The Legend of Euler's Series

"One of the great mathematical challenges of the early 18th century was to find an expression for the sum of reciprocal squares

$$(\star)\qquad 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots$$

Joh. Bernoulli eagerly sought for this expression for many decades."[1]

In 1689 Jac. Bernoulli proved the convergence of the series. In 1728–1729 Goldbach and D. Bernoulli evaluated the series with an accuracy of 0.01. Stirling in 1730 found eight digits of the sum.

L. Euler in 1734 calculated the first eighteen digits (!) after the decimal point of the sum ($\star$) and recognized $\pi^2/6$, which has the same eighteen digits. He conjectured that the infinite sum is equal to $\pi^2/6$. In 1735 Euler discovered an expansion of the sine function into an infinite product of polynomials:

$$(\star\star)\qquad \frac{\sin x}{x}=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\left(1-\frac{x^2}{4^2\pi^2}\right)\cdots$$

Comparing this presentation with the standard sine series expansion

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$$

Euler not only proved that the sum ($\star$) is equal to $\pi^2/6$, moreover he calculated all sums of the type

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\frac{1}{5^k}+\cdots$$

for even k.

Putting $x=\pi/2$ in ($\star\star$) he got the beautiful Wallis Product

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\frac{8\cdot 8}{7\cdot 9}\cdots$$

which had been known since 1655. But Euler's first proof of ($\star\star$) was not satisfactory. In 1748, in his famous *Introductio in Analysin Infinitorum*, he presented a proof which was sufficiently rigorous for the 18th century. The series of reciprocal squares was named the *Euler series*.

If somebody wants to understand all the details of the above legend he has to study a lot of things, up to complex contour integrals. This is why the detailed mathematical exposition of the legend of Euler's series turns into an entire course of Calculus. The fascinating history of Euler's series is the guiding thread of the present course, *On Euler's footsteps*.

[1] E. Hairer, G. Wanner, *Analysis by its History*, Springer, 1997.