2.6. Virtually monotone functions

Monotonization of the integrand. Let us say that a pair of functions f_1, f_2 *monotonize* a function f, if f_1 is non-negative and non-decreasing, f_2 is non-positive and non-increasing and $f = f_1 + f_2$.

LEMMA 2.6.1. *Let $f = f_1 + f_2$ and $f = f_1' + f_2'$ be two monotonizations of f. Then for any monotone h one has $f_1 dh + f_2 dh = f_1 dh + f_2' dh$.*

PROOF. Our equality is equivalent to $f_1 dh - f_2' dh = f_1' dh - f_2 dh$. By the sign rule this turns into $f_1 dh + (-f_2')dh = f_1' dh + (-f_2)dh$. Now all integrands are nonnegative and for non-decreasing h we can apply the Addition Theorem and transform the inequality into $(f_1 - f_2')dh = (f_1' - f_2)dh$. This is true because $(f_1 - f_2') = (f_1' - f_2)$.

The case of a non-increasing differand is reduced to the case of a non-decreasing one by the transformation $f_1 d(-h) + f_2 d(-h) = f_1' d(-h) + f_2' d(-h)$, which is based on the Sign Rule. □

A function which has a monotonization is called *virtually monotone.*

We define the integral $\int_a^b f\,dg$ for any virtually monotone integrand f and any continuous monotone differand g via a monotonization $f = f_1 + f_2$ by

$$\int_a^b f\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$$

Lemma 2.6.1 demonstrates that this definition does not depend on the choice of a monotonization.

LEMMA 2.6.2. *Let f and g be virtually monotone functions; then $f + g$ is virtually monotone and $f dh + g dh = (f + g)dh$ for any continuous monotone h.*

PROOF. Let h be nondecreasing. Consider monotonizations $f = f_1 + f_2$ and $g = g_1 + g_2$. Then $f dh + g dh = f_1 dh + f_2 dh + g_1 dh + g_2 dh$ by definition via monotonization of the integrand. By virtue of the Addition Theorem 2.3.3 this turns into $(f_1 + g_1)dh + (f_2 + g_2)dh$. But the pair of brackets monotonize $f + g$. Hence $f + g$ is proved to be virtually monotone and the latter expression is $(f + g)dh$ by definition, via monotonization of the integrand. The case of non-increasing h is reduced to the previous case via $-f d(-h) - g d(-h) = -(f + g)d(-h)$. □

Lemma on locally constant functions. Let us say that a function $f(x)$ is *locally constant* at a point x if $f(y) = f(x)$ for all y sufficiently close to x, i.e., for all y from an interval $(x - \varepsilon, x + \varepsilon)$.

LEMMA 2.6.3. *A function f which is locally constant at each point of an interval is constant.*

PROOF. Suppose $f(x)$ is not constant on $[a, b]$. We will construct by induction a sequence of intervals $I_k = [a_k, b_k]$, such that $I_0 = [a, b]$, $I_{k+1} \subset I_k$, $|b_k - a_k| \geq 2|b_{k+1} - a_{k+1}|$ and the function f is not constant on each I_k. First step: Let $c = (a + b)/2$, as f is not constant $f(x) \neq f(c)$ for some x. Then choose $[x, c]$ or $[c, x]$ as for $[a_1, b_1]$. On this interval f is not constant. The same are all further steps. The intersection of the sequence is a point such that any of its neighborhoods contains some interval of the sequence. Hence f is not locally constant at this point. □

LEMMA 2.6.4. *If $f(x)$ is a continuous monotone function and $a < f(x) < b$ then $a < f(y) < b$ for all y sufficiently close to x.*

PROOF. If f takes values greater than b, then it takes value b and if $f(x)$ takes values less than a then it takes value a due to continuity. Then $[f^{-1}(a), f^{-1}(b)]$ is the interval where inequalities hold. □

LEMMA 2.6.5. *Let g_1, g_2 be continuous* comonotone *functions. Then $g_1 + g_2$ is continuous and monotone, and for any virtually monotone f one has*

$$fdg_1 + fdg_2 = fd(g_1 + g_2). \tag{2.6.1}$$

PROOF. Suppose $g_1(x) + g_2(x) < p$, let $\varepsilon = p - g_1(x) - g_2(x)$. Then $g_1(y) < g_1(y) + \varepsilon/2$ and $g_2(y) < g_2(y) + \varepsilon/2$ for all y sufficiently close to x. Hence $g_(y) + g_2(y) < p$ for all y sufficiently close to x. The same is true for the opposite inequality. Hence $\operatorname{sgn}(g_1(x)+g_2(x)-p)$ is locally constant at all points where it is not 0. But it is not constant if p is an intermediate value, hence it is not locally constant, hence it takes value 0. At this point $g_1(x) + g_2(x) = p$ and the continuity of $g_1 + g_2$ is proved.

Consider a monotonization $f = f_1 + f_2$. Let g_i be nondecreasing. By definition via monotonization of the integrand, the left-hand side of (2.6.1) turns into $(f_1dg_1 + f_2dg_1) + (f_1dg_2 + f_2dg_2) = (f_1dg_1 + f_1dg_2) + (f_2dg_1 + f_2dg_2)$. By the Addition Theorem 2.3.3 $f_1dg_1 + f_1dg_2 = f_1d(g_1 + g_2)$. And the equality $f_2dg_1 + f_2dg_2 = f_2d(g_1 + g_2)$ follows from $(-f_2)dg_1 + (-f_2)dg_2 = (-f_2)d(g_1 + g_2)$ by the Sign Rule. Hence the left-hand side is equal to $f_1d(g_1 + g_2) + f_2d(g_1 + g_2)$, which coincides with the right-hand side of (2.6.1) by definition via monotonization of integrand. The case of non-increasing differands is taken care of via transformation of (2.6.1) by the Sign Rule into $fd(-g_1) + fd(-g_2) = fd(-g_1 - g_2)$. □

LEMMA 2.6.6. *Let $g_1 + g_2 = g_3 + g_4$ where all $(-1)^k g_k$ are non-increasing continuous functions. Then $fdg_1 + fdg_2 = fdg_3 + fdg_4$ for any virtually monotone f.*

PROOF. Our equality is equivalent to $fdg_1 - fdg_4 = fdg_3 - fdg_2$. By the Sign Rule it turns into $fdg_1 + fd(-g_4) = fdg_3 + fd(-g_2)$. Now all differands are nondecreasing and by Lemma 2.6.5 it transforms into $fd(g_1 - g_4) = fd(g_3 - g_2)$. This is true because $g_1 - g_4 = g_3 - g_2$. □

Monotonization of the differand. A monotonization by continuous functions is called continuous. A virtually monotone function which has a continuous monotonization is called *continuous*. The integral for any virtually monotone integrand f against a virtually monotone continuous differand g is defined via a continuous virtualization $g = g_1 + g_2$ of the differand

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

The integral is well-defined because of Lemma 2.6.6.

THEOREM 2.6.7 (Addition Theorem). *For any virtually monotone functions f, f' and any virtually monotone continuous g, g', $fdg + f'dg = (f + f')dg$ and $fdg + fdg' = fd(g + g')$*

PROOF. To prove $fdg + f'dg = (f + f')dg$, consider a continuous monotonization $g = g_1 + g_2$. Then by definition of the integral for virtually monotone differands this equality turns into $(fdg_1 + fdg_2) + (f'dg_1 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. After rearranging it turns into $(fdg_1 + f'dg_1) + (fdg_2 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. But this is true due to Lemma 2.6.2.

To prove $fdg + fdg' = fd(g + g')$, consider monotonizations $g = g_1 + g_2$, $g' = g'_1 + g'_2$. Then $(g_1 + g'_1) + (g_2 + g'_2)$ is a monotonization for $g + g'$. And by the definition of the integral for virtually monotone differands our equality turns into $fdg_1 + fdg_2 + fdg'_1 + fdg'_2$ □

Change of variable.

LEMMA 2.6.8. *If f is virtually monotone and g is monotone, then $f(g(x))$ is virtually monotone.*

PROOF. Let $f_1 + f_2$ be a monotonization of f. If h is non-decreasing then $f_1(h(x)) + f_2(h(x))$ gives a monotonization of $f(g(x))$. If h is decreasing then the monotonization is given by $(f_2(h(x)) + c) + (f_1(h(x)) - c)$ where c is a sufficiently large constant to provide positivity of the first brackets and negativity of the second one. □

The following natural convention is applied to define an integral with reversed limits: $\int_a^b f(x)\,dg(x) = -\int_b^a f(x)\,dg(x)$.

THEOREM 2.6.9 (on change of variable). *If $h\colon [a, b] \to [h(a), h(b)]$ is monotone, $f(x)$ is virtually monotone and $g(x)$ is virtually monotone continuous then*

$$\int_a^b f(h(t))\,dg(h(t)) = \int_{h(a)}^{h(b)} f(x)\,dg(x).$$

PROOF. Let $f = f_1 + f_2$ and $g = g_1 + g_2$ be a monotonization and a continuous monotonization of f and g respectively. The $\int_a^b f(h(t))\,dg(h(t))$ splits into sum of four integrals: $\int_a^b f_i(h(t))\,dg_j(h(t))$ where f_i are of constant sign and g_j are monotone continuous. These integrals coincide with the corresponding integrals $\int_{h(a)}^{h(b)} f_i(x)\,dg_i(x)$. Indeed their absolute values are the areas of the same curvilinear trapezia. And their signs determined by the Sign Rule are the same. □

Integration by parts. We have established the Integration by Parts formula for non-negative and non-decreasing differential forms. Now we extend it to the case of continuous monotone forms. In the first case f and g are non-decreasing. In this case choose a positive constant c sufficiently large to provide positivity of $f + c$ and $g + c$ on the interval of integration. Then $d(f + c)(g + c) = (f + c)d(g + c) + (g + c)d(f + c)$. On the other hand $d(f + c)(g + c) = dfg + cdf + cdg$ and $(f + c)d(g + c) + (g + c)d(f + c) = fdg + cdg + cdf$. Compare these results to get $dfg = fdg + gdf$. Now if f is increasing and g is decreasing then $-g$ is increasing and we get $-dfg = df(-g) = fd(-g) + (-g)df = -fdg - gdf$, which leads to $dfg = fdg + gdf$. The other cases: f decreasing, g increasing and both decreasing are proved by the same arguments. The extension of the Integration by Parts formula to piecewise monotone forms immediately follows by the Partition Rule.

Variation. Define the *variation of a sequence* of numbers $\{x_k\}_{k=1}^n$ as the sum $\sum_{k=1}^{\infty} |x_{k+1} - x_k|$. Define the variation of a function f along a sequence $\{x_k\}_{k=0}^n$

as the variation of sequence $\{f(x_k)\}_{k=0}^n$. Define a *chain* on an interval $[a,b]$ as a nondecreasing sequence $\{x_k\}_{k=0}^n$ such that $x_0 = a$ and $x_n = b$. Define the *partial variation* of f on an interval $[a,b]$ as its variation along a chain on the interval.

The least number surpassing all partial variations function f over $[a,b]$ is called *the (ultimate) variation of a function* $f(x)$ on an interval $[a,b]$ and is denoted by $\operatorname{var}_f[a,b]$.

LEMMA 2.6.10. *For any function f one has the inequality* $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$. *If f is a monotone function on $[a,b]$, then* $\operatorname{var}_f[a,b] = |f(b) - f(a)|$.

PROOF. The inequality $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$ follows immediately from the definition because $\{a,b\}$ is a chain. For monotone f, all partial variations are telescopic sums equal to $|f(b) - f(a)|$ □

THEOREM 2.6.11 (additivity of variation). $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] = \operatorname{var}_f[a,c]$.

PROOF. Consider a chain $\{x_k\}_{k=0}^n$ of $[a,c]$, which contains b. In this case the variation of f along $\{x_k\}_{k=0}^n$ splits into sums of partial variations of f along $[a,b]$ and along $[b,c]$. As a partial variations does not exceed an ultimate, we get that in this case the variation of f along $\{x_k\}_{k=0}^n$ does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$.

If $\{x_k\}_{k=0}^n$ does not contain b, let us add b to the chain. Then in the sum expressing the partial variation of f, the summand $|f(x_{i+1}) - f(x_i)|$ changes by the sum $|f(b) - f(x_i)| + |f(x_{i+1} - f(b)|$ which is greater or equal. Hence the variation does not decrease after such modification. But the variation along the modified chain does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$ as was proved above. As all partial variations of f over $[a,c]$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$, the same is true for the ultimate variation.

To prove the opposite inequality we consider a *relaxed* inequality $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] \leq \operatorname{var}_f[a,c] + \varepsilon$ where ε is an positive number. Choose chains $\{x_k\}_{k=0}^n$ on $[a,b]$ and $\{y_k\}_{k=0}^m$ on $[b,c]$ such that corresponding partial variations of f are $\geq \operatorname{var}_f[a,b] + \varepsilon/2$ and $\geq \operatorname{var}_f[b,c] + \varepsilon/2$ respectively. As the union of these chains is a chain on $[a,c]$ the sum of these partial variations is a partial variation of f on $[a,c]$. Consequently this sum is less or equal to $\operatorname{var}_f[a,c]$. On the other hand it is greater or equal to $\operatorname{var}_f[a,b] + \varepsilon/2 + \operatorname{var}_f[b,c] + \varepsilon/2$. Comparing these results gives just the relaxed inequality. As the relaxed inequality is proved for all $\varepsilon > 0$ it also holds for $\varepsilon = 0$. □

LEMMA 2.6.12. *For any functions f, g one has the inequality* $\operatorname{var}_{f+g}[a,b] \leq \operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$.

PROOF. Since $|f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \leq |f(x_{k+1}) - f(x_k)| + |g(x_{k+1}) - g(x_k)|$, the variation of $f + g$ along any sequence does not exceed the sum of the variations of f and g along the sequence. Hence all partial variations of $f + g$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$, and so the same is true for the ultimate variation. □

LEMMA 2.6.13. *For any function of finite variation on $[a,b]$, the functions* $\operatorname{var}_f[a,x]$ *and* $\operatorname{var}_f[a,x] - f(x)$ *are both nondecreasing functions of x.*

PROOF. That $\operatorname{var}_f[a,x]$ is nondecreasing follows from nonnegativity and additivity of variation. If $x > y$ then the inequality $\operatorname{var}_f[a,x] - f(x) \geq \operatorname{var}_f[a,y] - f(y)$

is equivalent to $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] \geq f(x)-f(y)$. This is true because $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] = \operatorname{var}_f[x,y] \geq |f(x)-f(y)|$. □

LEMMA 2.6.14. $\operatorname{var}_{f^2}[a,b] \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b]$.

PROOF. For all $x, y \in [a,b]$ one has

$$\begin{aligned} |f(x)+f(y)| &= |2f(a)+f(x)-f(a)+f(y)-f(a)| \\ &\leq 2|f(a)| + \operatorname{var}_f[a,x] + \operatorname{var}_f[a,y] \\ &\leq 2|f(a)| + 2\operatorname{var}_f[a,b]. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=0}^{n-1} |f^2(x_{k+1}) - f^2(x_k)| &= \sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)||f(x_{k+1})+f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b] \end{aligned}$$

□

LEMMA 2.6.15. *If* $\operatorname{var}_f[a,b] < \infty$ *and* $\operatorname{var}_g[a,b] < \infty$, *then* $\operatorname{var}_{fg}[a,b] < \infty$.

PROOF. $4fg = (f+g)^2 - (f-g)^2$. □

THEOREM 2.6.16. *The function f is virtually monotone on $[a,b]$ if and only if it has a finite variation.*

PROOF. Since monotone functions have finite variation on finite intervals, and the variation of a sum does not exceed the sum of variations, one gets that all virtually monotone functions have finite variation. On the other hand, if f has finite variation then $f = (\operatorname{var}_f[a,x]+c) + (f(x)-\operatorname{var}_f[a,x]-c)$, the functions in the brackets are monotone due to Lemma 2.6.13, and by choosing a constant c sufficiently large, one obtains that the second bracket is negative. □

Problems.

1. Evaluate $\int_1^i z^2\,dz$.
2. Prove that $1/f(x)$ has finite variation if it is bounded.
3. Prove $\int_a^b f(x)\,dg(x) \leq \max_{[a,b]} f \operatorname{var}_g[a,b]$.