1.3. Unordered Sums

On the contents of the lecture. Our summation theory culminates in the Sum Partition Theorem. This lecture will contribute towards evaluation of the Euler series in two ways: we prove its convergence, and even estimate its sum by 2. On the other hand, we will realize that evaluation of the Euler series with Euler's accuracy (10^{-18}) seems to be beyond a human being's strength.

Consider a family $\{a_i\}_{i\in I}$ of nonnegative numbers indexed by elements of an arbitrary set I. An important special case of I is the set of pairs of natural numbers $\mathbb{N}\times\mathbb{N}$. Families indexed by $\mathbb{N}\times\mathbb{N}$ are called double series. They arise when one multiplies one series by another one.

Any sum of the type $\sum_{i\in K} a_i$, where K is a finite subset of I is called a *subsum* of $\{a_i\}_{i\in I}$ over K.

DEFINITION. *The least number majorizing all subsums of $\{a_i\}_{i\in I}$ over finite subsets is called its (ultimate) sum and denoted by $\sum_{i\in I} a_i$*

The One-for-All and All-for-One principles for non-ordered sums are obtained from the corresponding principles for ordered sums by replacing "partial sums" by "finite subsums".

Commutativity. In case $I=\mathbb{N}$ we have a definition which apparently is new. But fortunately this definition is equivalent to the old one. Indeed, as any finite subsum of positive series does not exceed its ultimate (ordered) sum, the non-ordered sum also does not exceed it. On the other hand, any partial sum of the series is a finite subsum. This implies the opposite inequality. Therefore we have established the equality.

$$\sum_{k=1}^{\infty} a_k = \sum_{k\in\mathbb{N}} a_k$$

This means that positive series obey the Commutativity Law. Because the non-ordered sum obviously does not depend on the order of summands.

Partitions. A family of subsets $\{I_k\}_{k\in K}$ of a set I is called a *partition* of I and is written $\bigsqcup_{k\in K} I_k$ if $I=\bigcup_{k\in K} I_k$ and $I_k\cap I_j=\varnothing$ for all $k\neq j$.

THEOREM 1.3.1 (Sum Partition Theorem). *For any partition $I=\bigsqcup_{j\in J} I_j$ of the indexing set and any family $\{a_i\}_{i\in I}$ of nonnegative numbers,*

$$\sum_{i\in I} a_i = \sum_{j\in J}\sum_{i\in I_j} a_i. \tag{1.3.1}$$

Iverson notation. We will apply the following notation: a statement included into [ ] takes value 1, if the statement is true, and 0, if it is false. Prove the following simple lemmas to adjust to this notation. In both lemmas one has $K\subset I$.

LEMMA 1.3.2. $\sum_{i\in K} a_i = \sum_{i\in I} a_i[i\in K]$.

In particular, for $K=I$, Lemma 1.3.2 turns into

LEMMA 1.3.3. $\sum_{i\in I} a_i = \sum_{i\in I} a_i[i\in I]$.

LEMMA 1.3.4. $\sum_{k\in K}[i\in I_k]=[i\in I_K]$ *for all $i\in I$ iff $I_K=\bigsqcup_{k\in K} I_k$.*

Proof of Sum Partition Theorem. At first we prove the following Sum Transposition formula for finite J,

$$\sum_{i\in I}\sum_{j\in J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}. \tag{1.3.2}$$

Indeed, if J contains just two elements, this formula turns into the Termwise Addition formula. The proof of this formula is the same as for series. Suppose the formula is proved for any set which contains fewer elements than J does. Decompose J into a union of two nonempty subsets $J_1 \sqcup J_2$. Then applying only Termwise Addition and Lemmas 1.3.2, 1.3.3, 1.3.4, we get

$$\begin{aligned}\sum_{i\in I}\sum_{j\in J} a_{ij} &= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J]\\ &= \sum_{i\in I}\sum_{j\in J}(a_{ij}[j\in J_1] + a_{ij}[j\in J_2])\\ &= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_1] + \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_2]\\ &= \sum_{i\in I}\sum_{j\in J_1} a_{ij} + \sum_{i\in I}\sum_{j\in J_2} a_{ij}.\end{aligned}$$

But the last two sums can be transposed by the induction hypothesis. After such a transposition one gets

$$\begin{aligned}\sum_{j\in J_1}\sum_{i\in I} a_{ij} + \sum_{j\in J_2}\sum_{i\in I} a_{ij} &= \sum_{j\in J}[j\in J_1]\sum_{i\in I} a_{ij} + \sum_{j\in J}[j\in J_2]\sum_{i\in I} a_{ij}\\ &= \sum_{j\in J}([j\in J_1] + [j\in J_2])\sum_{i\in I} a_{ij}\\ &= \sum_{j\in J}[j\in J]\sum_{i\in I} a_{ij}\\ &= \sum_{j\in J}\sum_{i\in I} a_{ij}\end{aligned}$$

and the Sum Transposition formula for finite J is proved. Consider the general case. To prove $\leq$ in (1.3.2), consider a finite $K \subset I$. By the finite Sum Transposition formula the subsum $\sum_{i\in K}\sum_{j\in J} a_{ij}$ is equal to $\sum_{j\in J}\sum_{i\in K} a_{ij}$. But this sum is termwise majorized by the right-hand side sum in (1.3.2). Therefore the left-hand side does not exceed the right-hand side by All-for-One principle.

To derive the Sum Partition Theorem from the Sum Transposition formula, pose $a_{ij} = a_i[i\in I_j]$. Then $a_i = \sum_{j\in J} a_{ij}$ and (1.3.1) turns into (1.3.2). This completes the proof of the Sum Partition Theorem.

Blocking. For a given a series $\sum_{k=0}^{\infty} a_k$ and an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ starting with $n_0 = 0$ one defines a new series $\sum_{k=0}^{\infty} A_k$ by the rule $A_k = \sum_{i=n_k}^{n_{k+1}-1} a_i$. The series $\sum_{k=0}^{\infty} A_k$ is called *blocking of* $\sum_{k=0}^{\infty} a_k$ *by* $\{n_k\}$.

The Sum Partition Theorem implies that the sums of blocked and unblocked series coincide. Blocking formalizes putting of brackets. Therefore the Sum Partition Theorem implies the *Sequential Associativity Law*: *Placing brackets does not change the sum of series.*

Estimation of the Euler series. Let us compare the Euler series with the series $\sum_{k=0}^{\infty}\frac{1}{2^k}$, blocked by $\{2^n\}$ to $\sum_{k=1}^{\infty}a_k$. The sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ consists of 2^n summands, all of which are less then the first one, which is $\frac{1}{2^{2n}}$. As $2^n\frac{1}{2^{2n}}=\frac{1}{2^n}$, it follows that $a_n\le\frac{1}{2^n}$ for each n. Summing these inequalities, one gets $\sum_{k=1}^{\infty}a_k\le 2$.

Now let us estimate how many terms of Euler's series one needs to take into account to find its sum up to the eighteenth digit. To do this, we need to estimate its tail. The arguments above give $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\le\frac{1}{2^{n-1}}$. To obtain a lower estimate, let us remark that all terms of sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ exceed $\frac{1}{2^{2(n+1)}}$. As the number of summands is 2^n, one gets $a_n\ge\frac{1}{4\cdot 2^n}$. Hence $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\ge\frac{1}{2^{n+1}}$. Since $2^{10}=1024\simeq 10^3$, one gets $2^{60}\simeq 10^{18}$. So, to provide an accuracy of 10^{-18} one needs to sum approximately 10^{18} terms. This task is inaccessible even for a modern computer. How did Euler manage to do this? He invented a summation formula (Euler-MacLaurin formula) and transformed this slowly convergent series into non-positive divergent (!) one, whose partial sum containing as few as ten terms gave eighteen digit accuracy. The whole calculation took him an evening. To introduce this formula, one needs to know integrals and derivatives. We will do this later.

Problems.

1. Find $\sum_{k=1}^{\infty}\frac{1}{(2k)^2}$ and $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, assuming $\sum_{k=1}^{\infty}\frac{1}{k^2}=\pi^2/6$.
2. Prove the convergence of $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}}$.
3. Estimate how many terms of the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are necessary for calculation of its sum with precision 10^{-3}.
4. Estimate the value of $\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{k}$.
5. Prove the equality $\sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k=\sum_{j,k\in\mathbb{N}}a_jb_k$.
6. Estimate how many terms of the Harmonic series give the sum surpassing 1000.
7. Prove the Dirichlet formula $\sum_{k=1}^{n}\sum_{i=1}^{k}a_{ki}=\sum_{i=1}^{n}\sum_{k=i}^{n}a_{ki}$.
8. Evaluate $\sum_{i,j\in N}\frac{1}{2^i3^j}$.
9. Evaluate $\sum_{i,j\in N}\frac{i+j}{2^i3^j}$.
10. Represent an unordered sum $\sum_{i+j<n}a_{ij}$ as a double sum.
11. Evaluate $\sum_{i,j\in N}\frac{ij}{2^i3^j}$.
12. Change the summation order in $\sum_{i=0}^{\infty}\sum_{j=0}^{2i}a_{ij}$.
13. Define by Iverson notation the following functions:
- $[x]$ (integral part),
- $|x|$ (module),
- $\operatorname{sgn} x$ (signum),
- $n!$ (factorial).

14. Define only by formulas the expression $[p \text{ is prime}]$.