.

Irrationality of e. The expansion of the exponenta into a power series gives an expansion into a series for e which is $\exp 1$.

LEMMA 3.2.8. *For any natural n one has $\frac{1}{n+1} < en! - [en!] < \frac{1}{n}$.*

PROOF. $en! = \sum_{k=0}^\infty \frac{n!}{k!}$. The partial sum $\sum_{k=0}^n \frac{n!}{k!}$ is an integer. The tail $\sum_{k=n+1}^\infty \frac{n!}{k!}$ is termwise majorized by the geometric series $\sum_{k=1}^\infty \frac{1}{(n+1)^k} = \frac{1}{n}$. On the other hand the first summand of the tail is $\frac{1}{n+1}$. Consequently the tail has its sum between $\frac{1}{n+1}$ and $\frac{1}{n}$. □

THEOREM 3.2.9. *The number e is irrational.*

PROOF. Suppose $e = \frac{p}{q}$ where p and q are natural. Then $eq!$ is a natural number. But it is not an integer by Lemma 3.2.8. □

Problems.

1. Prove the inequalities $1 + x \leq \exp x \leq \frac{1}{1-x}$.
2. Prove the inequalities $\frac{x}{1+x} \leq \ln(1 + x) \leq x$.
3. Evaluate $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.
4. Evaluate $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$.
5. Evaluate $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$.
6. Find the derivative of x^x.
7. Prove: $x > y$ implies $\exp x > \exp y$.
8. Express via e: $\exp 2$, $\exp(1/2)$, $\exp(2/3)$, $\exp(-1)$.
9. Prove that $\exp(m/n) = e^{\frac{m}{n}}$.
10. Prove that $\exp x > 0$ for any x.
11. Prove the addition formulas $\operatorname{ch}(x + y) = \operatorname{ch}(x)\operatorname{ch}(y) + \operatorname{sh}(x)\operatorname{sh}(y)$, $\operatorname{sh}(x + y) = \operatorname{sh}(x)\operatorname{ch}(y) + \operatorname{sh}(y)\operatorname{ch}(x)$.
12. Prove that $\Delta \operatorname{sh}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{ch}(x)$, $\Delta \operatorname{ch}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{sh}(x)$.
13. Prove $\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$.
14. Prove $\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = 1$.
15. Solve the equation $\operatorname{sh} x = 4/5$.
16. Express via e the sum $\sum_{k=1}^{\infty} k/k!$.
17. Express via e the sum $\sum_{k=1}^{\infty} k^2/k!$.
18. Prove that $\{\frac{\exp k}{k^n}\}$ is unbounded.
19. Prove: The product $\prod(1 + p_n)$ converges if and only if the sum $\sum p_n$ $(p_n \geq 0)$ converges.
20. Determine the convergence of $\prod \frac{e^{1/n}}{1+\frac{1}{n}}$.
21. Does $\prod n(e^{1/n} - 1)$ converges?
22. Prove the divergence of $\sum_{k=1}^{\infty} \frac{[k-prime]}{k}$.
23. Expand a^x into a power series.
24. Determine the geometrical sense of $\operatorname{sh} x$ and $\operatorname{ch} x$.
25. Evaluate $\lim_{n\to\infty} \sin \pi e n!$.
26. Does the series $\sum_{k=1}^{\infty} \sin \pi e k!$ converge?
***27.** Prove the irrationality of e^2.