2.5. Quadrature of Circle

On the contents of the lecture. We extend the concept of the integral to complex functions. We evaluate a very important integral $\oint \frac{1}{z}dz$ by applying Archimedes' theorem on the area of circular sector. As a consequence, we evaluate the Wallis product and the Stirling constant.

Definition of a complex integral. To specify an integral of a complex function one has to indicate not only its limits, but also the *path of integration*. A path of integration is a mapping $p\colon [a,b] \to \mathbb{C}$, of an interval $[a,b]$ of the real line into complex plane. The integral of a complex differential form fdg (here f and g are complex functions of complex variable) along the path p is defined via separate integration of different combinations of real and imaginary parts in the following way:

$$\int_a^b \operatorname{Re} f(p(t))\, d \operatorname{Re} g(p(t)) - \int_a^b \operatorname{Im} f(p(t))\, d \operatorname{Im} g(p(t)) \\ + i \int_a^b \operatorname{Re} f(p(t))\, d \operatorname{Im} g(p(t)) + i \int_a^b \operatorname{Im} f(p(t))\, d \operatorname{Re} g(p(t))$$

Two complex differential forms are called equal if their integrals coincide for all paths. So, the definition above can be written shortly as $fdg = \operatorname{Re} f d \operatorname{Re} g - \operatorname{Im} f d \operatorname{Im} g + i \operatorname{Re} f d \operatorname{Im} g + i \operatorname{Im} f d \operatorname{Re} g$.

The integral $\int \frac{1}{z}dz$. The Integral is the principal concept of Calculus and $\int \frac{1}{z}dz$ is the principal integral. Let us evaluate it along the path $p(t) = \cos t + i \sin t$, $t \in [0,\phi]$, which goes along the arc of the circle of the length $\phi \le \pi/2$. Since $\frac{1}{\cos t + i \sin t} = \cos t - i \sin t$, one has

$$\begin{aligned} \int_p \frac{1}{z}\, dz = &\int_0^\phi \cos t \, d \cos t + \int_0^\phi \sin t \, d \sin t \\ &- i \int_0^\phi \sin t \, d \cos t + i \int_0^\phi \cos t \, d \sin t. \end{aligned} \tag{2.5.1}$$

Its real part transforms into $\int_0^\phi \frac{1}{2}\, d \cos^2 t + \int_0^\phi \frac{1}{2}\, d \sin^2 t = \int_0^\phi \frac{1}{2}\, d(\cos^2 t + \sin^2 t) = \int_0^\phi \frac{1}{2}\, d1 = 0$. An attentive reader has to object: integrals were defined only for differential forms with non-decreasing differands, while $\cos t$ decreases.

Sign rule. Let us define the integral for any differential form fdg with any continuous monotone differand g and any integrand f of a constant sign (i.e, non-positive or non-negative). The definition relies on the following *Sign Rule*.

$$\int_a^b -f\, dg = -\int_a^b f\, dg = \int_a^b f\, d(-g) \tag{2.5.2}$$

If f is of constant sign, and g is monotone, then among the forms fdg, $-fdg$, $fd(-g)$ and $-fd(-g)$ there is just one with non-negative integrand and non-decreasing differand. For this form, the integral was defined earlier, for the other cases it is defined by the Sign Rule.

Thus the integral of a negative function against an increasing differand and the integral of a positive function against a decreasing differand are negative. And the integral of a negative function against a decreasing differand is positive.

The Sign Rule agrees with the Constant Rule: the formula $\int_a^b c\,dg = c(g(b) - g(a))$ remains true either for negative c or decreasing g.

The Partition Rule also is not affected by this extension of the integral.

The Inequality Rule takes the following form: if $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$ then $\int_a^b f_1(x)\,dg(x) \leq \int_a^b f_2(x)\,dg(x)$ for non-decreasing g and $\int_a^b f_1(x)\,dg(x) \geq \int_a^b f_2(x)\,dg(x)$ for non-increasing g.

Change of variable. Now all integrals in (2.5.1) are defined. The next objection concerns transformation $\cos t d\cos t = \frac{1}{2} d\cos^2 t$. This transformation is based on a decreasing change of variable $x = \cos t$ in $dx^2/2 = xdx$. But what happens with an integral when one applies a decreasing change of variable? The curvilinear trapezium, which represents the integral, does not change at all under any change of variable, even for a non-monotone one. Hence the only thing that may happen is a change of sign. And the sign changes by the Sign Rule, simultaneously on both sides of equality $dx^2/2 = xdx$. If the integrals of xdx and dx^2 are positive, both integrals of $\cos t d\cos t$ and $\cos^2 t$ are negative and have the same absolute value. These arguments work in the general case:

A decreasing change of variable reverses the sign of the integral.

Addition Formula. The next question concerns the legitimacy of addition of differentials, which appeared in the calculation $d\cos^2 t + d\sin^2 t = d(\cos^2 t + \sin^2 t) = 0$, where differands are not *comonotone*: $\cos t$ decreases, while $\sin t$ increases. The addition formula in its full generality will be proved in the next lecture, but this special case is not difficult to prove. Our equality is equivalent to $d\sin^2 t = -d\cos^2 t$. By the Sign Rule $-d\cos^2 t = d(-\cos^2 t)$, but $-\cos^2 t$ is increasing. And by the Addition Theorem $d(-\cos^2 t + 1) = d(-\cos^2 t) + d1 = d(-\cos^2 t)$. But $-\cos^2 t + 1 = \sin^2 t$. Hence our evaluation of the real part of (2.5.1) is justified.

Trigonometric integrals. We proceed to the evaluation of the imaginary part of (2.5.1), which is $\cos t\,d\sin t - \sin t\,d\cos t$. This is a simple geometric problem.

The integral of $\sin t\,d\cos t$ is negative as $\cos t$ is decreasing on $[0, \frac{\pi}{2}]$, and its absolute value is equal to the area of the curvilinear triangle $A'BA$, which is obtained from the circular sector OBA with area $\phi/2$ by deletion of the triangle $OA'B$, which has area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \sin t\,d\cos t$ is $\phi/2 - \frac{1}{2}\cos\phi\sin\phi$.

The integral of $\cos t\,d\sin t$ is equal to the area of curvilinear trapezium $OB'BA$. The latter consists of a circular sector OBA with area $\phi/2$ and a triangle $OB'B$ with area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \cos t\,d\sin t = \phi/2 + \frac{1}{2}\cos\phi\sin\phi$.

As a result we get $\int_p \frac{1}{z}\,dz = i\phi$. This result has a lot of consequences. But today we restrict our attention to the integrals of $\sin t$ and $\cos t$.

Multiplication of differentials. We have proved

$$\cos t\,d\sin t - \sin t\,d\cos t = dt. \tag{2.5.3}$$

Multiplying this equality by $\cos t$, one gets

$$\cos^2 t\,d\sin t - \sin t\cos t\,d\cos t = \cos t\,dt.$$

Replacing $\cos^2 t$ by $(1 - \sin^2 t)$ and moving $\cos t$ into the differential, one transforms the left-hand side as

$$d\sin t - \sin^2 t\,d\sin t - \tfrac{1}{2}\sin t\,d\cos^2 t = d\sin t - \tfrac{1}{2}\sin t\,d\sin^2 t - \tfrac{1}{2}\sin t\,d\cos^2 t.$$

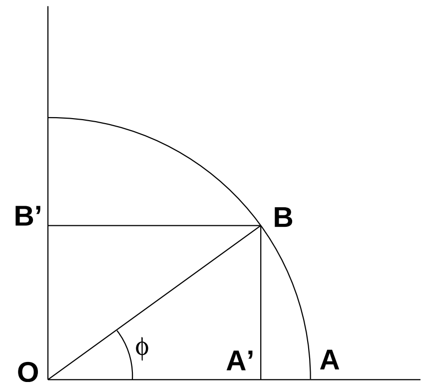

FIGURE 2.5.1. Trigonometric integrals

We already know that $d\sin^2 t + d\cos^2 t$ is zero. Now we have to prove the same for the product of this form by $\frac{1}{2}\sin t$. The arguments are the same: we multiply by $\frac{1}{2}\sin t$ the equivalent equality $d\sin^2 t = d(-\cos^2 t)$ whose differands are increasing. This is a general way to extend the theorem on multiplication of differentials to the case of any monotone functions. We will do it later. Now we get just $d\sin t = \cos t\, dt$.

Further, multiplication of the left-hand side of (2.5.3) by $\sin t$ gives

$$\sin t\cos t\, d\sin t - \sin^2 t\, d\cos t = \tfrac{1}{2}\cos t\, d\sin^2 t - d\cos t + \tfrac{1}{2}\cos t\, d\cos^2 t = -d\cos t.$$

So we get $d\cos t = -\sin t dt$.

THEOREM 2.5.1. *$d\sin t = \cos t\, dt$ and $d\cos t = -\sin t\, dt$.*

We have proved this equality only for $[0, \pi/2]$. But due to well-known symmetries this suffices.

Application of trigonometric integrals.

LEMMA 2.5.2. *For any convergent infinite product of factors ≥ 1 one has*

$$\lim \prod_{k=1}^{n} p_k = \prod_{k=1}^{\infty} p_k. \tag{2.5.4}$$

PROOF. Let ε be a positive number. Then $\prod_{k=1}^{\infty} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$, and by All-for-One there is n such that $\prod_{k=1}^{n} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Then for any $m > n$ one has the inequalities $\prod_{k=1}^{\infty} p_k \geq \prod_{k=1}^{m} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Therefore $|\prod_{k=1}^{m} p_k - \prod_{k=1}^{\infty} p_k| < \varepsilon$. □

Wallis product. Set $I_n = \int_0^\pi \sin^n x\, dx$. Then $I_0 = \int_0^\pi 1\, dx = \pi$ and $I_1 = \int_0^\pi \sin x\, dx = -\cos\pi + \cos 0 = 2$. For $n \geq 2$, let us replace the integrand $\sin^n x$ by

$\sin^{n-2} x(1-\cos^2 x)$ and obtain

$$\begin{aligned} I_n &= \int_0^\pi \sin^{n-2} x(1-\cos^2 x)\,dx \\ &= \int_0^\pi \sin^{n-2} x\,dx - \int_0^\pi \sin^{n-2} x \cos x\, d\sin x \\ &= I_{n-2} - \tfrac{1}{n-1}\int_0^\pi \cos x\, d\sin^{n-1}(x) \\ &= I_{n-2} - \int_0^\pi d(\cos x \sin^{n-1} x) + \int_0^\pi \sin^{n-1} x\, d\cos x \\ &= I_{n-2} - \tfrac{1}{n-1} I_n. \end{aligned}$$

We get the recurrence relation $I_n = \frac{n-1}{n} I_{n-2}$, which gives the formula

$$I_{2n} = \pi\frac{(2n-1)!!}{2n!!}, \quad I_{2n-1} = 2\frac{(2n-2)!!}{(2n-1)!!} \tag{2.5.5}$$

where $n!!$ denotes the product $n(n-2)(n-4)\cdots(n \mod 2+1)$. Since $\sin^n x \leq \sin^{n-1} x$ for all $x \in [0,\pi]$, the sequence $\{I_n\}$ decreases. Since $I_n \leq I_{n-1} \leq I_{n-2}$, one gets $\frac{n-1}{n} = \frac{I_n}{I_{n-2}} \leq \frac{I_{n-1}}{I_{n-2}} \leq 1$. Hence $\frac{I_{n-1}}{I_{n-2}}$ differs from 1 less than $\frac{1}{n}$. Consequently, $\lim \frac{I_{n-1}}{I_{n-2}} = 1$. In particular, $\lim \frac{I_{2n+1}}{I_{2n}} = 1$. Substituting in this last formula the expressions of I_n from (2.5.5) one gets

$$\lim \frac{\pi}{2}\frac{(2n+1)!!(2n-1)!!}{2n!!2n!!} = 1.$$

Therefore this is the famous Wallis Product

$$\frac{\pi}{2} = \lim \frac{2n!!2n!!}{(2n-1)!!(2n+1)!!} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}.$$

Stirling constant. In Lecture 2.4 we have proved that

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n, \tag{2.5.6}$$

where o_n is infinitesimally small and σ is a constant. Now we are ready to determine this constant. Consider the difference $\ln 2n! - 2\ln n!$. By (2.5.6) it expands into

$$\begin{aligned} &(2n\ln 2n - 2n + \tfrac{1}{2}\ln 2n + \sigma + o_{2n}) - 2(n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n) \\ &\qquad = 2n\ln 2 + \tfrac{1}{2}\ln 2n - \ln n - \sigma + o'_n, \end{aligned}$$

where $o'_n = o_{2n} - 2o_n$ is infinitesimally small. Then σ can be presented as

$$\sigma = 2\ln n! - \ln 2n! + 2n\ln 2 + \tfrac{1}{2}\ln n + \tfrac{1}{2}\ln 2 - \ln n + o'_n.$$

Multiplying by 2 one gets

$$2\sigma = 4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2 + 2o'_n.$$

Hence $2\sigma = \lim(4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2)$. Switching to product and keeping in mind the identities $n! = n!!(n-1)!!$ and $n!2^n = 2n!!$ one gets

$$\sigma^2 = \lim \frac{n!^4 2^{4n+1}}{(2n!)^2 n} = \lim \frac{2\cdot(2n!!)^4}{(2n!!)^2(2n-1)!!^2 n} \lim \frac{2\cdot(2n!!)^2(2n+1)}{(2n-1)!!(2n+1)!!n} = 2\pi.$$

Problems.

1. Evaluate $\int \sqrt{1-x^2}\,dx$.
2. Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx$.
3. Evaluate $\int \sqrt{5-x^2}\,dx$.
4. Evaluate $\int \cos^2 x\,dx$.
5. Evaluate $\int \tan x\,dx$.
6. Evaluate $\int \sin^4 x\,dx$.
7. Evaluate $\int \sin x^2\,dx$.
8. Evaluate $\int \tan x\,dx$.
9. Evaluate $\int x^2 \sin x\,dx$.
10. Evaluate $d \arcsin x$.
11. Evaluate $\int \arcsin x\,dx$.
12. Evaluate $\int e^x \cos x\,dx$.