1.6. Complex Series

On the contents of the lecture. Complex numbers hide the key to the Euler Series. The summation theory developed for positive series now extends to complex series. We will see that complex series can help to sum real series.

Cubic equation. Complex numbers arise in connection with the solution of the cubic equation. The substitution $x = y - \frac{a}{3}$ reduces the general cubic equation $x^3 + ax^2 + bx + c = 0$ to

$$y^3 + py + q = 0.$$

The reduced equation one solves by the following trick. One looks for a root in the form $y = \alpha+\beta$. Then $(\alpha+\beta)^3+p(\alpha+\beta)+q = 0$ or $\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)+p(\alpha+\beta)+q = 0$. The latter equality one reduces to the system

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 3\alpha\beta &= -p. \end{aligned} \tag{1.6.1}$$

Raising the second equation into a cube one gets

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 27\alpha^3\beta^3 &= -p^3. \end{aligned}$$

Now α^3, β^3 are roots of the quadratic equation

$$x^2 + qx - \tfrac{p^3}{27},$$

called the *resolution* of the original cubic equation. Sometimes the resolution has no roots, while the cubic equation always has a root. Nevertheless one can evaluate a root of the cubic equation with the help of its resolution. To do this one simply ignores that the numbers under the square roots are negative.

For example consider the following cubic equation

$$x^3 - \tfrac{3}{2}x - \tfrac{1}{2} = 0. \tag{1.6.2}$$

Then (1.6.1) turns into

$$\begin{aligned} \alpha^3 + \beta^3 &= \tfrac{1}{2}, \\ \alpha^3\beta^3 &= \tfrac{1}{8}, \end{aligned}$$

The corresponding resolution is $t^2 - \frac{t}{2} + \frac{1}{8} = 0$ and its roots are

$$t_{1,2} = \tfrac{1}{4} \pm \sqrt{\tfrac{1}{16} - \tfrac{1}{8}} = \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{-1}.$$

Then the desired root of the cubic equation is given by

$$\sqrt[3]{\tfrac{1}{4}(1+\sqrt{-1})} + \sqrt[3]{\tfrac{1}{4}(1-\sqrt{-1})} = \tfrac{1}{\sqrt[3]{4}}\left(\sqrt[3]{1+\sqrt{-1}} + \sqrt[3]{1-\sqrt{-1}}\right). \tag{1.6.3}$$

It turns out that the latter expression one uniquely interprets as a real number which is a root of the equation (1.6.2). To evaluate it consider the following expression

$$\sqrt[3]{(1+\sqrt{-1})^2} - \sqrt[3]{(1+\sqrt{-1})}\sqrt[3]{(1-\sqrt{-1})} + \sqrt[3]{(1-\sqrt{-1})^2}. \tag{1.6.4}$$

Since

$$(1+\sqrt{-1})^2 = 1^2 + 2\sqrt{-1} + \sqrt{-1}^2 = 1 + 2\sqrt{-1} - 1 = 2\sqrt{-1},$$

the left summand of (1.6.4) is equal to

$$\sqrt[3]{2\sqrt{-1}} = \sqrt[3]{2}\sqrt[3]{\sqrt{-1}} = \sqrt[3]{2}\sqrt{\sqrt[3]{-1}} = \sqrt[3]{2}\,\sqrt{-1}.$$

Similarly $(1-\sqrt{-1})^2 = -2\sqrt{-1}$, and the right summand of (1.6.4) turns into $-\sqrt[3]{2}\sqrt{-1}$. Finally $(1+\sqrt{-1})(1-\sqrt{-1}) = 1^2 - \sqrt{-1}^2 = 2$ and the central one is $-\sqrt[3]{2}$. As a result the whole expression (1.6.4) is evaluated as $-\sqrt[3]{2}$.

On the other hand one evaluates the product of (1.6.3) and (1.6.4) by the usual formula as the sum of cubes

$$\tfrac{1}{\sqrt[3]{4}}((1+\sqrt{-1})+(1-\sqrt{-1})) = \tfrac{1}{\sqrt[3]{4}}((1+1)+(\sqrt{-1})-\sqrt{-1})) = \tfrac{1}{\sqrt[3]{4}}(2+0) = \sqrt[3]{2}.$$

Consequently (1.6.3) is equal to $\frac{\sqrt[3]{2}}{-\sqrt[3]{2}} = -1$. And -1 is a true root of (1.6.2).

Arithmetic of complex numbers. In the sequel we use i instead of $\sqrt{-1}$. There are two basic ways to represent a complex number. The representation $z = a + ib$, where a and b are real numbers we call the *Cartesian form* of z. The numbers a and b are called respectively the *real* and the *imaginary* parts of z and are denoted by $\operatorname{Re} z$ and by $\operatorname{Im} z$ respectively. Addition and multiplication of complex numbers are defined via their real and imaginary parts as follows

$$\begin{aligned}
\operatorname{Re}(z_1+z_2) &= \operatorname{Re} z_1 + \operatorname{Re} z_2,\\
\operatorname{Im}(z_1+z_2) &= \operatorname{Im} z_1 + \operatorname{Im} z_2,\\
\operatorname{Re}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2,\\
\operatorname{Im}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Im} z_1 \operatorname{Re} z_2.
\end{aligned}$$

The *trigonometric form* of a complex number is $z = \rho(\cos\phi + i\sin\phi)$, where $\rho \geq 0$ is called the *module* or the *absolute value* of a complex number z and is denoted $|z|$, and ϕ is called its *argument*. The argument of a complex number is defined modulo 2π. We denote by $\operatorname{Arg} z$ the set of all arguments of z, and by $\arg z$ the element of $\operatorname{Arg} z$ which satisfies the inequalities $-\pi < \arg z \leq \pi$. So $\arg z$ is uniquely defined for all complex numbers. $\arg z$ is called the *principal argument* of z.

The number $a - bi$ is called the *conjugate* to $z = a + bi$ and denoted $\overline{z}$. One has $z\overline{z} = |z|^2$. This allows us to express z^{-1} as $\frac{\overline{z}}{|z|^2}$.

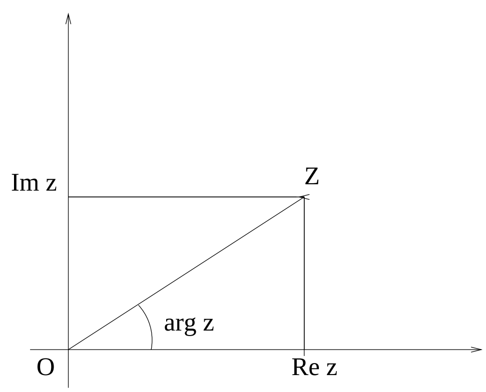

FIGURE 1.6.1. The representation of a complex number

If $z = a+ib$ then $|z| = \sqrt{a^2+b^2}$ and $\arg z = \operatorname{arctg}\frac{b}{a}$. One represents a complex number $z = a+bi$ as a point Z of the plane with coordinates (a,b). Then $|z|$ is equal to the distance from Z to the origin O. And $\arg z$ represents the angle between the axis of abscises and the ray $\overrightarrow{OZ}$. Addition of complex numbers corresponds to usual vector addition. And the usual triangle inequality turns into the *module inequality*:

$$|z+\zeta| \leq |z| + |\zeta|.$$

The multiplication formula for complex numbers in the trigonometric form is especially simple:

$$\begin{aligned} r(\cos\phi + i\sin\phi)r'(\cos\psi + i\sin\psi) \\ = rr'(\cos(\phi+\psi) + i\sin(\phi+\psi)). \end{aligned} \tag{1.6.5}$$

Indeed, the left-hand side and the right-hand side of (1.6.5) transform to

$$rr'(\cos\phi\cos\psi - \sin\phi\sin\psi) + irr'(\sin\phi\cos\psi + \sin\psi\cos\phi).$$

That is, the module of the product is equal to the product of modules and the argument of product is equal to the sum of arguments:

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 \oplus \operatorname{Arg} z_2.$$

Any complex number is uniquely defined by its module and argument.

The multiplication formula allows us to prove by induction the following:

(Moivre Formula) $$(\cos\phi + i\sin\phi)^n = (\cos n\phi + i\sin n\phi).$$

Sum of a complex series. Now is the time to extend our summation theory to series made of complex numbers. We extend the whole theory without any losses to so-called absolutely convergent series. The series $\sum_{k=1}^{\infty} z_k$ with arbitrary complex terms is called *absolutely convergent*, if the series $\sum_{k=1}^{\infty} |z_k|$ of absolute values converges.

For any real number x one defines two nonnegative numbers: its *positive* x^+ *and negative* x^- *parts* as $x^+ = x[x \geq 0]$ and $x^- = -x[x < 0]$. The following identities characterize the positive and negative parts of x

$$x^+ + x^- = |x|, \qquad x^+ - x^- = x.$$

Now the sum of an absolutely convergent series of real numbers is defined as follows:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-. \tag{1.6.6}$$

That is, from the sum of all positive summands one subtracts the sum of modules of all negative summands. The two series on the right-hand side converge, because $a_k^+ \leq |a_k|$, $a_k^- \leq |a_k|$ and $\sum_{k=1}^{\infty} |a_k| < \infty$.

For an absolutely convergent complex series $\sum_{k=1}^{\infty} z_k$ we define the real and imaginary parts of its sum separately by the formulas

$$\operatorname{Re}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Re} z_k, \qquad \operatorname{Im}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Im} z_k. \tag{1.6.7}$$

The series in the right-hand sides of these formulas are absolutely convergent, since $|\operatorname{Re} z_k| \leq |z_k|$ and $|\operatorname{Im} z_k| \leq |z_k|$.

THEOREM 1.6.1. *For any pair of absolutely convergent series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *its termwise sum* $\sum_{k=1}^{\infty}(a_k+b_k)$ *absolutely converges and*

$$\sum_{k=1}^{\infty}(a_k+b_k)=\sum_{k=1}^{\infty}a_k+\sum_{k=1}^{\infty}b_k. \tag{1.6.8}$$

PROOF. First, remark that the absolute convergence of the series on the left-hand side follows from the Module Inequality $|a_k+b_k|\le|a_k|+|b_k|$ and the absolute convergence of the series on the right-hand side.

Now consider the case of real numbers. Representing all sums in (1.6.8) as differences of their positive and negative parts and separating positive and negative terms in different sides one transforms (1.6.8) into

$$\sum_{k=1}^{\infty}a_k^+ +\sum_{k=1}^{\infty}b_k^+ +\sum_{k=1}^{\infty}(a_k+b_k)^- =\sum_{k=1}^{\infty}a_k^- +\sum_{k=1}^{\infty}b_k^- +\sum_{k=1}^{\infty}(a_k+b_k)^+.$$

But this equality is true due to termwise addition for positive series and the following identity,

$$x^- + y^- + (x+y)^+ = x^+ + y^+ + (x+y)^-.$$

Moving terms around turns this identity into

$$(x+y)^+ - (x+y)^- = (x^+ - x^-) + (y^+ - y^-),$$

which is true due to the identity $x^+ - +x^- = x$.

In the complex case the equality (1.6.8) splits into two equalities, one for real parts and another for imaginary parts. As for real series the termwise addition is already proved, we can write the following chain of equalities,

$$\begin{aligned}\operatorname{Re}\left(\textstyle\sum_{k=1}^{\infty}a_k+\sum_{k=1}^{\infty}b_k\right)&=\operatorname{Re}\textstyle\sum_{k=1}^{\infty}a_k+\operatorname{Re}\sum_{k=1}^{\infty}b_k\\&=\textstyle\sum_{k=1}^{\infty}\operatorname{Re}a_k+\sum_{k=1}^{\infty}\operatorname{Re}b_k\\&=\textstyle\sum_{k=1}^{\infty}(\operatorname{Re}a_k+\operatorname{Re}b_k)\\&=\textstyle\sum_{k=1}^{\infty}\operatorname{Re}(a_k+b_k)\\&=\operatorname{Re}\textstyle\sum_{k=1}^{\infty}(a_k+b_k),\end{aligned}$$

which proves the equality of real parts in (1.6.8). The same proof works for the imaginary parts. □

Sum Partition Theorem. An unordered sum of a family of complex numbers is defined by the same formulas (1.6.6) and (1.6.7). Since for positive series non-ordered sums coincide with the ordered sums, we get the same coincidence for all absolutely convergent series. Hence the commutativity law holds for all absolutely convergence series.

THEOREM 1.6.2. *If* $I=\bigsqcup_{j\in J}I_j$ *and* $\sum_{k=1}^{\infty}|a_k|<\infty$ *then* $\sum_{j\in J}\left|\sum_{i\in I_j}a_i\right|<\infty$ *and* $\sum_{j\in J}\sum_{i\in I_j}a_i=\sum_{i\in I}a_i$.

PROOF. At first consider the case of real summands. By definition $\sum_{i\in I}a_i=\sum_{i\in I}a_i^+ - \sum_{i\in I}a_i^-$. By Sum Partition Theorem positive series one transforms the original sum into

$$\textstyle\sum_{j\in J}\sum_{i\in I_j}a_i^+ - \sum_{j\in J}\sum_{i\in I_j}a_i^-.$$

Now by the Termwise Addition applied at first to external and after to internal sums one gets

$$\sum_{j\in J}\left(\sum_{i\in I_j} a_i^+ - \sum_{i\in I_j} a_i^-\right) = \sum_{j\in J}\sum_{i\in I_j}(a_i^+ - a_i^-) = \sum_{j\in J}\sum_{i\in I_j} a_i.$$

So the Sum Partition Theorem is proved for all absolutely convergent real series. And it immediately extends to absolutely convergent complex series by its splitting into real and imaginary parts. □

THEOREM 1.6.3 (Termwise Multiplication). *If* $\sum_{k=1}^\infty |z_k| < \infty$ *then for any (complex)* c, $\sum_{k=1}^\infty |cz_k| < \infty$ *and* $\sum_{k=1}^\infty cz_k = c\sum_{k=1}^\infty z_k$.

PROOF. Termwise Multiplication for positive numbers gives the first statement of the theorem $\sum_{k=1}^\infty |cz_k| = \sum_{k=1}^\infty |c||z_k| = |c|\sum_{k=1}^\infty |z_k|$. The further proof is divided into five cases.

At first suppose c is positive and z_k real. Then $cz_k^+ = cz_k^+$ and by virtue of Termwise Multiplication for positive series we get

$$\begin{aligned}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty cz_k^+ - \sum_{k=1}^\infty cz_k^- \\ &= c\sum_{k=1}^\infty z_k^+ - c\sum_{k=1}^\infty z_k^- \\ &= c\left(\sum_{k=1}^\infty z_k^+ - \sum_{k=1}^\infty z_k^-\right) \\ &= c\sum_{k=1}^\infty z_k.\end{aligned}$$

The second case. Let $c = -1$ and z_k be real. In this case

$$\sum_{k=1}^\infty -z_k = \sum_{k=1}^\infty (-z_k)^+ - \sum_{k=1}^\infty (-z_k)^- = \sum_{k=1}^\infty z_k^- - \sum_{k=1}^\infty z_k^+ = -\sum_{k=1}^\infty z_k.$$

The third case. Let c be real and z_k complex. In this case $\operatorname{Re} cz_k = c\operatorname{Re} z_k$ and the two cases above imply the Termwise Multiplication for any real c. Hence

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty \operatorname{Re} cz_k \\ &= \sum_{k=1}^\infty c\operatorname{Re} z_k \\ &= c\sum_{k=1}^\infty \operatorname{Re} z_k \\ &= c\operatorname{Re}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} c\sum_{k=1}^\infty z_k.\end{aligned}$$

The same is true for imaginary parts.

The fourth case. Let $c = i$ and z_k be complex. Then $\operatorname{Re} iz_k = -\operatorname{Im} z_k$ and $\operatorname{Im} iz_k = \operatorname{Re} z_k$. So one gets for real parts

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty iz_k &= \sum_{k=1}^\infty \operatorname{Re}(iz_k) \\ &= \sum_{k=1}^\infty -\operatorname{Im} z_k \\ &= -\sum_{k=1}^\infty \operatorname{Im} z_k \\ &= -\operatorname{Im}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} i\sum_{k=1}^\infty z_k.\end{aligned}$$

The general case. Let $c = a + bi$ with real a, b. Then

$$\begin{aligned} c\sum_{k=1}^\infty z_k &= a\sum_{k=1}^\infty z_k + ib\sum_{k=1}^\infty z_k \\ &= \sum_{k=1}^\infty az_k + \sum_{k=1}^\infty ibz_k \\ &= \sum_{k=1}^\infty (az_k + ibz_k) \\ &= \sum_{k=1}^\infty cz_k. \end{aligned}$$

□

Multiplication of Series. For two given series $\sum_{k=0}^\infty a_k$ and $\sum_{k=0}^\infty b_k$, one defines their *convolution* as a series $\sum_{n=0}^\infty c_n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$.

THEOREM 1.6.4 (Cauchy). *For any pair of absolutely convergent series* $\sum_{k=0}^\infty a_k$ *and* $\sum_{k=0}^\infty b_k$ *their convolution* $\sum_{k=0}^\infty c_k$ *absolutely converges and*

$$\sum_{k=0}^\infty c_k = \sum_{k=0}^\infty a_k \sum_{k=0}^\infty b_k.$$

PROOF. Consider the double series $\sum_{i,j} a_i b_j$. Then by the Sum Partition Theorem its sum is equal to

$$\sum_{j=0}^\infty \left(\sum_{i=0}^\infty a_i b_j\right) = \sum_{j=0}^\infty b_j \left(\sum_{i=0}^\infty a_i\right) = \left(\sum_{i=0}^\infty a_i\right)\left(\sum_{j=0}^\infty b_j\right).$$

On the other hand, $\sum_{i,j} a_i b_j = \sum_{n=0}^\infty \sum_{k=0}^{n+1-1} a_k b_{n-k}$. But the last sum is just the convolution.

This proof goes through for positive series. In the generalcase we have to prove absolute convergence of the double series. But this follows from

$$\left(\sum_{k=0}^\infty |a_k|\right)\left(\sum_{k=0}^\infty |b_k|\right) = \sum_{k=0}^\infty |c_k|.$$

□

Module Inequality.

$$\left|\sum_{k=1}^\infty z_k\right| \le \sum_{k=1}^\infty |z_k|. \tag{1.6.9}$$

Let $z_k = x_k + iy_k$. Summation of the inequalities $-|x_k| \le x_k \le |x_k|$ gives $-\sum_{k=1}^\infty |x_k| \le \sum_{k=1}^\infty x_k \le \sum_{k=1}^\infty |x_k|$, which means $|\sum_{k=1}^\infty x_k| \le \sum_{k=1}^\infty |x_k|$. The same inequality is true for y_k. Consider $z'_k = |x_k| + i|y_k|$. Then $|z_k| = |z'_k|$ and $|\sum_{k=1}^\infty z_k| \le |\sum_{k=1}^\infty z'_k|$. Therefore it is sufficient to prove the inequality (1.6.9) for z'_k, that is, for numbers with non-negative real and imaginary parts. Now supposing x_k, y_k to be nonnegative one gets the following chain of equivalent transformations of (1.6.9):

$$\begin{aligned} \left(\sum_{k=1}^\infty x_k\right)^2 + \left(\sum_{k=1}^\infty y_k\right)^2 &\le \left(\sum_{k=1}^\infty |z_k|\right)^2 \\ \sum_{k=1}^\infty x_k &\le \sqrt{\left(\sum_{k=1}^\infty |z_k|\right)^2 - \left(\sum_{k=1}^\infty y_k\right)^2} \\ \sum_{k=1}^n x_k &\le \sqrt{\left(\sum_{k=1}^\infty |z_k|\right)^2 - \left(\sum_{k=1}^\infty y_k\right)^2}, \quad \forall n = 1, 2, \dots \\ \sum_{k=1}^\infty y_k &\le \sqrt{\left(\sum_{k=1}^\infty |z_k|\right)^2 - \left(\operatorname{Re}\sum_{k=1}^n x_k\right)^2}, \quad \forall n = 1, 2, \dots \\ \sum_{k=1}^m y_k &\le \sqrt{\left(\sum_{k=1}^\infty |z_k|\right)^2 - \left(\sum_{k=1}^n x_k\right)^2}, \quad \forall n, m = 1, 2, \dots \\ \left(\sum_{k=1}^n x_k\right)^2 + \left(\sum_{k=1}^m y_k\right)^2 &\le \left(\sum_{k=1}^\infty |z_k|\right)^2, \quad \forall m, n = 1, 2, \dots \end{aligned}$$

$$\sqrt{\left(\sum_{k=1}^N x_k\right)^2 + \left(\sum_{k=1}^N y_k\right)^2} \leq \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$
$$\left|\sum_{k=1}^N z_k\right| \leq \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$

The inequalities of the last system hold because $\left|\sum_{k=1}^N z_k\right| \leq \sum_{k=1}^N |z_k| \leq \sum_{k=1}^\infty |z_k|$.

Complex geometric progressions. The sum of a geometric progression with a complex ratio is given by the same formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}. \tag{1.6.10}$$

And the proof is the same as in the case of real numbers. But the meaning of this formula is different. Any complex formula is in fact a pair of formulas. Any complex equation is in fact a pair of equations.

In particular, for $z = q(\sin\phi + i\cos\phi)$ the real part of the left-hand side of (1.6.10) owing to the Moivre Formula turns into $\sum_{k=0}^{n-1} q^k \sin k\phi$ and the right-hand side turns into $\sum_{k=0}^{n-1} q^k \cos k\phi$. So the formula for a geometric progression splits into two formulas which allow us to telescope some trigonometric series.

Especially interesting is the case with the ratio $\varepsilon_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. In this case the geometric progression cyclically takes the same values, because $\varepsilon_n^n = 1$. The terms of this sequence are called the *roots of unity*, because they satisfy the equation $z^n - 1 = 0$.

LEMMA 1.6.5. $(z^n - 1) = \prod_{k=1}^n (z - \varepsilon_n^k)$.

PROOF. Denote by $P(z)$ the right-hand side product. This polynomial has degree n, has major coefficient 1 and has all ε_n^k as its roots. Then the difference $(z^n - 1) - P(z)$ is a polynomial of degree $< n$ which has n different roots. Such a polynomial has to be 0 by virtue of the following general theorem. □

THEOREM 1.6.6. *The number of roots of any nonzero complex polynomial does not exceed its degree.*

PROOF. The proof is by induction on the degree of $P(z)$. A polynomial of degree 1 has the form $az+b$ and the only root is $-\frac{b}{a}$. Suppose our theorem is proved for any polynomial of degree $< n$. Consider a polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$ of degree n, where the coefficients are complex numbers. Suppose it has at least n roots $z_1, \dots, z_n$. Consider the polynomial $P^*(z) = a_n \prod_{k=1}^n (z - z_k)$. The difference $P(z) - P^*(z)$ has degree $< n$ and has at least n roots (all z_k). By the induction hypothesis this difference is zero. Hence, $P(z) = P^*(z)$. But $P^*(z)$ has only n roots. Indeed, for any z different from all z_k one has $|z - z_k| > 0$. Therefore $|P^*(z)| = |a_n| \prod_{k=1}^n |z - z_k| > 0$. □

By blocking conjugated roots one gets a pure real formula:

$$z^n - 1 = (z-1) \prod_{k=1}^{(n-1)/2} \left(z^2 - 2z\cos\frac{2k\pi}{n} + 1\right).$$

Complexification of series. Complex numbers are effectively applied to sum up so-called *trigonometric series*, i.e., series of the type $\sum_{k=0}^{\infty} a_k \cos kx$ and $\sum_{k=0}^{\infty} a_k \sin kx$. For example, to sum the series $\sum_{k=1}^{\infty} q^k \sin k\phi$ one couples it with its dual $\sum_{k=0}^{\infty} q^k \cos k\phi$ to form a complex series $\sum_{k=0}^{\infty} q^k(\cos k\phi + i \sin k\phi)$. The last is a complex geometric series. Its sum is $\frac{1}{1-z}$, where $z = \cos\phi + i\sin\phi$. Now the sum of the sine series $\sum_{k=1}^{\infty} q^k \sin k\phi$ is equal to $\operatorname{Im}\frac{1}{1-z}$, the imaginary part of the complex series, and the real part of the complex series coincides with the cosine series. In particular, for $q = 1$, one has $\frac{1}{1-z} = \frac{1}{1+\cos\phi + i\sin\phi}$. To evaluate the real and imaginary parts one multiplies both numerator and denominator by $1 + \cos\phi - i\sin\phi$. Then one gets $(1-\cos\phi)^2 + \sin^2\phi = 1 - 2\cos^2\phi + \cos^2\phi + \sin^2\phi = 2 - 2\cos\phi$ as the denominator. Hence $\frac{1}{1-z} = \frac{1-\cos\phi + i\sin\phi}{2-2\cos\phi} = \frac{1}{2} + \frac{1}{2}\cot\frac{\phi}{2}$. And we get two remarkable formulas for the sum of the divergent series

$$\sum_{k=0}^{\infty} \cos k\phi = \frac{1}{2}, \qquad \sum_{k=1}^{\infty} \sin k\phi = \frac{1}{2}\cot\frac{\phi}{2}.$$

For $\phi = 0$ the left series turns into $\sum_{k=0}^{\infty}(-1)^k$. The evaluation of the Euler series via this cosine series is remarkably short, it takes one line. But one has to know integrals and a something else to justify this evaluation.

Problems.

1. Find real and imaginary parts for $\frac{1}{1-i}$, $(\frac{1-i}{1+i})^3$, $\frac{i^5+2}{i^{19}+1}$, $\frac{(1+i)^5}{(1-i)^3}$.
2. Find trigonometric form for -1, $1+i$, $\sqrt{3}+i$.
3. Prove that $z_1 z_2 = 0$ implies either $z_1 = 0$ or $z_2 = 0$.
4. Prove the distributivity law for complex numbers.
5. Analytically prove the inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.
6. Evaluate $\sum_{k=1}^{n-1} \frac{1}{z_k(z_k+1)}$, where $z_k = 1 + kz$.
7. Evaluate $\sum_{k=1}^{n-1} z_k^2$, where $z_k = 1 + kz$.
8. Evaluate $\sum_{k=1}^{n-1} \dfrac{\sin k}{2^k}$.
9. Solve $z^2 = i$.
10. Solve $z^2 = 3 - 4i$.
11. Telescope $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
12. Prove that the conjugated to a root of polynomial with real coefficient is the root of the polynomial.
13. Prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
14. Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
*__15.__ Solve $8x^3 - 6x - 1 = 0$.
16. Evaluate $\sum_{k=1}^{\infty} \dfrac{\sin k}{2^k}$.
17. Evaluate $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
18. Prove absolute convergence of $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ for any z.
19. For which z the series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ absolutely converges?
20. Multiply a geometric series onto itself several times applying Cauchy formula.
21. Find series for $\sqrt{1+x}$ by method of indefinite coefficients.
22. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ absolutely converge?
23. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ absolutely converge?