1.5. Telescopic Sums

On the content of this lecture. In this lecture we learn the main secret of elementary summation theory. We will evaluate series via their partial sums. We introduce *factorial powers*, which are easy to sum. Following Stirling we expand $\frac{1}{1+x^2}$ into a series of negative factorial powers and apply this expansion to evaluate the Euler series with Stirling's accuracy of 10^{-8}.

The series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$. In the first lecture we calculated infinite sums directly without invoking partial sums. Now we present a dual approach to summing series. According to this approach, at first one finds a formula for the n-th partial sum and then substitutes in this formula infinity instead of n. The series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ gives a simple example for this method. The key to sum it up is the following identity

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Because of this identity $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ turns into the sum of differences

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\dots. \tag{1.5.1}$$

Its n-th partial sum is equal to $1-\frac{1}{n+1}$. Substituting in this formula $n=+\infty$, one gets 1 as its ultimate sum.

Telescopic sums. The sum (1.5.1) represents a *telescopic sum*. This name is used for sums of the form $\sum_{k=0}^{n}(a_k-a_{k+1})$. The value of such a telescopic sum is determined by the values of the first and the last of a_k, similarly to a telescope, whose thickness is determined by the radii of the external and internal rings. Indeed,

$$\sum_{k=0}^{n}(a_k-a_{k+1})=\sum_{k=0}^{n}a_k-\sum_{k=0}^{n}a_{k+1}=a_0+\sum_{k=1}^{n}a_k-\sum_{k=0}^{n-1}a_{k+1}-a_{n+1}=a_0-a_{n+1}.$$

The same arguments for infinite telescopic sums give

$$\sum_{k=0}^{\infty}(a_k-a_{k+1})=a_0. \tag{1.5.2}$$

But this proof works only if $\sum_{k=0}^{\infty}a_k<\infty$. This is untrue for $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$, owing to the divergence of the Harmonic series. But the equality (1.5.2) holds also if a_k tends to 0 as k tends to infinity. Indeed, in this case a_0 is the least number majorizing all a_0-a_n, the n-th partial sums of $\sum_{k=0}^{\infty}a_k$.

Differences. For a given sequence $\{a_k\}$ one denotes by $\{\Delta a_k\}$ the sequence of differences $\Delta a_k=a_{k+1}-a_k$ and calls the latter sequence the *difference* of $\{a_k\}$. This is the main formula of elementary summation theory.

$$\boxed{\sum_{k=0}^{n-1}\Delta a_k=a_n-a_0}$$

To telescope a series $\sum_{k=0}^{\infty}a_k$ it is sufficient to find a sequence $\{A_k\}$ such that $\Delta A_k=a_k$. On the other hand the sequence of sums $A_n=\sum_{k=0}^{n-1}a_k$ has difference $\Delta A_n=a_n$. Therefore, we see that to telescope a sum is equivalent to find a formula for partial sums. This lead to concept of a *telescopic function*. For a function $f(x)$ we introduce its difference $\Delta f(x)$ as $f(x+1) - f(x)$. A function $f(x)$ telescopes $\sum a_k$ if $\Delta f(k) = a_k$ for all k.

Often the sequence $\{a_k\}$ that we would like to telescope has the form $a_k = f(k)$ for some function. Then we are searching for a *telescopic function* $F(x)$ for $f(x)$, i.e., a function such that $\Delta F(x) = f(x)$.

To evaluate the difference of a function is usually much easier than to telescope it. For this reason one has evaluated the differences of all basic functions and organized a *table of differences*. In order to telescope a given function, look in this table to find a table function whose difference coincides with or is close to given function.

For example, the differences of x^n for $n \le 3$ are $\Delta x = 1$, $\Delta x^2 = 2x + 1$, $\Delta x^3 = 3x^2 + 3x + 1$. To telescope $\sum_{k=1}^{\infty} k^2$ we choose in this table x^3. Then $\frac{\Delta x^3}{3} - x^2 = x + \frac{1}{3} = \frac{\Delta x^2}{2} - \Delta \frac{x}{6}$. Therefore, $x^2 = \Delta\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}\right)$. This immediately implies the following formula for sums of squares:

$$\sum_{k=1}^{n-1} k^2 = \frac{2n^3 - 3n^2 + n}{6}. \tag{1.5.3}$$

Factorial powers. The usual powers x^n have complicated differences. The so-called *factorial powers* $x^{\underline{k}}$ have simpler differences. For any number x and any natural number k, let $x^{\underline{k}}$ denote $x(x-1)(x-2)\dots(x-k+1)$, and by $x^{\underline{-k}}$ we denote $\frac{1}{(x+1)(x+2)\dots(x+k)}$. At last we define $x^{\underline{0}} = 1$. The factorial power satisfies the following *addition law*.

$$\boxed{x^{\underline{k+m}} = x^{\underline{k}}(x-k)^{\underline{m}}}$$

We leave to the reader to check this rule for all integers m, k. The power $n^{\underline{n}}$ for a natural n coincides with the factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$. The main property of factorial powers is given by:

$$\boxed{\Delta x^{\underline{n}} = n x^{\underline{n-1}}}$$

The proof is straightforward:

$$\begin{aligned}(x+1)^{\underline{k}} - x^{\underline{k}} &= (x+1)^{\underline{1+(k-1)}} - x^{\underline{(k-1)+1}} \\ &= (x+1)x^{\underline{k-1}} - x^{\underline{k-1}}(x-k+1) \\ &= kx^{\underline{k-1}}.\end{aligned}$$

Applying this formula one can easily telescope any *factorial polynomial*, i.e., an expression of the form

$$a_0 + a_1 x^{\underline{1}} + a_2 x^{\underline{2}} + a_3 x^{\underline{3}} + \dots + a_n x^{\underline{n}}.$$

Indeed, the explicit formula for the telescoping function is

$$a_0 x^{\underline{1}} + \tfrac{a_1}{2} x^{\underline{2}} + \tfrac{a_2}{3} x^{\underline{3}} + \tfrac{a_3}{4} x^{\underline{4}} + \dots + \tfrac{a_n}{n+1} x^{\underline{n+1}}.$$

Therefore, another strategy to telescope x^k is to represent it as a factorial polynomial.

For example, to represent x^2 as factorial polynomial, consider $a + bx + cx^{\underline{2}}$, a general factorial polynomial of degree 2. We are looking for $x^2 = a + bx + cx^{\underline{2}}$. Substituting $x = 0$ in this equality one gets $a = 0$. Substituting $x = 1$, one gets

$1 = b$, and finally for $x = 2$ one has $4 = 2 + 2c$. Hence $c = 1$. As result $x^2 = x + x^{\underline{2}}$. And the telescoping function is given by

$$\tfrac{1}{2}x^{\underline{2}} + \tfrac{1}{3}x^{\underline{3}} = \tfrac{1}{2}(x^2 - x) + \tfrac{1}{3}(x(x^2 - 3x + 2)) = \tfrac{1}{6}(2x^3 - 3x^2 + x).$$

And we have once again proved the formula (1.5.3).

Stirling Estimation of the Euler series. We will expand $\frac{1}{(1+x)^2}$ into a series of negative factorial powers in order to telescope it. A natural first approximation to $\frac{1}{(1+x^2)}$ is $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$. We represent $\frac{1}{(1+x)^2}$ as $x^{\underline{-2}} + R_1(x)$, where

$$R_1(x) = \frac{1}{(1+x)^2} - x^{\underline{-2}} = \frac{1}{(x+1)^2(x+2)}.$$

The remainder $R_1(x)$ is in a natural way approximated by $x^{\underline{-3}}$. If $R_1(x) = x^{\underline{-3}} + R_2(x)$ then $R_2(x) = \frac{2}{(x+1)^2(x+2)(x+3)}$. Further, $R_2(x) = 2x^{\underline{-4}} + R_3(x)$, where

$$R_3(x) = \frac{2 \cdot 3}{(x+1)^2(x+2)(x+3)(x+4)} = \frac{3!}{x+1}x^{\underline{-4}}.$$

The above calculations lead to the conjecture

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{n-1} k!x^{\underline{-k-2}} + \frac{n!}{x+1}x^{\underline{-n-1}}. \tag{1.5.4}$$

This conjecture is easily proved by induction. The remainder $R_n(x) = \frac{n!}{x+1}x^{\underline{-n-1}}$ represents the difference $\frac{1}{(1+x)^2} - \sum_{k=0}^{n-1} k!x^{\underline{-2-k}}$. Owing to the inequality $x^{\underline{-1-n}} \leq \frac{1}{(n+1)!}$, which is valid for all $x \geq 0$, the remainder decreases to 0 as n increases to infinity. This implies

THEOREM 1.5.1. *For all $x \geq 0$ one has*

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} k!x^{\underline{-2-k}}.$$

To calculate $\sum_{k=p}^{\infty} \frac{1}{(1+k)^2}$, replace all summands by the expressions (1.5.4). We will get

$$\sum_{k=p}^{\infty} \left(\sum_{m=0}^{n-1} m!k^{\underline{-2-m}} + \frac{n!}{k+1}k^{\underline{-1-n}} \right).$$

Changing the order of summation we have

$$\sum_{m=0}^{n-1} m! \sum_{k=p}^{\infty} k^{\underline{-2-m}} + \sum_{k=p}^{\infty} \frac{n!}{k+1}k^{\underline{-1-n}}.$$

Since $\frac{1}{1+m}x^{\underline{-1-m}}$ telescopes the sequence $\{k^{\underline{-2-m}}\}$, $\sum_{k=p}^{\infty} k^{\underline{-2-m}} = \frac{1}{1+m}p^{\underline{-1-m}}$, Denote the sum of remainders $\sum_{k=p}^{\infty} \frac{n!}{k+1}k^{\underline{-1-n}}$ by $R(n,p)$. Then for all natural p and n one has

$$\sum_{k=p}^{\infty} \frac{1}{(1+k)^2} = \sum_{m=0}^{n-1} \frac{m!}{1+m}p^{\underline{-1-m}} + R(n,p)$$

For $p = 0$ and $n = +\infty$, the right-hand side turns into the Euler series, and one could get a false impression that we get nothing new. But $k^{\underline{-2-n}} \leq \frac{1}{k+1}k^{\underline{-1-n}} \leq (k-1)^{\underline{-2-n}}$, hence

$$\frac{n!}{1+n}p^{\underline{-1-n}} = \sum_{k=p}^{\infty} n!k^{\underline{-2-n}} \leq R(n,p) \leq \sum_{k=p}^{\infty} n!(k-1)^{\underline{-2-n}} = \frac{n!}{1+n}(p-1)^{\underline{-1-n}}.$$

Since $(p-1)^{\underline{-1-n}} - p^{\underline{-1-n}} = (1+n)(p-1)^{\underline{-2-n}}$, there is a $\theta \in (0,1)$ such that

$$R(n,p) = \frac{n!}{1+n}p^{\underline{-1-n}} + \theta n!(p-1)^{\underline{-2-n}}.$$

Finally we get:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=0}^{p-1}\frac{1}{(1+k)^2} + \sum_{k=0}^{n-1}\frac{k!}{1+k}p^{\underline{-1-k}} + \theta n!(p-1)^{\underline{-2-n}}.$$

For $p = n = 3$ this formula turns into

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{40} + \frac{1}{180} + \frac{\theta}{420}.$$

For $p = n = 10$ one gets $R(10,10) \leq 10!9^{\underline{-12}}$. After cancellations one has $\frac{1}{2\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 17\cdot 19}$. This is approximately $2\cdot 10^{-8}$. Therefore

$$\sum_{k=0}^{10-1}\frac{1}{(k+1)^2} + \sum_{k=0}^{10-1}\frac{k!}{1+k}10^{\underline{-1-k}}$$

is less than the sum of the Euler series by only $2\cdot 10^{-8}$. In such a way one can in one hour calculate eight digits of $\sum_{k=1}^{\infty}\frac{1}{k^2}$ after the decimal point. It is not a bad result, but it is still far from Euler's eighteen digits. For $p = 10$, to provide eighteen digits one has to sum essentially more than one hundred terms of the series. This is a bit too much for a person, but is possible for a computer.

Problems.

1. Telescope $\sum k^3$.
2. Represent x^4 as a factorial polynomial.
3. Evaluate $\sum_{k=1}^{\infty}\frac{1}{k(k+2)}$.
4. Evaluate $\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}$.
5. Prove: If $\Delta a_k \geq \Delta b_k$ for all k and $a_1 \geq b_1$ then $a_k \geq b_k$ for all k.
6. $\Delta(x+a)^{\underline{n}} = n(x+a)^{\underline{n-1}}$.
7. Prove Archimedes's inequality $\frac{n^3}{3} \leq \sum_{k=1}^{n-1}k^2 \leq \frac{(n+1)^3}{3}$.
8. Telescope $\sum_{k=1}^{\infty}\frac{k}{2^k}$.
9. Prove the inequalities $\frac{1}{n} \geq \sum_{k=n+1}^{\infty}\frac{1}{k^2} \geq \frac{1}{n+1}$.
10. Prove that the degree of $\Delta P(x)$ is less than the degree of $P(x)$ for any polynomial $P(x)$.
11. Relying on $\Delta 2^n = 2^n$, prove that $P(n) < 2^n$ eventually for any polynomial $P(x)$.
12. Prove $\sum_{k=0}^{\infty} k!(x-1)^{\underline{-1-k}} = \frac{1}{x}$.