2.4. Asymptotics of Sums

On the contents of the lecture. We become at last acquainted with the fundamental concept of a *limit*. We extend the notion of the sum of a series and discover that a change of order of summands can affect the ultimate sum. Finally we derive the famous Stirling formula for $n!$.

Asymptotic formulas. The Mercator series shows how useful series can be for evaluating integrals. In this lecture we will use integrals to evaluate both partial and ultimate sums of series. Rarely one has an explicit formula for partial sums of a series. There are lots of important cases where such a formula does not exist. For example, it is known that partial sums of the Euler series cannot be expressed as a finite combination of elementary functions. When an explicit formula is not available, one tries to find a so-called *asymptotic formula*. An asymptotic formula for a partial sum S_n of a series is a formula of the type $S_n = f(n) + R(n)$ where f is a known function called the *principal part* and $R(n)$ is a *remainder*, which is small, in some sense, with respect to the principal part. Today we will get an asymptotic formula for partial sums of the harmonic series.

Infinitesimally small sequences. The simplest asymptotic formula has a constant as its principal part and an infinitesimally small remainder. One says that a sequence $\{z_k\}$ is *infinitesimally small* and writes $\lim z_k = 0$, if z_k tends to 0 as n tends to infinity. That is for any positive ε eventually (i.e., beginning with some n) $|z_k| < \varepsilon$. With Iverson notation, this definition can be expressed in the following clear form:

$$[\{z_k\}_{k=1}^{\infty} \text{ is infinitesimally small}] = \prod_{m=1}^{\infty} 2\left|\sum_{n=1}^{\infty}(-1)^n \prod_{k=1}^{\infty}[m[k>n]|z_k|<1]\right|.$$

Three basic properties of infinitesimally small sequences immediately follow from the definition:

- if $\lim a_k = \lim b_k = 0$ then $\lim(a_k + b_k) = 0$;
- if $\lim a_k = 0$ then $\lim a_k b_k = 0$ for any bounded sequence $\{b_k\}$;
- if $a_k \leq b_k \leq c_k$ for all k and $\lim a_k = \lim c_k = 0$, then $\lim b_k = 0$.

The third property is called the *squeeze rule*.

Today we need just one property of infinitesimally small sequences:

THEOREM 2.4.1 (Addition theorem). *If the sequences $\{a_k\}$ and $\{b_k\}$ are infinitesimally small, than their sum and their difference are infinitesimally small too.*

PROOF. Let ε be a positive number. Then $\varepsilon/2$ also is positive number. And by definition of infinitesimally small, the inequalities $|a_k| < \varepsilon/2$ and $|b_k| < \varepsilon/2$ hold eventually beginning with some n. Then for $k > n$ one has $|a_k \pm b_k| \leq |a_k| + |b_k| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Limit of sequence.

DEFINITION. *A sequence $\{z_k\}$ of (complex) numbers* converges *to a number z if* $\lim z - z_k = 0$. *The number z is called the* limit *of the sequence $\{z_k\}$ and denoted by* $\lim z_k$.

An infinite sum represents a particular case of a limit as demonstrated by the following.

THEOREM 2.4.2. *The partial sums of an absolutely convergent series* $\sum_{k=1}^{\infty} z_k$ *converge to its sum.*

PROOF. $|\sum_{k=1}^{n-1} z_k - \sum_{k=1}^{\infty} z_k| = |\sum_{k=n}^{\infty} z_k| \le \sum_{k=n}^{\infty} |z_k|$. Since $\sum_{k=1}^{\infty} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$, there is a partial sum such that $\sum_{k=1}^{n-1} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$. Then for all $m \ge n$ one has $\sum_{k=m}^{\infty} |z_k| \le \sum_{k=n}^{\infty} |z_k| < \varepsilon$. □

Conditional convergence. The concept of the limit of sequence leads to a notion of convergence generalizing absolute convergence.

A series $\sum_{k=1}^{\infty} a_k$ is called (conditionally) *convergent* if $\lim \sum_{k=1}^{n} a_k = A + \alpha_n$, where $\lim \alpha_n = 0$. The number A is called its ultimate sum.

The following theorem gives a lot of examples of conditionally convergent series which are not absolutely convergent. By $[[n]]$ we denote the even part of the number n, i.e., $[[n]] = 2[n/2]$.

THEOREM 2.4.3 (Leibniz). *For any of positive decreasing infinitesimally small sequence* $\{a_n\}$, *the series* $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges.*

PROOF. Denote the difference $a_k - a_{k+1}$ by δa_k. The series $\sum_{k=1}^{\infty} \delta a_{2k-1}$ and $\sum_{k=1}^{\infty} \delta a_{2k}$ are positive and convergent, because their termwise sum is $\sum_{k=1}^{\infty} \delta a_k = a_1$. Hence $S = \sum_{k=1}^{\infty} \delta a_{2k-1} \le a_1$. Denote by S_n the partial sum $\sum_{k=1}^{n-1} (-1)^{k+1} a_k$. Then $S_{2n} = \sum_{k=1}^{n-1} \delta a_{2n-1} = S + \alpha_n$, where $\lim \alpha_n = 0$. Then $S_n = S_{[[n]]} + a_n[n \text{ is odd}] + \alpha_{[[n]]}$. As $a_n[n \text{ is odd}] + \alpha_{[[n]]}$ is infinitesimally small, this implies the theorem. □

LEMMA 2.4.4. *Let f be a non-increasing nonnegative function. Then the series* $\sum_{k=1}^{\infty} (f(k) - \int_k^{k+1} f(x)\,dx)$ *is positive and convergent and has sum* $c_f \le f(1)$.

PROOF. Integration of the inequalities $f(k) \ge f(x) \ge f(k+1)$ over $[k, k+1]$ gives $f(k) \ge \int_k^{k+1} f(x)\,dx \ge f(n+1)$. This proves the positivity of the series and allows us to majorize it by the telescopic series $\sum_{k=1}^{\infty} (f(k) - f(k+1)) = f(1)$. □

THEOREM 2.4.5 (Integral Test on Convergence). *If a nonnegative function* $f(x)$ *decreases monotonically on* $[1, +\infty)$, *then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if* $\int_1^{\infty} f(x)\,dx < \infty$.

PROOF. Since $\int_1^{\infty} f(x)\,dx = \sum_{k=1}^{\infty} \int_k^{k+1} f(x)\,dx$, one has $\sum_{k=1}^{\infty} f(k) = c_f + \int_1^{\infty} f(x)\,dx$. □

Euler constant. The sum $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right)$, which is c_f for $f(x) = \frac{1}{x}$, is called *Euler's constant* and denoted by γ. Its first ten digits are $0.5772156649\ldots$.

Harmonic numbers. The sum $\sum_{k=1}^{n} \frac{1}{k}$ is denoted H_n and is called the n-th *harmonic number.*

THEOREM 2.4.6. $H_n = \ln n + \gamma + o_n$ *where* $\lim o_n = 0$.

PROOF. Since $\ln n = \sum_{k=1}^{n-1} (\ln(k+1) - \ln k) = \sum_{k=1}^{n-1} \ln(1 + \frac{1}{k})$, one has $\ln n + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = H_{n-1}$. But $\sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = \gamma + \alpha_n$, where $\lim \alpha_n = 0$. Therefore $H_n = \ln n + \gamma + (\frac{1}{n} + \alpha_n)$. □

Alternating harmonic series. The *alternating harmonic series* $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is a conditionally convergent series due to the Leibniz Theorem 2.4.3, and it is not absolutely convergent. To find its sum we apply our Theorem 2.4.6 on asymptotics of harmonic numbers.

Denote by $S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$ the partial sum. Then $S_n = H'_n - H''_n$, where $H'_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is odd}]$ and $H''_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is even}]$. Since $H''_{2n} = \frac{1}{2}H_n$ and $H'_{2n} = H_{2n} - H''_{2n} = H_{2n} - \frac{1}{2}H_n$ one gets

$$\begin{aligned}
S_{2n} &= H_{2n} - \tfrac{1}{2}H_n - \tfrac{1}{2}H_n \\
&= H_{2n} - H_n \\
&= \ln 2n + \gamma + o_{2n} - \ln n - \gamma - o_n \\
&= \ln 2 + (o_{2n} - o_n).
\end{aligned}$$

Consequently $S_n = \ln 2 + (o_{[[n]]} - o_{[n/2]} + \frac{(-1)^{n+1}}{n}[n \text{ is odd}])$. As the sum in brackets is infinitesimally small, one gets

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \ln 2.$$

The same arguments for a permutated alternating harmonic series give

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots = \tfrac{3}{2}\ln 2. \tag{2.4.1}$$

Indeed, in this case its $3n$-th partial sum is

$$\begin{aligned}
S_{3n} &= H'_{4n} - H''_{2n} \\
&= H_{4n} - \tfrac{1}{2}H_{2n} - \tfrac{1}{2}H_n \\
&= \ln 4n + \gamma + o_{4n} - \tfrac{1}{2}(\ln 2n + \gamma + o_{2n} + \ln n + \gamma + o_n) \\
&= \ln 4 - \tfrac{1}{2}\ln 2 + o'_n \\
&= \tfrac{3}{2}\ln 2 + o'_n,
\end{aligned}$$

where $\lim o'_n = 0$. Since the difference between S_n and S_{3m} where $m = [n/3]$ is infinitesimally small, this proves (2.4.1).

Stirling's Formula. We will try to estimate $\ln n!$. Integration of the inequalities $\ln[x] \le \ln x \le \ln[x+1]$ over $[1,n]$ gives $\ln(n-1)! \le \int_1^n \ln x\,dx \le \ln n!$. Let us estimate the difference D between $\int_1^n \ln x\,dx$ and $\frac{1}{2}(\ln n! + \ln(n-1)!)$.

$$\begin{aligned}
D &= \int_1^n (\ln x - \tfrac{1}{2}(\ln[x] + \ln[x+1]))\,dx \\
&= \sum_{k=1}^{n-1}\int_0^1 \left(\ln(k+x) - \ln\sqrt{k(k+1)}\right)\,dx.
\end{aligned} \tag{2.4.2}$$

To prove that all summands on the left-hand side are nonnegative, we apply the following general lemma.

LEMMA 2.4.7. *$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ for any function.*

PROOF. The reflection of the plane across the line $y = \frac{1}{2}$ transforms the curvilinear trapezium of $f(x)$ over $[0,1]$ into curvilinear trapezium of $f(1-x)$ over $[0,1]$. □

LEMMA 2.4.8. $\int_0^1 \ln(k+x)\,dx \geq \ln\sqrt{k(k+1)}$.

PROOF. Due to Lemma 2.4.7 one has

$$\begin{aligned}\int_0^1 \ln(k+x)\,dx &= \int_0^1 \ln(k+1-x)\,dx \\ &= \int_0^1 \tfrac{1}{2}(\ln(k+x)+\ln(k+1-x))\,dx \\ &= \int_0^1 \ln\sqrt{(k+x)(k+1-x)}\,dx \\ &= \int_0^1 \ln\sqrt{k(k+1)+x-x^2}\,dx \\ &\geq \int_0^1 \ln\sqrt{k(k+1)}\,dx \\ &= \ln\sqrt{k(k+1)}.\end{aligned}$$

□

Integration of the inequality $\ln(1+x/k) \leq x/k$ over $[0,1]$ gives

$$\int_0^1 \ln(1+x/k)\,dx \leq \int_0^1 \frac{x}{k}\,dx = \frac{1}{2k}.$$

This estimate together with the inequality $\ln(1+1/k) \geq 1/(k+1)$ allows us to estimate the summands from the right-hand side of (2.4.2) in the following way:

$$\begin{aligned}\int_0^1 \ln(k+x) - \ln\sqrt{k(k+1)}\,dx &= \int_0^1 \ln(k+x) - \ln k - \tfrac{1}{2}(\ln(k+1)-\ln k)\,dx \\ &= \int_0^1 \ln\left(1+\tfrac{x}{k}\right) - \tfrac{1}{2}\ln\left(1+\tfrac{1}{k}\right)\,dx \\ &\leq \tfrac{1}{2k} - \tfrac{1}{2(k+1)}.\end{aligned}$$

We see that $D_n \leq \sum_{k=1}^{\infty} \frac{1}{2k} - \frac{1}{2(k+1)} = \frac{1}{2}$ for all n. Denote by D_∞ the sum (2.4.2) for infinite n. Then $R_n = D_\infty - D_n = \frac{\theta}{2n}$ for some nonnegative $\theta < 1$, and we get

$$\begin{aligned}D_\infty - \tfrac{\theta}{2n} &= \int_1^n \ln x\,dx - \tfrac{1}{2}\left(\ln n! + \ln(n-1)!\right) \\ &= \int_1^n \ln x\,dx - \ln n! + \tfrac{1}{2}\ln n.\end{aligned} \tag{2.4.3}$$

Substituting in (2.4.3) the value of the integral $\int_1^n \ln x\,dx = \int_1^n d(x\ln x - x) = (n\ln n - n) - (1\ln 1 - 1) = n\ln n - n + 1$, one gets

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + (1 - D_\infty) + \tfrac{\theta}{2n}.$$

Now we know that $1 \geq (1 - D_\infty) \geq \frac{1}{2}$, but it is possible to evaluate the value of D_∞ with more accuracy. Later we will prove that $1 - D_\infty = \sqrt{2\pi}$.

Problems.

1. Does $\sum_{k=1}^{\infty} \sin k$ converge?
2. Does $\sum_{k=1}^{\infty} \sin k^2$ converge?
3. Evaluate $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots - \frac{2}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} - \ldots$.
4. Prove: If $\lim \frac{a_{n+1}}{a_n} < 1$, then $\sum_{k=1}^{\infty} a_k$ converge.
5. Prove: If $\sum_{k=1}^{\infty} |a_k - a_{k-1}| < \infty$, then $\{a_k\}$ converges.
6. Prove the convergence of $\sum_{k=1}^{\infty} \dfrac{(-1)^{[\sqrt{k}]}}{k}$.
7. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{\ln^3 k}$.
8. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k \sqrt{\ln \ln k}}$.
9. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k (\ln \ln k)^2}$.
10. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ and find its asymptotic formula.
11. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$.
12. Which partial sum of the above series is 0.01 close to its ultimate sum?
13. Evaluate $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$ with precision 0.01.
14. Evaluate $\int_1^3 \ln x \, d[x]$.
15. Express the Stirling constant via the Wallis product $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1}$.