2.2. Definite Integral

On the contents of the lecture. Areas of curvilinear trapezia play an extraordinary important role in mathematics. They generate a key concept of Calculus — the concept of the *integral.*

Three basic rules. For a nonnegative function f its integral $\int_a^b f(x)\,dx$ along the interval $[a,b]$ is defined just as the area of the curvilinear trapezium below the graph of f over $[a,b]$. We allow a function to take infinite values. Let us remark that changing of the value of function in one point does not affect the integral, because the area of the line is zero. That is why we allow the functions under consideration to be undefined in a finite number of points of the interval.

Immediately from the definition one gets the following three *basic rules of integration*:

Rule of constant	$\int_a^b f(x)\,dx = c(b-a),$ if $f(x)=c$	for $x \in (a,b)$,
Rule of inequality	$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx,$ if $f(x) \leq g(x)$	for $x \in (a,b)$,
Rule of partition	$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	for $b \in (a,c)$.

Partition. Let $|J|$ denote the length of an interval J. Let us say that a sequence $\{J_k\}_{k=1}^n$ of disjoint open subintervals of an interval I is a *partition* of I, if $\sum_{k=1}^n |I_k| = |I|$. The boundary of a partition $P = \{J_k\}_{k=1}^n$ is defined as the difference $I \setminus \bigcup_{k=1}^n J_k$ and is denoted ∂P.

For any finite subset S of an interval I, which contains the ends of I, there is a unique partition of I which has this set as the boundary. Such a partition is called *generated* by S. For a monotone sequence $\{x_k\}_{k=0}^n$ the generated partition is $\{(x_{k-1}, x_k)\}_{k=1}^n$.

Piecewise constant functions. A function $f(x)$ is called *partially constant* on a partition $\{J_k\}_{k=1}^n$ of $[a,b]$ if it is constant on each J_k. The Rules of Constant and Partition immediately imply:

$$\int_a^b f(x)\,dx = \sum_{k=1}^n f(J_k)|J_k|. \tag{2.2.1}$$

Proof. Indeed, the integral splits into a sum of integrals over $J_k = [x_{k-1}, x_k]$, and the function takes the value $f(J_k)$ in (x_{k-1}, x_k). □

A function is called *piecewise constant* over an interval if it is partially constant with respect to some finite partition of the interval.

Lemma 2.2.1. *Let f and g be piecewise constant functions over $[a,b]$. Then $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.*

Proof. First, suppose $f(x) = c$ is constant on the interval (a,b). Let g take the value g_k over the interval (x_k, x_{k+1}) for an exhausting $\{x_k\}_{k=0}^n$. Then $f(x)+g(x)$ takes values $(c+g_k)$ over (x_k, x_{k+1}). Hence $\int_a^b (f(x)+g(x))\,dx = \sum_{k=0}^{n-1}(c+g_k)|\delta x_k|$ due to (2.2.1). Splitting this sum and applying (2.2.1) to both summands, one gets $\sum_{k=0}^{n-1} c|\delta x_k| + \sum_{k=0}^{n-1} g_k|\delta x_k| = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. This proves the case of a constant f.

Now let f be partially constant on the partition generated by $\{x_k\}_{k=0}^n$. Then, by the partition rule, $\int_a^b (f(x)+g(x))\,dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx$. As f is constant on any (x_{k-1}, x_k), for any k one gets $\int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx = \int_{x_{k-1}}^{x_k} f(x)\,dx + \int_{x_{k-1}}^{x_k} g(x)\,dx$. Summing up these equalities one completes the proof of Lemma 2.2.1 for the sum.

The statement about differences follows from the addition formula applied to $g(x)$ and $f(x) - g(x)$. ☐

LEMMA 2.2.2. *For any monotone nonnegative function f on the interval $[a, b]$ and for any $\varepsilon > 0$ there is such piecewise constant function f_ε such that $f_\varepsilon \le f(x) \le f_\varepsilon(x) + \varepsilon$.*

PROOF. $f_\varepsilon(x) = \sum_{k=0}^\infty k\varepsilon[k\varepsilon \le f(x) < (k+1)\varepsilon]$. ☐

THEOREM 2.2.3 (Addition Theorem). *Let f and g be nonnegative monotone functions defined on $[a, b]$. Then*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

PROOF. Let f_ε and g_ε be ε-approximations of f and g respectively provided by Lemma 2.2.2. Set $f^\varepsilon(x) = f_\varepsilon(x) + \varepsilon$ and $g^\varepsilon(x) = g_\varepsilon(x) + \varepsilon$. Then $f_\varepsilon(x) \le f(x) \le f^\varepsilon(x)$ and $g_\varepsilon(x) \le g(x) \le g^\varepsilon(x)$ for $x \in (a, b)$. Summing and integrating these inequalities in different order gives

$$\int_a^b (f_\varepsilon(x) + g_\varepsilon(x))\,dx \le \int_a^b (f(x) + g(x))\,dx \le \int_a^b (f^\varepsilon(x) + g^\varepsilon(x))\,dx$$

$$\int_a^b f_\varepsilon(x)\,dx + \int_a^b g_\varepsilon(x)\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le \int_a^b f^\varepsilon(x)\,dx + \int_a^b g^\varepsilon(x)\,dx.$$

Due to Lemma 2.2.1, the left-hand sides of these inequalities coincide, as well as the right-hand sides. Hence the difference between the central parts does not exceed

$$\int_a^b (f^\varepsilon(x) - f_\varepsilon(x))\,dx + \int_a^b (g^\varepsilon(x) - g_\varepsilon(x))\,dx \le 2\varepsilon(b - a).$$

Hence, for any positive ε

$$\left|\int_a^b (f(x) + g(x))\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| < 2\varepsilon(b - a).$$

This implies that the left-hand side vanishes. ☐

Term by term integration of a functional series.

LEMMA 2.2.4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of nonnegative nondecreasing functions and let p be a piecewise constant function. If $\sum_{k=1}^\infty f_k(x) \ge p(x)$ for all $x \in [a, b]$ then $\sum_{k=1}^\infty \int_a^b f_k(x)\,dx \ge \int_a^b p(x)\,dx$.*

PROOF. Let p be a piecewise constant function with respect to $\{x_i\}_{i=0}^n$. Choose any positive ε. Since $\sum_{k=1}^\infty f_k(x_i) \ge p(c)$, eventually one has $\sum_{k=1}^m f_k(x_i) > p(x_i) - \varepsilon$. Fix m such that this inequality holds simultaneously for all $\{x_i\}_{i=0}^n$. Let $[x_i, x_{i+1}]$ be an interval where $p(x)$ is constant. Then for any $x \in [x_i, x_{i+1}]$ one has these inequalities: $\sum_{k=1}^m f_k(x) \ge \sum_{k=1}^m f_k(x_k) > p(x_k) - \varepsilon = p(x) - \varepsilon$. Consequently for all $x \in [a,b]$ one has the inequality $\sum_{k=1}^{m} f_k(x) > p(x) - \varepsilon$. Taking integrals gives $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx \geq \int_a^b (p(x)-\varepsilon)\,dx = \int_a^b p(x)\,dx - \varepsilon(b-a)$. By the Addition Theorem $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx = \sum_{k=1}^{m} \int_a^b f_k(x)\,dx \leq \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Therefore $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx - \varepsilon(b-a)$ for any positive ε. This implies the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx$. □

THEOREM 2.2.5. *For any sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative nondecreasing functions on an interval $[a,b]$*

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

PROOF. Since $\sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{\infty} f_k(x)$ for all x, by integrating one gets

$$\int_a^b \sum_{k=1}^{n} f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx.$$

By the the Addition Theorem the left-hand side is equal to $\sum_{k=1}^{n} \int_a^b f_k(x)\,dx$, which is a partial sum of $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Then by All-for-One one gets the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$.

To prove the opposite inequality for any positive ε, we apply Lemma 2.2.2 to find a piecewise constant function F_ε, such that $F_\varepsilon(x) \leq \sum_{k=1}^{\infty} f_k(x)dx$ and $\int_a^b \sum_{k=1}^{\infty} (f_k(x) - F_\varepsilon(x))\,dx < \varepsilon$. On the other hand, by Lemma 2.2.4 one gets

$$\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b F_\varepsilon(x)\,dx.$$

Together these inequalities imply $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx + \varepsilon \geq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$. As the last inequality holds for all $\varepsilon > 0$, it holds also for $\varepsilon = 0$ □

THEOREM 2.2.6 (Mercator,1668). *For any $x \in (-1,1]$ one has*

$$\boxed{\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}} \tag{2.2.2}$$

PROOF. Consider $x \in [0,1)$. Since $\int_0^x t^k\,dt = \frac{t^{k+1}}{k+1}$ due to the Fermat Theorem 2.1.2, termwise integration of the geometric series $\sum_{k=0}^{\infty} t^k$ over the interval $[0,x]$ for $x < 1$ gives $\int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$.

LEMMA 2.2.7. $\int_0^x \frac{1}{1-t}\,dt = \ln(1-x)$.

PROOF OF LEMMA. Construct a translation of the plane which transforms the curvilinear trapezium below $\frac{1}{1-t}$ over $[0,x]$ into the trapezium for $\ln(1-x)$. Indeed, the reflection of the plane $((x,y) \to (2-x,y))$ along the line $x=1$ transforms this trapezium to the curvilinear trapezium under $\frac{1}{x-1}$ over $[2-x,2]$. The parallel translation by 1 to the left of the latter trapezium $(x,y) \to (x-1,y)$ transforms it just in to the ogarithmic trapezium for $\ln(1-x)$. □

The Lemma proves the Mercator Theorem for negative x. To prove it for positive x, set $f_k(x) = x^{2k-1} - x^{2k}$. All functions f_k are nonnegative on $[0,1]$ and

$\sum_{k=1}^{\infty} f_k(x) = \frac{1}{1+x}$. Termwise integration of this equality over $[0,x]$ gives (2.2.2), modulo the equality $\int_0^x \frac{1}{1+t}\,dt = \int_1^x \frac{1}{t}\,dt$. The latter is proved by parallel translation of the plane. Let us remark, that in the case $x = 1$ the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$ is not absolutely convergent, and under its sum we mean $\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = \sum_{k=1}^{\infty}(\frac{1}{2k-1} - \frac{1}{2k})$. And the above proof proves just this fact. □

The arithmetic mean of Mercator's series evaluated at x and $-x$ gives *Gregory's Series*

$$\boxed{\frac{1}{2}\ln\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots.} \tag{2.2.3}$$

Gregory's series converges much faster than Mercator's one. For example, putting $x = \frac{1}{3}$ in (2.2.3) one gets

$$\ln 2 = \frac{2}{3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \frac{2}{7\cdot 3^7} + \dots.$$

Problems.

1. Prove that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

2. Prove the following formulas via piecewise constant approximations:

(multiplication formula) $$\int_a^b \lambda f(x)\,dx = \lambda\int_a^b f(x)\,dx$$

(shift formula) $$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

(reflection formula) $$\int_0^a f(x)\,dx = \int_{-a}^0 f(-x)\,dx$$

(compression formula) $$\int_0^a f(x)\,dx = \frac{1}{k}\int_0^{ka} f\left(\frac{x}{k}\right)\,dx$$

3. Evaluate $\int_0^{2\pi}(\sin x + 1)\,dx$.

4. Prove the inequality $\int_{-2}^{2}(2 + x^3 2^x)\,dx > 8$.

5. Prove $\int_0^{2\pi} x(\sin x + 1)\,dx < 2\pi$.

6. Prove $\int_{100\pi}^{200\pi} \frac{x+\sin(x)}{x}\,dx \le 100\pi + \frac{1}{50}\pi$.

7. Denote by s_n the area of $\{(x,y) \mid 0 \le x \le 1, (1-x)\ln n + x\ln(n+1) \le y \le \ln(1+x)\}$. Prove that $\sum_{k=1}^{\infty} s_k < \infty$.

8. Prove that $\sum_{k=1}^{2n}(-1)^{k+1}\frac{x^k}{k} < \ln(1+x) < \sum_{k=1}^{2n+1}(-1)^{k+1}\frac{x^k}{k}$ for $x > 0$.

9. Compute the logarithms of the primes $2, 3, 5, 7$ with accuracy 0.01.

10. Evaluate $\int_0^1 \sqrt{x}\,dx$.

***11.** Evaluate $\int_0^{\pi} \sin x\,dx$.