Uppsala Lectures on Calculus

On Euler's footsteps

E.V. Shchepin

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Topology Atlas
Toronto, Ontario, Canada
http://at.yorku.ca/topology/
atlas@at.yorku.ca

E.V. Shchepin
V.A. Steklov Institute of Mathematics
Russian Academy of Sciences
scepin@mi.ras.ru

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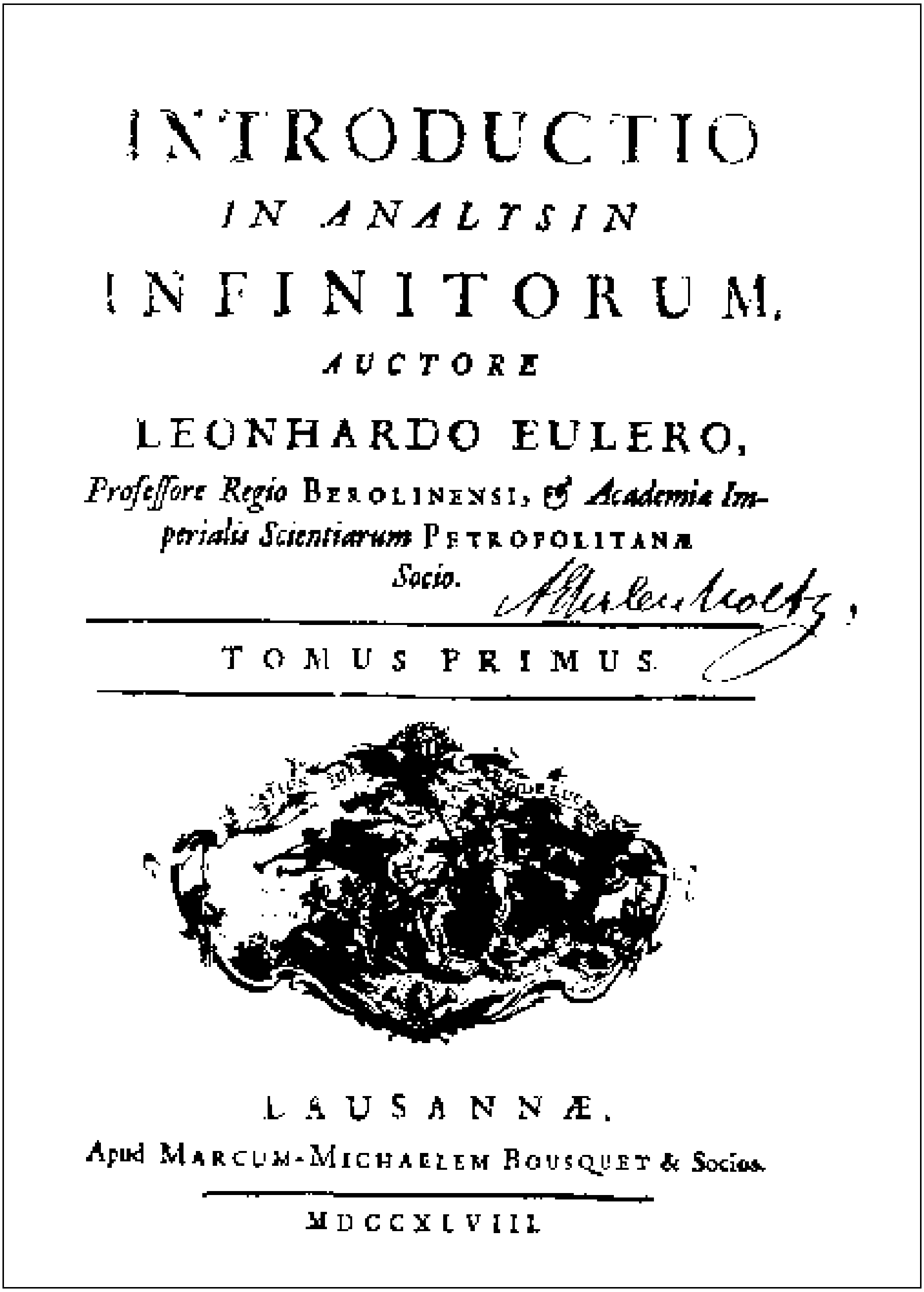

INTRODUCTIO

IN ANALYSIN

INFINITORUM,

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

Euler's *Introductio in Analysin Infinitorum*, 1748.

Preface

This book represents an introductory course of Calculus. The course evolved from the lectures, which the author had given in the Kolmogorov School in years 1986–1998 for the one-year stream. The Kolmogorov School is a special physics-mathematical undergraduate school for gifted children. Most of the graduates of Kolmogorov School continue their education in Moscow University, where they have to learn the Calculus from the beginning.

This motivates the author efforts to create a course of Calculus, which on the one hand facilitates to the students the perception of the standard one, but on the other hand misses the maximum possible of the standard material to provide the freshness of perception of the customary course. In the present form the course was given in Uppsala University in the autumn semester of 2001 for a group of advanced first-year students.

The material of the course covers the standard Calculus of the first-year, covers the essential part of the standard course of the complex Calculus, in particular, it includes the theory of residues. Moreover it contains an essential part of the theory of finite differences. Such topics presented here as Newton interpolation formula, Bernoulli polynomials, Gamma-function and Euler-Maclaurin summation formula one usually learns only beyond the common programs of a mathematical faculty. And the last lecture of the course is devoted to divergent series—a subject unfamiliar to the most of modern mathematicians.

The presence of a number of material exceeding the bounds of the standard course is accompanied with the absence of some of "inevitable" topics[1] and concepts. There is no a theory of real numbers. There is no theory of the integral neither Riemann nor Lebesgue. The present course even does not contain the Cauchy criterion of convergence. Such achievements of the ninetieth century as *uniform convergence* and *uniform continuity* are avoided. Nevertheless the level of rigor in the book is modern. In the first chapter the greek principle of exhaustion works instead of the theory of limits.

"Less words, more actions" this is the motto of the present course. Under "words" we mean "concepts and definitions" and under "actions" we mean "calculations and formulas". Every lecture gives a new recipe for the evaluation of series or integrals and is equipped with problems for independent solution. More difficult problems are marked with an asterisk. The course has a lot to do with the *Concrete Mathematics* of Graham, Knuth, Patashnik.[2]

The order of exposition in the course is far from the standard one. The standard modern course of Calculus starts with sequences and their limits. This course, following to Euler's *Introductio in Analysin Infinitorum*,[3] starts with series. The introduction of the concept of the limits is delayed up to tenth lecture. The Newton-Leibniz formula appears after all elementary integrals are already evaluated. And power series for elementary functions are obtained without help of Taylor series.

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[1] A.Ya. Hinchin wrote: "The modern course of Calculus has to begin with the theory of real numbers".

[2] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[3] L. Euler, *Introductio in Analysin Infinitorum*, 1748. Available in *Opera Omnia*, Series I, Volume 8, Springer, 1922.

The course demonstrates the unity of real, complex and discrete Calculus. For example, complex numbers immediately after their introduction are applied to evaluate a real series.

Two persons play a crucial role in appearance of these lectures. These are Alexandre Rusakov and Oleg Viro. Alexandre Rusakov several years was an assistent of the author in the Kolmogorov School, he had written the first conspectus of the course and forced the author to publish it. Oleg Viro has invited the author to Uppsala University. Many hours the author and Oleg spent in "correcting of English" in these lectures. But his influence on this course is far more then a simple correction of English. This is Oleg who convinces the author not to construct the integral, and simply reduces it to the concept of the area. The realization of this idea ascending to Oleg's teacher Rokhlin is one of characteristic features of the course.

The main motivation of the author was to present the power and the beauty of the Calculus. The author understand that this course is somewhere difficult, but he believes that it is nowhere tiresome. The course gives a new approach to exposition of Calculus, which may be interesting for students as well as for teachers. Moreover it may be interesting for mathematicians as a "mathematical roman".

The Legend of Euler's Series

"One of the great mathematical challenges of the early 18th century was to find an expression for the sum of reciprocal squares

$$(\star) \qquad 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots$$

Joh. Bernoulli eagerly sought for this expression for many decades."[1]

In 1689 Jac. Bernoulli proved the convergence of the series. In 1728–1729 Goldbach and D. Bernoulli evaluated the series with an accuracy of 0.01. Stirling in 1730 found eight digits of the sum.

L. Euler in 1734 calculated the first eighteen digits (!) after the decimal point of the sum ($\star$) and recognized $\pi^2/6$, which has the same eighteen digits. He conjectured that the infinite sum is equal to $\pi^2/6$. In 1735 Euler discovered an expansion of the sine function into an infinite product of polynomials:

$$(\star\star) \qquad \frac{\sin x}{x}=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\left(1-\frac{x^2}{4^2\pi^2}\right)\cdots$$

Comparing this presentation with the standard sine series expansion

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$$

Euler not only proved that the sum ($\star$) is equal to $\pi^2/6$, moreover he calculated all sums of the type

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\frac{1}{5^k}+\cdots$$

for even k.

Putting $x=\pi/2$ in ($\star\star$) he got the beautiful Wallis Product

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\frac{8\cdot 8}{7\cdot 9}\cdots$$

which had been known since 1655. But Euler's first proof of ($\star\star$) was not satisfactory. In 1748, in his famous *Introductio in Analysin Infinitorum*, he presented a proof which was sufficiently rigorous for the 18th century. The series of reciprocal squares was named the *Euler series.*

If somebody wants to understand all the details of the above legend he has to study a lot of things, up to complex contour integrals. This is why the detailed mathematical exposition of the legend of Euler's series turns into an entire course of Calculus. The fascinating history of Euler's series is the guiding thread of the present course, *On Euler's footsteps.*

[1] E. Hairer, G. Wanner, *Analysis by its History*, Springer, 1997.

CHAPTER 1

Series

1.1. Autorecursion of Infinite Expressions

On the contents of the lecture. The lecture presents a romantic style of early analytics. The motto of the lecture could be "infinity, equality and no definitions!". *Infinity* is the main personage we will play with today. We demonstrate how infinite expressions (i.e., infinite sums, products, fractions) arise in solutions of simple equations, how it is possible to calculate them, and how the results of such calculations apply to finite mathematics. In particular, we will deduce the Euler-Binet formula for Fibonacci numbers, the first Euler's formula of the course. We become acquainted with geometric series and the golden section.

Achilles and the turtle. The ancient Greek philosopher Zeno claimed that Achilles pursuing a turtle could never pass it by, in spite of the fact that his velocity was much greater than the velocity of the turtle. His arguments adopted to our purposes are the following.

First Zeno proposed a pursuing algorithm for Achilles:

Initialization. Assign to the variable *goal* the original position of the turtle.

Action. Reach the *goal*.

Correction. If the current turtle's position is *goal*, then stop, else reassign to the variable *goal* the current position of the turtle and go to **Action**.

Secondly, Zeno remarks that this algorithm never stops if the turtle constantly moves in one direction.

And finally, he notes that Achilles has to follow his algorithm if he want pass the turtle by. He may be not aware of this algorithm, but unconsciously he must perform it. Because he cannot run the turtle down without reaching the original position of the turtle and then all positions of the turtle which the variable *goal* takes.

Zeno's algorithm generates a sequence of times $\{t_k\}$, where t_k is the time of execution of the k-th action of the algorithm. And the whole time of work of the algorithm is the infinite sum $\sum_{k=1}^{\infty} t_k$; and this sum expresses the time Achilles needs to run the turtle down. (The corrections take zero time, because Achilles really does not think about them.) Let us name this sum the *Zeno series.*

Assume that both Achilles and the turtle run with constant velocities v and w, respectively. Denote the initial distance between Achilles and the turtle by d_0. Then $t_1 = \frac{d_0}{v}$. The turtle in this time moves by the distance $d_1 = t_1 w = \frac{w}{v} d_0$. By his second action Achilles overcomes this distance in time $t_2 = \frac{d_1}{v} = \frac{w}{v} t_1$, while the turtle moves away by the distance $d_2 = t_2 w = \frac{w}{v} d_1$. So we see that the sequences of times $\{t_k\}$ and distances $\{d_k\}$ satisfy the following *recurrence relations*: $t_k = \frac{w}{v} t_{k-1}$, $d_k = \frac{w}{v} d_{k-1}$.

Hence $\{t_k\}$ as well as $\{d_k\}$ are *geometric progressions* with ratio $\frac{w}{v}$. And the time t which Achilles needs to run the turtle down is

$$t = t_1 + t_2 + t_3 + \cdots = t_1 + \tfrac{w}{v} t_1 + \tfrac{w^2}{v^2} t_1 + \cdots = t_1 \left(1 + \tfrac{w}{v} + \tfrac{w^2}{v^2} + \cdots\right).$$

In spite of Zeno, we know that Achilles does catch up with the turtle. And one easily gets the time t he needs to do it by the following argument: the distance between Achilles and the turtle permanently decreases with the velocity $v - w$. Consequently it becomes 0 in the time $t = \frac{d_0}{v-w} = t_1 \frac{v}{v-w}$. Comparing the results we come to the following conclusion

$$\tfrac{v}{v-w} = 1 + \tfrac{w}{v} + \tfrac{w^2}{v^2} + \tfrac{w^3}{v^3} + \cdots . \quad (1.1.1)$$

Infinite substitution. We see that some infinite expressions represent finite values. The fraction in the left-hand side of (1.1.1) expands into the infinite series on the right-hand side. Infinite expressions play a key rôle in mathematics and physics. Solutions of equations quite often are presented as infinite expressions.

For example let us consider the following simple equation

$$t = 1 + qt. \tag{1.1.2}$$

Substituting on the right-hand side $1 + qt$ instead of t, one gets a new equation $t = 1 + q(1 + qt) = 1 + q + q^2t$. Any solution of the original equation satisfies this one. Repeating this trick, one gets $t = 1 + q(1 + q(1 + qt)) = 1 + q + q^2 + q^3t$. Repeating this infinitely many times, one eliminates t on the right hand side and gets a solution of (1.1.2) in an infinite form

$$t = 1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{\infty} q^k.$$

On the other hand, the equation (1.1.2) solved in the usual way gives $t = \frac{1}{1-q}$. As a result, we obtain the following formula

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \cdots = \sum_{k=0}^{\infty} q^k. \tag{1.1.3}$$

which represents a special case of (1.1.1) for $v = 1$, $w = q$.

Autorecursion. An infinite expression of the form $a_1 + a_2 + a_3 + \dots$ is called a *series* and is concisely denoted by $\sum_{k=1}^{\infty} a_k$. Now we consider a summation method for series which is inverse to the above method of infinite substitution. To find the sum of a series we shall construct an equation which is satisfied by its sum. We name this method *autorecursion*. Recursion means "return to something known". Autorecursion is "return to oneself".

The series $a_2 + a_3 + \cdots = \sum_{k=2}^{\infty} a_k$ obtained from $\sum_{k=1}^{\infty} a_k$ by dropping its first term is called the *shift* of $\sum_{k=1}^{\infty} a_k$.

We will call the following equality the *shift formula*:

$$\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} a_k.$$

Another basic formula we need is the following *multiplication formula*:

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

These two formulas are all one needs to find the sum of geometric series $\sum_{k=0}^{\infty} q^k$. To be exact, the multiplication formula gives the equality $\sum_{k=1}^{\infty} q^k = q \sum_{k=0}^{\infty} q^k$. Hence the shift formula turns into equation $x = 1 + qx$, where x is $\sum_{k=0}^{\infty} q^k$. The solution of this equation gives us the formula (1.1.3) for the sum of the geometric series again.

From this formula, one can deduce the formula for the sum of a finite geometric progression. By $\sum_{k=0}^{n} a_k$ is denoted the sum $a_0 + a_1 + a_2 + \cdots + a_n$. One has

$$\sum_{k=0}^{n-1} q^k = \sum_{k=0}^{\infty} q^k - \sum_{k=n}^{\infty} q^k = \frac{1}{1-q} - \frac{q^n}{1-q} = \frac{1-q^n}{1-q}.$$

This is an important formula which was traditionally studied in school.

The series $\sum_{k=0}^{\infty} kx^k$. To find the sum of $\sum_{k=1}^{\infty} kx^k$ we have to apply additionally the following *addition formula*,

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

which is the last general formula for series we introduce in the first lecture.

Reindexing the shift $\sum_{k=2}^{\infty} kx^k$ we give it the form $\sum_{k=1}^{\infty}(k+1)x^{k+1}$. Further it splits into two parts

$$x\sum_{k=1}^{\infty}(k+1)x^k = x\sum_{k=1}^{\infty} kx^k + x\sum_{k=1}^{\infty} x^k = x\sum_{k=1}^{\infty} kx^k + x\frac{x}{1-x}$$

by the addition formula. The first summand is the original sum multiplied by x. The second is a geometric series. We already know its sum. Now the shift formula for the sum $s(x)$ of the original series turns into the equation $s(x) = x + x\frac{x}{1-x} + xs(x)$. Its solution is $s(x) = \frac{x}{(1-x)^2}$.

Fibonacci Numbers. Starting with $\phi_0 = 0$, $\phi_1 = 1$ and applying the recurrence relation

$$\phi_{n+1} = \phi_n + \phi_{n-1},$$

one constructs an infinite sequence of numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$, called *Fibonacci numbers.* We are going to get a formula for ϕ_n.

To do this let us consider the following function $\Phi(x) = \sum_{k=0}^{\infty} \phi_k x^k$, which is called the *generating function* for the sequence $\{\phi_k\}$. Since $\phi_0 = 0$, the sum $\Phi(x) + x\Phi(x)$ transforms in the following way:

$$\sum_{k=1}^{\infty} \phi_k x^k + \sum_{k=1}^{\infty} \phi_{k-1} x^k = \sum_{k=1}^{\infty} \phi_{k+1} x^k = \frac{\Phi(x) - x}{x}.$$

Multiplying both sides of the above equation by x and collecting all terms containing $\Phi(x)$ on the right-hand side, one gets $x = \Phi(x) - x\Phi(x) - x^2\Phi(x) = x$. It leads to

$$\Phi(x) = \frac{x}{1 - x - x^2}.$$

The roots of the equation $1 - x - x^2 = 0$ are $\frac{-1\pm\sqrt{5}}{2}$. More famous is the pair of their inverses $\frac{1\pm\sqrt{5}}{2}$. The number $\phi = \frac{-1+\sqrt{5}}{2}$ is the so-called *golden section* or *golden mean*. It plays a significant rôle in mathematics, architecture and biology. Its dual is $\hat{\phi} = \frac{-1-\sqrt{5}}{2}$. Then $\phi\hat{\phi} = -1$, and $\phi + \hat{\phi} = 1$. Hence $(1 - x\phi)(1 - x\hat{\phi}) = 1 - x - x^2$, which in turn leads to the following decomposition:

$$\frac{x}{x^2 + x - 1} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x}\right).$$

We expand both fractions on the right hand side into geometric series:

$$\frac{1}{1 - \phi x} = \sum_{k=0}^{\infty} \phi^k x^k, \qquad \frac{1}{1 - \hat{\phi} x} = \sum_{k=0}^{\infty} \hat{\phi}^k x^k.$$

This gives the following representation for the generating function

$$\Phi(x) = \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}(\phi^k - \hat{\phi}^k)x^k.$$

On the other hand the coefficient at x^k in the original presentation of $\Phi(x)$ is ϕ_k. Hence

$$\phi_k = \frac{1}{\sqrt{5}}(\phi^k - \hat{\phi}^k) = \frac{(\sqrt{5}+1)^k + (-1)^k(\sqrt{5}-1)^k}{2^k\sqrt{5}}. \tag{1.1.4}$$

This is called the *Euler-Binet* formula. It is possible to check it for small k and then prove it by induction using Fibonacci recurrence.

Continued fractions. The application of the method of infinite substitution to the solution of quadratic equation leads us to a new type of infinite expressions, the so-called *continued fractions.* Let us consider the golden mean equation $x^2 - x - 1 = 0$. Rewrite it as $x = 1 + \frac{1}{x}$. Substituting $1 + \frac{1}{x}$ instead of x on the right-hand side we get $x = 1 + \frac{1}{1+\frac{1}{x}}$. Repeating the substitution infinitely many times we obtain a solution in the form of the *continued fraction*:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1+\dots}}} \tag{1.1.5}$$

As this fraction seems to represent a positive number and the golden mean is the unique positive root of the golden mean equation, it is natural to conclude that this fraction is equal to $\phi = \frac{1+\sqrt{5}}{2}$. This is true and this representation allows one to calculate the golden mean and $\sqrt{5}$ effectively with great precision.

To be precise, consider the sequence

$$1, \quad 1 + \tfrac{1}{1}, \quad 1 + \cfrac{1}{1 + \frac{1}{1}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}, \quad \dots \tag{1.1.6}$$

of so-called *convergents* of the continued fraction (1.1.5). Let us remark that all odd convergents are less than ϕ and all even convergents are greater than ϕ. To see this, compare the n-th convergent with the corresponding term of the following sequence of fractions:

$$1 + \tfrac{1}{x}, \quad 1 + \cfrac{1}{1 + \frac{1}{x}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{x}}}, \quad \dots. \tag{1.1.7}$$

We know that for $x = \phi$ all terms of the above sequence are equal to ϕ. Hence all we need is to observe how the removal of $\frac{1}{x}$ affects the value of the considered fraction. The value of the first fraction of the sequence decreases, the value of the second fraction increases. If we denote the value of n-th fraction by f_n, then the value of the next fraction is given by the following recurrence relation:

$$f_{n+1} = 1 + \frac{1}{f_n}. \tag{1.1.8}$$

Hence increasing f_n decreases f_{n+1} and decreasing f_n increases f_{n+1}. Consequently in general all odd fractions of the sequence (1.1.7) are less than the corresponding convergent, and all even are greater. The recurrence relation (1.1.8) is valid for the golden mean convergent. By this recurrence relation one can quickly calculate the first ten convergents $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$. The golden mean lies between last two fractions, which have the difference $\frac{1}{34\cdot 55}$. This allows us to determine the first four decimal digits after the decimal point of it and of $\sqrt{5}$.

Problems.

1. Evaluate $\sum_{k=0}^{\infty} \frac{2^{2k}}{3^{3k}}$.
2. Evaluate $1 - 1 + 1 - 1 + \cdots$.
3. Evaluate $1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + \cdots$.
4. Evaluate $\sum_{k=1}^{\infty} \frac{k}{3^k}$.
5. Evaluate $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$.
6. Decompose the fraction $\frac{1}{a+x}$ into a power series.
7. Find the generating function for the sequence $\{2^k\}$.
8. Find sum the $\sum_{k=1}^{\infty} \phi_k 3^{-k}$.
9. Prove by induction the Euler-Binet formula.
***10.** Evaluate $1 - 2 + 1 + 1 - 2 + 1 + \cdots$.
11. Approximate $\sqrt{2}$ by a rational with precision 0.0001.
12. Find the value of $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}$.
13. Find the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$.
14. By infinite substitution, solve the equation $x^2 - 2x - 1 = 0$, and represent $\sqrt{2}$ by a continued fraction.
15. Find the value of the infinite product $2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot \cdots$.
16. Find a formula for n-th term of the recurrent sequence $x_{n+1} = 2x_n + x_{n-1}$, $x_0 = x_1 = 1$.
17. Find the sum of the Fibonacci numbers $\sum_{k=1}^{\infty} \phi_k$.
18. Find sum $1 + 0 - 1 + 1 + 0 - 1 + \cdots$.
19. Decompose into the sum of partial fractions $\frac{1}{x^2-3x+2}$.

1.2. Positive Series

On the contents of the lecture. Infinity is pregnant with paradoxes. Paradoxes throw us down from the heavens to the earth. We leave the poetry for prose, and rationalize the *infinity and equality* by working with *finiteness and inequality.* We shall lay a solid foundation for a summation theory for positive series. And the reader will find out what $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ precisely means.

Divergent series paradox. Let us consider the series $\sum_{k=0}^{\infty}2^k$. This is a geometric series. We know how to sum it up by autorecursion. The autorecursion equation is $s=1+2s$. The only number satisfying this equation is -1. The sum of positive numbers turns to be negative!? Something is wrong!

A way to save the situation is to admit infinity as a feasible solution. Infinity is an obvious solution of $s=1+2s$. The sum of any geometric series $\sum_{k=0}^{\infty}q^k$ with denominator $q\geq 1$ is obviously infinite, isn't it?

Indeed, this sum is greater than $1+1+1+1+\dots$, which symbolizes infinity. (The autorecursion equation for $1+1+1+\dots$ is $s=s+1$. Infinity is the unique solution of this equation.)

The series $\sum_{k=0}^{\infty}2^k$ represents Zeno's series in the case of the *Mighty Turtle*, which is faster than Achilles. To be precise, this series arises if $v=d_0=1$ and $w=2$. As the velocity of the turtle is greater than the velocity of Achilles he never reaches it. So the infinity is right answer for this problem. But the negative solution -1 also makes sense. One could interpret it as an event in the past. Just the point in time when *the turtle passed Achilles.*

Oscillating series paradoxes. The philosopher Gvido Grandy in 1703 attracted public attention to the series $1-1+1-1+\dots$. He claimed this series symbolized the Creation of Universe from Nothing. Namely, insertion of brackets in one way gives Nothing (that is 0), in another way, gives 1.

$$(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\cdots=0,$$
$$1-(1-1)-(1-1)-(1-1)-\cdots=1-0-0-0-\cdots=1.$$

On the other hand, this series $1-1+1-1+1-1+\dots$ is geometric with negative ratio $q=-1$. Its autorecursion equation $s=1-s$ has the unique solution $s=\frac{1}{2}$. Neither $+\infty$ nor $-\infty$ satisfy it. So $\frac{1}{2}$ seems to be its true sum.

Hence we see the Associativity Law dethroned by $1-1+1-1+\dots$. The next victim is the Commutativity Law. The sum $-1+1-1+1-1+\dots$ is equal to $-\frac{1}{2}$. But the last series is obtained from $1-1+1-1+\dots$ by transposition of odd and even terms.

And the third amazing thing: diluting it by zeroes changes its sum. The sum $1+0-1+1+0-1+1+0-1+\dots$ by no means is $\frac{1}{2}$. It is $\frac{2}{3}$. Indeed, if we denote this sum by s then by shift formulas one gets

$$\begin{aligned} s &= 1+0-1+1+0-1+1+0-1+1+0-1+\dots,\\ s-1 &= 0-1+1+0-1+1+0-1+1+0-1+1+\dots,\\ s-1-0 &= -1+1+0-1+1+0-1+1+0-1+1+0+\dots. \end{aligned}$$

Summing the numbers column-wise (i.e., by the Termwise Addition Formula), we get

$$s+(s-1)+(s-1-0)=(1+0-1)+(0-1+1)+(-1+1+0)$$
$$+(1+0-1)+(0-1+1)+(-1+1+0)+\dots.$$

The left-hand side is $3s-2$. The right-hand side is the zero series. That is why $s=\frac{2}{3}$.

The series $1-1+1-1+\dots$ arises as Zeno's series in the case of a blind Achilles directed by a cruel Zeno, who is interested, as always, only in proving his claim, and a foolish, but merciful turtle. The blind Achilles is not fast, his velocity equals the velocity of the turtle. At the first moment Zeno tells the blind Achilles where the turtle is. Achilles starts the rally. But the merciful turtle wishing to help him goes towards him instead of running away. Achilles meets the turtle half-way. But he misses it, being busy to perform the first step of the algorithm. When he accomplishes this step, Zeno orders: “Turn about!” and surprises Achilles by saying that the turtle is on Achilles' initial position. The turtle discovers that Achilles turns about and does the same. The situation repeats ad infinitum. Now we see that assigning the sum $\frac{1}{2}$ to the series $1-1+1-1+\dots$ makes sense. It predicts accurately the time of the first meeting of Achilles and turtle.

Positivity. The paradoxes discussed above are discouraging. Our intuition based on handling finite sums fails when we turn to infinite ones. Observe that all paradoxes above involve negative numbers. And to eliminate the evil in its root, let us consider only nonnegative numbers.

We return to the ancient Greeks. They simply did not know what a negative number is. But in contrast to the Greeks, we will retain zero. A series with nonnegative terms will be called a *positive* series. We will show that for positive series all familiar laws, including associativity and commutativity, hold true and zero terms do not affect the sum.

Definition of Infinite Sum. Let us consider what Euler's equality could mean:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

The natural answer is: the *partial sums* $\sum_{k=1}^{n}\frac{1}{k^2}$, which contain more and more reciprocal squares, approach closer and closer the value $\frac{\pi^2}{6}$. Consequently, all partial sums have to be less than $\frac{\pi^2}{6}$, its *ultimate sum.* Indeed, if some partial sum exceeds or coincides with $\frac{\pi^2}{6}$ then all subsequent sums will move away from $\frac{\pi^2}{6}$. Furthermore, any number c which is less than $\frac{\pi^2}{6}$ has to be surpassed by partial sums eventually, when they approach $\frac{\pi^2}{6}$ closer than by $\frac{\pi^2}{6}-c$. Hence the ultimate sum majorizes all partial ones, and any lesser number does not. This means that the ultimate sum is the smallest number which majorizes all partial sums.

Geometric motivation. Imagine a sequence $[a_{i-1},a_i]$ of intervals of the real line. Denote by l_i the length of i-th interval. Let $a_0=0$ be the left end point of the first interval. Let $[0,A]$ be the smallest interval containing the whole sequence. Its length is naturally interpreted as the sum $\sum_{i=1}^{\infty}l_i$

This motivates the following definition.

DEFINITION. *If the partial sums of the positive series* $\sum_{k=1}^{\infty} a_k$ *increase without bound, its sum is defined to be* ∞ *and the series is called* divergent*. In the opposite case the series called* convergent*, and its sum is defined as the smallest number* A *such that* $A \geq \sum_{k=1}^{n} a_k$ *for all* n*.*

This Definition is equivalent to the following couple of principles. The first principle limits the ultimate sum from below:

PRINCIPLE (One-for-All). *The ultimate sum of a positive series majorizes all partial sums.*

And the second principle limits the ultimate sum from above:

PRINCIPLE (All-for-One). *If all partial sums of a positive series do not exceed a number, then the ultimate sum also does not exceed it.*

THEOREM 1.2.1 (Termwise Addition Formula).

$$\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k).$$

PROOF. The inequality $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k)$ is equivalent to $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k$. By All-for-One, the last is equivalent to the system of inequalities

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} b_k \quad N = 1, 2, \dots.$$

This system is equivalent to the following system

$$\sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad N = 1, 2, \dots.$$

Each inequality of the last system, in its turn, is equivalent to the system of inequalities

$$\sum_{k=1}^{M} b_k \leq \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{N} a_k \quad M = 1, 2, \dots.$$

But these inequalities are true for all N and M, as the following computations show.

$$\sum_{k=1}^{M} b_k + \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{M+N} b_k + \sum_{k=1}^{M+N} a_k = \sum_{k=1}^{M+N} (a_k + b_k) \leq \sum_{k=1}^{\infty}(a_k + b_k).$$

In the opposite direction, we see that any partial sum on the right-hand side $\sum_{k=1}^{n}(a_k + b_k)$ splits into $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. And by virtue of the One-for-All principle, this does not exceed $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$. Now, the All-for-One principle provides the inequality in the opposite direction. □

THEOREM 1.2.2 (Shift Formula).

$$\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k.$$

PROOF. The Shift Formula immediately follows from the Termwise Addition formula. To be precise, immediately from the definition, one gets the following: $a_0 + 0 + 0 + 0 + 0 + \cdots = a_0$ and that $0 + a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$. Termwise Addition of these series gives

$$a_0 + \sum_{k=1}^{\infty} a_k = (a_0 + 0) + (0 + a_1) + (0 + a_2) + (0 + a_3) + \cdots = \sum_{k=0}^{\infty} a_k.$$

□

THEOREM 1.2.3 (Termwise Multiplication Formula).

$$\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k.$$

PROOF. For any partial sum from the right-hand side one has

$$\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k \leq \lambda \sum_{k=1}^{\infty} a_k$$

by the Distributivity Law for finite sums and One-for-All. This implies the inequality $\lambda \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} \lambda a_k$ by All-for-One. The opposite inequality is equivalent to $\sum_{k=1}^{\infty} a_k \geq \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$. As any partial sum $\sum_{k=1}^{n} a_k$ is equal to $\frac{1}{\lambda} \sum_{k=1}^{n} \lambda a_k$, which does not exceed $\frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$, one gets the opposite inequality. □

Geometric series. We have to return to the geometric series, because the autorecursion equation produced by shift and multiplication formulas says nothing about convergence. So we have to prove convergence for $\sum_{k=0}^{\infty} q^k$ with positive $q < 1$. It is sufficient to prove the following inequality for all n

$$1 + q + q^2 + q^3 + \cdots + q^n < \tfrac{1}{1-q}.$$

Multiplying both sides by $1 - q$ one gets on the left-hand side

$$\begin{aligned}
&(1 - q) + (q - q^2) + (q^2 - q^3) + \cdots + (q^{n-1} - q^n) + (q^n - q^{n+1}) \\
&= 1 - q + q - q^2 + q^2 - q^3 + q^3 - \cdots - q^n + q^n - q^{n+1} \\
&= 1 - q^{n+1}
\end{aligned}$$

and 1 on the right-hand side. The inequality $1 - q^{n+1} < 1$ is obvious. Hence we have proved the convergence. Now the autorecursion equation $x = 1 + qx$ for $\sum_{k=0}^{\infty} q^k$ is constructed in usual way by the shift formula and termwise multiplication. It leaves only two possibilities for $\sum_{k=0}^{\infty} q^k$, either $\frac{1}{q-1}$ or ∞. For $q < 1$ we have proved convergence, and for $q \geq 1$ infinity is the true answer.

Let us pay special attention to the case $q = 0$. We adopt a common convention:

$$0^0 = 1.$$

This means that the series $\sum_{k=0}^{\infty} 0^k$ satisfies the common formula for a convergent geometric series $\sum_{k=0}^{\infty} 0^k = \frac{1}{1-0} = 1$. Finally we state the theorem, which is essentially due to Eudoxus, who proved the convergence of the geometric series with ratio $q < 1$.

THEOREM 1.2.4 (Eudoxus). *For every nonnegative q one has*

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \text{for } q < 1, \text{ and} \quad \sum_{k=0}^{\infty} q^k = \infty \quad \text{for } q \geq 1.$$

Comparison of series. Quite often exact summation of series is too difficult, and for practical purposes it is enough to know the sum approximatively. In this case one usually compares the series with another one whose sum is known. Such a comparison is based on the following *Termwise Comparison Principle*, which immediately follows from the definition of a sum.

PRINCIPLE (Termwise Comparison). *If $a_k \leq b_k$ for k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

The only series we have so far to compare with are the geometric ones. The following lemma is very useful for this purposes.

LEMMA 1.2.5 (Ratio Test). *If $a_{k+1} \leq q a_k$ for k holds for some $q < 1$ then*

$$\sum_{k=0}^{\infty} a_k \leq \frac{a_0}{1-q}.$$

PROOF. By induction one proves the inequality $a_k \leq a_0 q^k$. Now by Termwise Comparison one estimates $\sum_{k=0}^{\infty} a_k$ from above by the geometric series $\sum_{k=0}^{\infty} a_0 q^k = \frac{a_0}{1-q}$ □

If the series under consideration satisfies an autorecursion equation, to prove its convergence usually means to evaluate it exactly. For proving convergence, the Termwise Comparison Principle can be strengthened. Let us say that the series $\sum_{k=1}^{\infty} a_k$ is *eventually* majorized by the series $\sum_{k=1}^{\infty} b_k$, if the inequality $b_k \geq a_k$ holds for each k starting from $k = n$ for some n. The following lemma is very useful to prove convergence.

PRINCIPLE (Eventual Comparison). *A series $\sum_{k=1}^{\infty} a_k$, which is eventually majorized by a convergent series $\sum_{k=1}^{\infty} b_k$, is convergent.*

PROOF. Consider a tail $\sum_{k=n}^{\infty} b_k$, which termwise majorizes $\sum_{k=n}^{\infty} a_k$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} a_k \\ &\leq \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} b_k \\ &\leq \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{\infty} b_k \\ &< \infty.\end{aligned}$$

□

Consider the series $\sum_{k=1}^{\infty} k2^{-k}$. The ratio of two successive terms $\frac{a_{k+1}}{a_k}$ of the series is $\frac{k+1}{2k}$. This ratio is less or equal to $\frac{2}{3}$ starting with $k = 3$. Hence this series is eventually majorizes by the geometric series $\sum_{k=0}^{\infty} a_3 \frac{2^k}{3^k}$, $(a_3 = \frac{2}{3})$. This proves its convergence. And now by autorecursion equation one gets its sum.

Harmonic series paradox. Now we have a solid background to evaluate positive series. Nevertheless, we must be careful about infinity! Consider the following calculation:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} &= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{2k} \right) \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} \right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}.
\end{aligned}$$

We get that the sum $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k}$ satisfies the equation $s = \frac{s}{2}$. This equation has two roots 0 and ∞. But s satisfies the inequalities $\frac{1}{2} < s < \frac{\pi^2}{6}$. What is wrong?

Problems.

1. Prove $\sum_{k=1}^{\infty} 0 = 0$.
2. Prove $\sum_{k=1}^{\infty} 0^k = 1$.
3. Prove $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}$.
4. Prove $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$ for convergent series.
5. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.
6. Prove $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots = 1 - [(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + \cdots$.
7. Prove the convergence of $\sum_{k=0}^{\infty} \frac{2^k}{k!}$.
8. Prove the convergence of $\sum_{k=1}^{\infty} \frac{1000^k}{k!}$.
9. Prove the convergence of $\sum_{k=1}^{\infty} \frac{k^{1000}}{2^k}$.
10. Prove that $q^n < \frac{1}{n(1-q)}$ for $0 < q < 1$.
11. Prove that for any positive $q < 1$ there is an n that $q^n < \frac{1}{2}$.
12. Prove $\sum_{k=1}^{\infty} \frac{1}{k!} \leq 2$.
13. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.
14. Prove the convergence of the Euler series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.
***15.** Prove that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ for $a_{ij} \geq 0$.

1.3. Unordered Sums

On the contents of the lecture. Our summation theory culminates in the Sum Partition Theorem. This lecture will contribute towards evaluation of the Euler series in two ways: we prove its convergence, and even estimate its sum by 2. On the other hand, we will realize that evaluation of the Euler series with Euler's accuracy (10^{-18}) seems to be beyond a human being's strength.

. Consider a family $\{a_i\}_{i\in I}$ of nonnegative numbers indexed by elements of an arbitrary set I. An important special case of I is the set of pairs of natural numbers $\mathbb{N}\times\mathbb{N}$. Families indexed by $\mathbb{N}\times\mathbb{N}$ are called double series. They arise when one multiplies one series by another one.

Any sum of the type $\sum_{i\in K} a_i$, where K is a finite subset of I is called a *subsum* of $\{a_i\}_{i\in I}$ over K.

Definition. *The least number majorizing all subsums of $\{a_i\}_{i\in I}$ over finite subsets is called its (ultimate) sum and denoted by $\sum_{i\in I} a_i$*

The One-for-All and All-for-One principles for non-ordered sums are obtained from the corresponding principles for ordered sums by replacing "partial sums" by "finite subsums".

Commutativity. In case $I=\mathbb{N}$ we have a definition which apparently is new. But fortunately this definition is equivalent to the old one. Indeed, as any finite subsum of positive series does not exceed its ultimate (ordered) sum, the non-ordered sum also does not exceed it. On the other hand, any partial sum of the series is a finite subsum. This implies the opposite inequality. Therefore we have established the equality.

$$\sum_{k=1}^{\infty} a_k = \sum_{k\in\mathbb{N}} a_k$$

This means that positive series obey the Commutativity Law. Because the non-ordered sum obviously does not depend on the order of summands.

Partitions. A family of subsets $\{I_k\}_{k\in K}$ of a set I is called a *partition* of I and is written $\bigsqcup_{k\in K} I_k$ if $I=\bigcup_{k\in K} I_k$ and $I_k\cap I_j=\varnothing$ for all $k\neq j$.

Theorem 1.3.1 (Sum Partition Theorem). *For any partition $I=\bigsqcup_{j\in J} I_j$ of the indexing set and any family $\{a_i\}_{i\in I}$ of nonnegative numbers,*

$$\sum_{i\in I} a_i = \sum_{j\in J}\sum_{i\in I_j} a_i. \tag{1.3.1}$$

Iverson notation. We will apply the following notation: a statement included into [ ] takes value 1, if the statement is true, and 0, if it is false. Prove the following simple lemmas to adjust to this notation. In both lemmas one has $K\subset I$.

Lemma 1.3.2. $\sum_{i\in K} a_i = \sum_{i\in I} a_i[i\in K]$.

In particular, for $K=I$, Lemma 1.3.2 turns into

Lemma 1.3.3. $\sum_{i\in I} a_i = \sum_{i\in I} a_i[i\in I]$.

Lemma 1.3.4. $\sum_{k\in K}[i\in I_k]=[i\in I_K]$ *for all $i\in I$ iff $I_K=\bigsqcup_{k\in K} I_k$.*

Proof of Sum Partition Theorem. At first we prove the following Sum Transposition formula for finite J,

$$\sum_{i\in I}\sum_{j\in J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}. \tag{1.3.2}$$

Indeed, if J contains just two elements, this formula turns into the Termwise Addition formula. The proof of this formula is the same as for series. Suppose the formula is proved for any set which contains fewer elements than J does. Decompose J into a union of two nonempty subsets $J_1 \sqcup J_2$. Then applying only Termwise Addition and Lemmas 1.3.2, 1.3.3, 1.3.4, we get

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in J} a_{ij} &= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J]\\
&= \sum_{i\in I}\sum_{j\in J}(a_{ij}[j\in J_1] + a_{ij}[j\in J_2])\\
&= \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_1] + \sum_{i\in I}\sum_{j\in J} a_{ij}[j\in J_2]\\
&= \sum_{i\in I}\sum_{j\in J_1} a_{ij} + \sum_{i\in I}\sum_{j\in J_2} a_{ij}.
\end{aligned}$$

But the last two sums can be transposed by the induction hypothesis. After such a transposition one gets

$$\begin{aligned}
\sum_{j\in J_1}\sum_{i\in I} a_{ij} + \sum_{j\in J_2}\sum_{i\in I} a_{ij} &= \sum_{j\in J}[j\in J_1]\sum_{i\in I} a_{ij} + \sum_{j\in J}[j\in J_2]\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}([j\in J_1] + [j\in J_2])\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}[j\in J]\sum_{i\in I} a_{ij}\\
&= \sum_{j\in J}\sum_{i\in I} a_{ij}
\end{aligned}$$

and the Sum Transposition formula for finite J is proved. Consider the general case. To prove $\leq$ in (1.3.2), consider a finite $K \subset I$. By the finite Sum Transposition formula the subsum $\sum_{i\in K}\sum_{j\in J} a_{ij}$ is equal to $\sum_{j\in J}\sum_{i\in K} a_{ij}$. But this sum is termwise majorized by the right-hand side sum in (1.3.2). Therefore the left-hand side does not exceed the right-hand side by All-for-One principle.

To derive the Sum Partition Theorem from the Sum Transposition formula, pose $a_{ij} = a_i[i\in I_j]$. Then $a_i = \sum_{j\in J} a_{ij}$ and (1.3.1) turns into (1.3.2). This completes the proof of the Sum Partition Theorem.

Blocking. For a given a series $\sum_{k=0}^{\infty} a_k$ and an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ starting with $n_0 = 0$ one defines a new series $\sum_{k=0}^{\infty} A_k$ by the rule $A_k = \sum_{i=n_k}^{n_{k+1}-1} a_i$. The series $\sum_{k=0}^{\infty} A_k$ is called *blocking of* $\sum_{k=0}^{\infty} a_k$ *by* $\{n_k\}$.

The Sum Partition Theorem implies that the sums of blocked and unblocked series coincide. Blocking formalizes putting of brackets. Therefore the Sum Partition Theorem implies the *Sequential Associativity Law*: *Placing brackets does not change the sum of series.*

Estimation of the Euler series. Let us compare the Euler series with the series $\sum_{k=0}^{\infty}\frac{1}{2^k}$, blocked by $\{2^n\}$ to $\sum_{k=1}^{\infty}a_k$. The sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ consists of 2^n summands, all of which are less then the first one, which is $\frac{1}{2^{2n}}$. As $2^n\frac{1}{2^{2n}}=\frac{1}{2^n}$, it follows that $a_n\leq\frac{1}{2^n}$ for each n. Summing these inequalities, one gets $\sum_{k=1}^{\infty}a_k\leq 2$.

Now let us estimate how many terms of Euler's series one needs to take into account to find its sum up to the eighteenth digit. To do this, we need to estimate its tail. The arguments above give $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\leq\frac{1}{2^{n-1}}$. To obtain a lower estimate, let us remark that all terms of sum $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}$ exceed $\frac{1}{2^{2(n+1)}}$. As the number of summands is 2^n, one gets $a_n\geq\frac{1}{4\cdot 2^n}$. Hence $\sum_{k=2^n}^{\infty}\frac{1}{k^2}\geq\frac{1}{2^{n+1}}$. Since $2^{10}=1024\simeq 10^3$, one gets $2^{60}\simeq 10^{18}$. So, to provide an accuracy of 10^{-18} one needs to sum approximately 10^{18} terms. This task is inaccessible even for a modern computer. How did Euler manage to do this? He invented a summation formula (Euler-MacLaurin formula) and transformed this slowly convergent series into non-positive divergent (!) one, whose partial sum containing as few as ten terms gave eighteen digit accuracy. The whole calculation took him an evening. To introduce this formula, one needs to know integrals and derivatives. We will do this later.

Problems.

1. Find $\sum_{k=1}^{\infty}\frac{1}{(2k)^2}$ and $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, assuming $\sum_{k=1}^{\infty}\frac{1}{k^2}=\pi^2/6$.
2. Prove the convergence of $\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}}$.
3. Estimate how many terms of the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are necessary for calculation of its sum with precision 10^{-3}.
4. Estimate the value of $\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{k}$.
5. Prove the equality $\sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k=\sum_{j,k\in\mathbb{N}}a_jb_k$.
6. Estimate how many terms of the Harmonic series give the sum surpassing 1000.
7. Prove the Dirichlet formula $\sum_{k=1}^{n}\sum_{i=1}^{k}a_{ki}=\sum_{i=1}^{n}\sum_{k=i}^{n}a_{ki}$.
8. Evaluate $\sum_{i,j\in N}\frac{1}{2^i3^j}$.
9. Evaluate $\sum_{i,j\in N}\frac{i+j}{2^i3^j}$.
10. Represent an unordered sum $\sum_{i+j<n}a_{ij}$ as a double sum.
11. Evaluate $\sum_{i,j\in N}\frac{ij}{2^i3^j}$.
12. Change the summation order in $\sum_{i=0}^{\infty}\sum_{j=0}^{2i}a_{ij}$.
13. Define by Iverson notation the following functions:
- $[x]$ (integral part),
- $|x|$ (module),
- $\operatorname{sgn} x$ (signum),
- $n!$ (factorial).

14. Define only by formulas the expression $[p \text{ is prime}]$.

1.4. Infinite Products

On the contents of the lecture. In this lecture we become acquainted with infinite products. The famous Euler Identity will be proved. We will find out that $\zeta(2)$ is another name for the Euler series. And we will see how Euler's decomposition of the sine function into a product works to sum up the Euler Series.

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DEFINITION. *The product of an infinite sequence of numbers* $\{a_k\}$*, such that* $a_k \geq 1$ *for all* k*, is defined as the least number majorizing all partial products* $\prod_{k=1}^{n} a_k = a_1 a_2 \dots a_n$.

A sequence of natural numbers is called *essentially finite* if all but finitely many of its elements are equal to zero. Denote by $\mathbb{N}^\infty$ the set of all essentially finite sequences of natural numbers.

THEOREM 1.4.1. *For any given sequence of positive series* $\sum_{k=0}^{\infty} a_k^j$, $j = 1, 2, \dots$ *such that* $a_0^j = 1$ *for all* j *one has*

$$\prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^\infty} \prod_{j=1}^{\infty} a_{k_j}^j. \tag{1.4.1}$$

The summands on the right-hand side of (1.4.1) usually contain factors which are less than one. But each of the summands contains only finitely many factors different from 1. So the summands are in fact finite products.

PROOF. For a sequence $\{k_j\} \in \mathbb{N}^\infty$ define its *length* as maximal j for which $k_j \neq 0$ and its *maximum* as the value of its maximal term. The length of the zero sequence is defined as 0.

Consider a finite subset $S \subset \mathbb{N}^\infty$. Consider the partial sum

$$\sum_{\{k_j\} \in S} \prod_{k=1}^{\infty} a_{k_j}^j.$$

To estimate it, denote by L the maximal length of elements of S and denote by M the greatest of maxima of $\{k_j\} \in S$. In this case

$$\sum_{\{k_j\} \in S} \prod_{j=1}^{\infty} a_{k_j}^j = \sum_{\{k_j\} \in S} \prod_{j=1}^{L} a_{k_j}^j \leq \sum_{\{k_j\} \in \mathbb{N}_M^L} \prod_{j=1}^{L} a_{k_j}^j = \prod_{j=1}^{L} \sum_{k=0}^{M} a_k^j \leq \prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j,$$

where $\mathbb{N}_M^L$ denotes the set of all finite sequences $\{k_1, k_2, \dots, k_L\}$ of natural numbers such that $k_i \leq M$. By All-for-One this implies one of the required inequalities, namely, $\geq$.

To prove the opposite inequality, we prove that for any natural L one has

$$\prod_{j=1}^{L} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^L} \prod_{j=1}^{L} a_{k_j}^j, \tag{1.4.2}$$

where $\mathbb{N}^L$ denotes the set of all finite sequences $\{k_1, \dots k_L\}$ of natural numbers. The proof is by induction on L.

LEMMA 1.4.2. *For any families $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$ of nonnegative numbers, one has*

$$\sum_{i\in I} a_i \sum_{j\in J} b_j = \sum_{(i,j)\in I\times J} a_i b_j.$$

PROOF OF LEMMA 1.4.2. Since $I \times J = \bigsqcup_{i\in I}\{i\} \times J$ by the Sum Partition Theorem one gets:

$$\begin{aligned}\sum_{(i,j)\in I\times J} a_i b_j &= \sum_{i\in I} \sum_{(i,j)\in\{i\}\times J} a_i b_j \\ &= \sum_{i\in I}\sum_{j\in J} a_i b_j \\ &= \sum_{i\in I} a_i \sum_{j\in J} b_j \\ &= \sum_{j\in J} b_j \sum_{i\in I} a_i.\end{aligned}$$

□

Case $L = 2$ follows from Lemma 1.4.2, because $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. The induction step is done as follows

$$\begin{aligned}\prod_{j=1}^{L+1}\sum_{k=0}^{\infty} a_k^j &= \sum_{k=0}^{\infty} a_k^{L+1} \prod_{j=1}^{L}\sum_{k=0}^{\infty} a_k^j \\ &= \sum_{k\in\mathbb{N}} a_k^{L+1} \sum_{\{k_j\}\in\mathbb{N}^L} \prod_{j=1}^{L} a_{k_j}^j \\ &= \sum_{\{k_j\}\in\mathbb{N}^{L+1}} \prod_{j=1}^{L+1} a_{k_j}^j.\end{aligned}$$

The left-hand side of (1.4.2) is a partial product for the left-hand side of (1.4.1) and the right-hand side of (1.4.2) is a subsum of the right-hand side of (1.4.1). Consequently, all partial products of the right-hand side in (1.4.1) do not exceed its left-hand side. This proves the inequality $\leq$. □

Euler's Identity. Our next goal is to prove the *Euler Identity.*

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^\alpha} = \prod_{p=1}^{\infty}\left(1 - \frac{1}{p^\alpha}\right)^{-[p \text{ is prime}]}}$$

Here α is any rational (or even irrational) positive number.

The product on the right-hand side is called the *Euler Product.* The series on the left-hand side is called the *Dirichlet series.* Each factor of the Euler Product expands into the geometric series $\sum_{k=0}^{\infty} \frac{1}{p^{k\alpha}}$. By Theorem 1.4.1, the product of these geometric series is equal to the sum of products of the type $p_1^{-k_1\alpha} p_2^{-k_2\alpha} \dots p_n^{-k_n\alpha}$ $= N^{-\alpha}$. Here $\{p_i\}$ are different prime numbers, $\{k_i\}$ are positive natural numbers and $p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} = N$. But each product $p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} = N$ is a natural number, different products represent different numbers and any natural number has a unique representation of this sort. This is exactly what is called Principal Theorem of

Arithmetic. That is, the above decomposition of the Euler product expands in the Dirichlet series.

Convergence of the Dirichlet series.

THEOREM 1.4.3. *The Dirichlet series* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *converges if and only if* $s > 1$.

PROOF. Consider a $\{2^k\}$ packing of the series. Then the n-th term of the packed series one estimates from above as

$$\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^s} \leq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{(2^n)^s} = 2^n \frac{1}{2^{ns}} = 2^{n-ns} = (2^{1-s})^n.$$

If $s > 1$ then $2^{1-s} < 1$ and the packed series is termwise majorized by a convergent geometric progression. Hence it converges. In the case of the Harmonic series ($s = 1$) the n-th term of its packing one estimates from below as

$$\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \geq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{2^{n+1}} = 2^n \frac{1}{2^{n+1}} = \frac{1}{2}.$$

That is why the harmonic series diverges. A Dirichlet series for $s < 1$ termwise majorizes the Harmonic series and so diverges. □

The Riemann ζ-function. The function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is called the *Riemann ζ-function*. It is of great importance in number theory.

The simplest application of Euler's Identity represents Euler's proof of the infinity of the set of primes. The divergence of the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ implies the Euler Product has to contain infinitely many factors to diverge.

Euler proved an essentially more exact result: the series of reciprocal primes diverges $\sum \frac{1}{p} = \infty$.

Summing via multiplication. Multiplication of series gives rise to a new approach to evaluating their sums. Consider the geometric series $\sum_{k=0}^{\infty} x^k$. Then

$$\left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{j,k \in \mathbb{N}^2} x^j x^k = \sum_{m=0}^{\infty} \sum_{j+k=m} x^j x^k = \sum_{m=0}^{\infty} (m+1) x^m.$$

As $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ one gets $\sum_{k=0}^{\infty} (k+1) x^k = \frac{1}{(1-x)^2}$.

Sine-product. Now we are ready to understand how two formulas

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right), \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{1.4.3}$$

which appeared in the Legends, yield an evaluation of the Euler Series. Since at the moment we do not know how to multiply infinite sequences of numbers which are less than one, we invert the product in the first formula. We get

$$\frac{x}{\sin x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right)^{-1} = \prod_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{x^{2j}}{k^{2j} \pi^{2j}}. \tag{1.4.4}$$

To avoid negative numbers, we interpret the series

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

in the second formula of (1.4.3) as the difference

$$\sum_{k=0}^{\infty}\frac{x^{4k+1}}{(4k+1)!}-\sum_{k=0}^{\infty}\frac{x^{4k+3}}{(4k+3)!}.$$

Substituting this expression for $\sin x$ in $\frac{x}{\sin x}$ and cancelling out x, we get

$$\frac{x}{\sin x}=\frac{1}{1-\sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}}=\sum_{j=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k+1)!}\right)^j.$$

All terms on the right-hand side starting with $j=2$ are divisible by x^4. Consequently the only summand with x^2 on the right-hand side is $\frac{x^2}{6}$. On the other hand in (1.4.4) after an expansion into a sum by Theorem 1.4.1, the terms with x^2 give the series $\sum_{k=1}^{\infty}\frac{x^2}{k^2\pi^2}$. Comparing these results, one gets $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

Problems.

1. Prove $\prod_{n=1}^{\infty}1.1=\infty$.
2. Prove the identity $\prod_{n=1}^{\infty}a_n^2=(\prod_{n=1}^{\infty}a_n)^2$ $(a_n\geq 1)$.
3. Does $\prod_{n=1}^{\infty}(1+\frac{1}{n})$ converge?
4. Evaluate $\prod_{n=2}^{\infty}\frac{n^2}{n^2-1}$.
5. Prove the divergence of $\prod_1^{\infty}(1+\frac{1}{k})^{[k\text{ is prime}]}$.
6. Evaluate $\prod_{n=3}^{\infty}\frac{n(n+1)}{(n-2)(n+3)}$.
7. Evaluate $\prod_{n=3}^{\infty}\frac{n^2-1}{n^2-4}$.
8. Evaluate $\prod_{n=1}^{\infty}(1+\frac{1}{n(n+2)})$.
9. Evaluate $\prod_{n=1}^{\infty}\frac{(2n+1)(2n+7)}{(2n+3)(2n+5)}$.
10. Evaluate $\prod_{n=2}^{\infty}\frac{n^3+1}{n^3-1}$.
11. Prove the inequality $\prod_{k=2}^{\infty}(1+\frac{1}{k^2})\geq\sum_{k=2}^{\infty}\frac{1}{k^2}$.
12. Prove the convergence of the Wallis product $\prod\frac{4k^2}{4k^2-1}$.
13. Evaluate $\sum_{k=1}^{\infty}\frac{1}{k^4}$ by applying (1.4.3).
14. Prove $\prod_{n=2}^{\infty}\frac{n^2+1}{n^2}<\infty$.
15. Multiply a geometric series by itself and get a power series expansion for $(1-x)^{-2}$.
16. Define $\tau(n)$ as the number of divisors of n. Prove $\zeta^2(x)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^x}$.
17. Define $\phi(n)$ as the number of numbers which are less than n are relatively prime to n. Prove $\frac{\zeta(x-1)}{\zeta(x)}=\sum_{n=1}^{\infty}\frac{\phi(n)}{n^x}$.
18. Define $\mu(n)$ (Möbius function) as follows: $\mu(1)=1$, $\mu(n)=0$, if n is divisible by the square of a prime number, $\mu(n)=(-1)^k$, if n is the product of k different prime numbers. Prove $\frac{1}{\zeta(x)}=\sum_{k=1}^{\infty}\frac{\mu(n)}{n^x}$.
***19.** Prove $\sum_{k=1}^{\infty}\frac{[k\text{ is prime}]}{k}=\infty$.
***20.** Prove the identity $\prod_{n=0}^{\infty}(1+x^{2^n})=\frac{1}{1-x}$.

1.5. Telescopic Sums

On the content of this lecture. In this lecture we learn the main secret of elementary summation theory. We will evaluate series via their partial sums. We introduce *factorial powers*, which are easy to sum. Following Stirling we expand $\frac{1}{1+x^2}$ into a series of negative factorial powers and apply this expansion to evaluate the Euler series with Stirling's accuracy of 10^{-8}.

The series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$. In the first lecture we calculated infinite sums directly without invoking partial sums. Now we present a dual approach to summing series. According to this approach, at first one finds a formula for the n-th partial sum and then substitutes in this formula infinity instead of n. The series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ gives a simple example for this method. The key to sum it up is the following identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Because of this identity $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ turns into the sum of differences

$$(1.5.1) \qquad \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\dots.$$

Its n-th partial sum is equal to $1-\frac{1}{n+1}$. Substituting in this formula $n=+\infty$, one gets 1 as its ultimate sum.

Telescopic sums. The sum (1.5.1) represents a *telescopic sum*. This name is used for sums of the form $\sum_{k=0}^{n}(a_k - a_{k+1})$. The value of such a telescopic sum is determined by the values of the first and the last of a_k, similarly to a telescope, whose thickness is determined by the radii of the external and internal rings. Indeed,

$$\sum_{k=0}^{n}(a_k-a_{k+1}) = \sum_{k=0}^{n}a_k - \sum_{k=0}^{n}a_{k+1} = a_0 + \sum_{k=1}^{n}a_k - \sum_{k=0}^{n-1}a_{k+1} - a_{n+1} = a_0 - a_{n+1}.$$

The same arguments for infinite telescopic sums give

$$(1.5.2) \qquad \sum_{k=0}^{\infty}(a_k - a_{k+1}) = a_0.$$

But this proof works only if $\sum_{k=0}^{\infty}a_k<\infty$. This is untrue for $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$, owing to the divergence of the Harmonic series. But the equality (1.5.2) holds also if a_k tends to 0 as k tends to infinity. Indeed, in this case a_0 is the least number majorizing all a_0-a_n, the n-th partial sums of $\sum_{k=0}^{\infty}a_k$.

Differences. For a given sequence $\{a_k\}$ one denotes by $\{\Delta a_k\}$ the sequence of differences $\Delta a_k = a_{k+1}-a_k$ and calls the latter sequence the *difference* of $\{a_k\}$. This is the main formula of elementary summation theory.

$$\boxed{\sum_{k=0}^{n-1}\Delta a_k = a_n - a_0}$$

To telescope a series $\sum_{k=0}^{\infty}a_k$ it is sufficient to find a sequence $\{A_k\}$ such that $\Delta A_k = a_k$. On the other hand the sequence of sums $A_n=\sum_{k=0}^{n-1}a_k$ has difference $\Delta A_n = a_n$. Therefore, we see that to telescope a sum is equivalent to find a formula for partial sums. This lead to concept of a *telescopic function*. For a function $f(x)$ we introduce its difference $\Delta f(x)$ as $f(x+1) - f(x)$. A function $f(x)$ telescopes $\sum a_k$ if $\Delta f(k) = a_k$ for all k.

Often the sequence $\{a_k\}$ that we would like to telescope has the form $a_k = f(k)$ for some function. Then we are searching for a *telescopic function* $F(x)$ for $f(x)$, i.e., a function such that $\Delta F(x) = f(x)$.

To evaluate the difference of a function is usually much easier than to telescope it. For this reason one has evaluated the differences of all basic functions and organized a *table of differences*. In order to telescope a given function, look in this table to find a table function whose difference coincides with or is close to given function.

For example, the differences of x^n for $n \leq 3$ are $\Delta x = 1$, $\Delta x^2 = 2x + 1$, $\Delta x^3 = 3x^2 + 3x + 1$. To telescope $\sum_{k=1}^{\infty} k^2$ we choose in this table x^3. Then $\frac{\Delta x^3}{3} - x^2 = x + \frac{1}{3} = \frac{\Delta x^2}{2} - \Delta \frac{x}{6}$. Therefore, $x^2 = \Delta\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}\right)$. This immediately implies the following formula for sums of squares:

$$\sum_{k=1}^{n-1} k^2 = \frac{2n^3 - 3n^2 + n}{6}. \tag{1.5.3}$$

Factorial powers. The usual powers x^n have complicated differences. The so-called *factorial powers* $x^{\underline{k}}$ have simpler differences. For any number x and any natural number k, let $x^{\underline{k}}$ denote $x(x-1)(x-2)\dots(x-k+1)$, and by $x^{\underline{-k}}$ we denote $\frac{1}{(x+1)(x+2)\dots(x+k)}$. At last we define $x^{\underline{0}} = 1$. The factorial power satisfies the following *addition law*.

$$\boxed{x^{\underline{k+m}} = x^{\underline{k}}(x-k)^{\underline{m}}}$$

We leave to the reader to check this rule for all integers m, k. The power $n^{\underline{n}}$ for a natural n coincides with the factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$. The main property of factorial powers is given by:

$$\boxed{\Delta x^{\underline{n}} = n x^{\underline{n-1}}}$$

The proof is straightforward:

$$\begin{aligned}
(x+1)^{\underline{k}} - x^{\underline{k}} &= (x+1)^{\underline{1+(k-1)}} - x^{\underline{(k-1)+1}} \\
&= (x+1)x^{\underline{k-1}} - x^{\underline{k-1}}(x-k+1) \\
&= kx^{\underline{k-1}}.
\end{aligned}$$

Applying this formula one can easily telescope any *factorial polynomial*, i.e., an expression of the form

$$a_0 + a_1 x^{\underline{1}} + a_2 x^{\underline{2}} + a_3 x^{\underline{3}} + \cdots + a_n x^{\underline{n}}.$$

Indeed, the explicit formula for the telescoping function is

$$a_0 x^{\underline{1}} + \tfrac{a_1}{2} x^{\underline{2}} + \tfrac{a_2}{3} x^{\underline{3}} + \tfrac{a_3}{4} x^{\underline{4}} + \cdots + \tfrac{a_n}{n+1} x^{\underline{n+1}}.$$

Therefore, another strategy to telescope x^k is to represent it as a factorial polynomial.

For example, to represent x^2 as factorial polynomial, consider $a + bx + cx^{\underline{2}}$, a general factorial polynomial of degree 2. We are looking for $x^2 = a + bx + cx^{\underline{2}}$. Substituting $x = 0$ in this equality one gets $a = 0$. Substituting $x = 1$, one gets

$1 = b$, and finally for $x = 2$ one has $4 = 2+2c$. Hence $c = 1$. As result $x^2 = x + x^{\underline{2}}$. And the telescoping function is given by

$$\tfrac{1}{2}x^{\underline{2}} + \tfrac{1}{3}x^{\underline{3}} = \tfrac{1}{2}(x^2 - x) + \tfrac{1}{3}(x(x^2 - 3x + 2)) = \tfrac{1}{6}(2x^3 - 3x^2 + x).$$

And we have once again proved the formula (1.5.3).

Stirling Estimation of the Euler series. We will expand $\frac{1}{(1+x)^2}$ into a series of negative factorial powers in order to telescope it. A natural first approximation to $\frac{1}{(1+x^2)}$ is $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$. We represent $\frac{1}{(1+x)^2}$ as $x^{\underline{-2}} + R_1(x)$, where

$$R_1(x) = \frac{1}{(1+x)^2} - x^{\underline{-2}} = \frac{1}{(x+1)^2(x+2)}.$$

The remainder $R_1(x)$ is in a natural way approximated by $x^{\underline{-3}}$. If $R_1(x) = x^{\underline{-3}} + R_2(x)$ then $R_2(x) = \frac{2}{(x+1)^2(x+2)(x+3)}$. Further, $R_2(x) = 2x^{\underline{-4}} + R_3(x)$, where

$$R_3(x) = \frac{2\cdot 3}{(x+1)^2(x+2)(x+3)(x+4)} = \frac{3!}{x+1}x^{\underline{-4}}.$$

The above calculations lead to the conjecture

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{n-1} k!x^{\underline{-k-2}} + \frac{n!}{x+1}x^{\underline{-n-1}}. \tag{1.5.4}$$

This conjecture is easily proved by induction. The remainder $R_n(x) = \frac{n!}{x+1}x^{\underline{-n-1}}$ represents the difference $\frac{1}{(1+x)^2} - \sum_{k=0}^{n-1} k!x^{\underline{-2-k}}$. Owing to the inequality $x^{\underline{-1-n}} \le \frac{1}{(n+1)!}$, which is valid for all $x \ge 0$, the remainder decreases to 0 as n increases to infinity. This implies

THEOREM 1.5.1. *For all $x \ge 0$ one has*

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} k!x^{\underline{-2-k}}.$$

To calculate $\sum_{k=p}^{\infty} \frac{1}{(1+k)^2}$, replace all summands by the expressions (1.5.4). We will get

$$\sum_{k=p}^{\infty}\left(\sum_{m=0}^{n-1} m!k^{\underline{-2-m}} + \frac{n!}{k+1}k^{\underline{-1-n}}\right).$$

Changing the order of summation we have

$$\sum_{m=0}^{n-1} m!\sum_{k=p}^{\infty} k^{\underline{-2-m}} + \sum_{k=p}^{\infty}\frac{n!}{k+1}k^{\underline{-1-n}}.$$

Since $\frac{1}{1+m}x^{\underline{-1-m}}$ telescopes the sequence $\{k^{\underline{-2-m}}\}$, $\sum_{k=p}^{\infty} k^{\underline{-2-m}} = \frac{1}{1+m}p^{\underline{-1-m}}$, Denote the sum of remainders $\sum_{k=p}^{\infty}\frac{n!}{k+1}k^{\underline{-1-n}}$ by $R(n,p)$. Then for all natural p and n one has

$$\sum_{k=p}^{\infty}\frac{1}{(1+k)^2} = \sum_{m=0}^{n-1}\frac{m!}{1+m}p^{\underline{-1-m}} + R(n,p)$$

For $p = 0$ and $n = +\infty$, the right-hand side turns into the Euler series, and one could get a false impression that we get nothing new. But $k^{\underline{-2-n}} \leq \frac{1}{k+1}k^{\underline{-1-n}} \leq (k-1)^{\underline{-2-n}}$, hence

$$\frac{n!}{1+n}p^{\underline{-1-n}} = \sum_{k=p}^{\infty} n!k^{\underline{-2-n}} \leq R(n,p) \leq \sum_{k=p}^{\infty} n!(k-1)^{\underline{-2-n}} = \frac{n!}{1+n}(p-1)^{\underline{-1-n}}.$$

Since $(p-1)^{\underline{-1-n}} - p^{\underline{-1-n}} = (1+n)(p-1)^{\underline{-2-n}}$, there is a $\theta \in (0,1)$ such that

$$R(n,p) = \frac{n!}{1+n}p^{\underline{-1-n}} + \theta n!(p-1)^{\underline{-2-n}}.$$

Finally we get:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{p-1} \frac{1}{(1+k)^2} + \sum_{k=0}^{n-1} \frac{k!}{1+k}p^{\underline{-1-k}} + \theta n!(p-1)^{\underline{-2-n}}.$$

For $p = n = 3$ this formula turns into

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{40} + \frac{1}{180} + \frac{\theta}{420}.$$

For $p = n = 10$ one gets $R(10,10) \leq 10!9^{\underline{-12}}$. After cancellations one has $\frac{1}{2\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 17\cdot 19}$. This is approximately $2 \cdot 10^{-8}$. Therefore

$$\sum_{k=0}^{10-1} \frac{1}{(k+1)^2} + \sum_{k=0}^{10-1} \frac{k!}{1+k}10^{\underline{-1-k}}$$

is less than the sum of the Euler series by only $2 \cdot 10^{-8}$. In such a way one can in one hour calculate eight digits of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ after the decimal point. It is not a bad result, but it is still far from Euler's eighteen digits. For $p = 10$, to provide eighteen digits one has to sum essentially more than one hundred terms of the series. This is a bit too much for a person, but is possible for a computer.

Problems.

1. Telescope $\sum k^3$.
2. Represent x^4 as a factorial polynomial.
3. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.
4. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}$.
5. Prove: If $\Delta a_k \geq \Delta b_k$ for all k and $a_1 \geq b_1$ then $a_k \geq b_k$ for all k.
6. $\Delta(x+a)^{\underline{n}} = n(x+a)^{\underline{n-1}}$.
7. Prove Archimedes's inequality $\frac{n^3}{3} \leq \sum_{k=1}^{n-1} k^2 \leq \frac{(n+1)^3}{3}$.
8. Telescope $\sum_{k=1}^{\infty} \frac{k}{2^k}$.
9. Prove the inequalities $\frac{1}{n} \geq \sum_{k=n+1}^{\infty} \frac{1}{k^2} \geq \frac{1}{n+1}$.
10. Prove that the degree of $\Delta P(x)$ is less than the degree of $P(x)$ for any polynomial $P(x)$.
11. Relying on $\Delta 2^n = 2^n$, prove that $P(n) < 2^n$ eventually for any polynomial $P(x)$.
12. Prove $\sum_{k=0}^{\infty} k!(x-1)^{\underline{-1-k}} = \frac{1}{x}$.

1.6. Complex Series

On the contents of the lecture. Complex numbers hide the key to the Euler Series. The summation theory developed for positive series now extends to complex series. We will see that complex series can help to sum real series.

Cubic equation. Complex numbers arise in connection with the solution of the cubic equation. The substitution $x = y - \frac{a}{3}$ reduces the general cubic equation $x^3 + ax^2 + bx + c = 0$ to

$$y^3 + py + q = 0.$$

The reduced equation one solves by the following trick. One looks for a root in the form $y = \alpha+\beta$. Then $(\alpha+\beta)^3+p(\alpha+\beta)+q = 0$ or $\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)+p(\alpha+\beta)+q = 0$. The latter equality one reduces to the system

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 3\alpha\beta &= -p. \end{aligned} \tag{1.6.1}$$

Raising the second equation into a cube one gets

$$\begin{aligned} \alpha^3 + \beta^3 &= -q, \\ 27\alpha^3\beta^3 &= -p^3. \end{aligned}$$

Now α^3, β^3 are roots of the quadratic equation

$$x^2 + qx - \tfrac{p^3}{27},$$

called the *resolution* of the original cubic equation. Sometimes the resolution has no roots, while the cubic equation always has a root. Nevertheless one can evaluate a root of the cubic equation with the help of its resolution. To do this one simply ignores that the numbers under the square roots are negative.

For example consider the following cubic equation

$$x^3 - \tfrac{3}{2}x - \tfrac{1}{2} = 0. \tag{1.6.2}$$

Then (1.6.1) turns into

$$\begin{aligned} \alpha^3 + \beta^3 &= \tfrac{1}{2}, \\ \alpha^3\beta^3 &= \tfrac{1}{8}, \end{aligned}$$

The corresponding resolution is $t^2 - \frac{t}{2} + \frac{1}{8} = 0$ and its roots are

$$t_{1,2} = \tfrac{1}{4} \pm \sqrt{\tfrac{1}{16} - \tfrac{1}{8}} = \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{-1}.$$

Then the desired root of the cubic equation is given by

$$\sqrt[3]{\tfrac{1}{4}(1+\sqrt{-1})} + \sqrt[3]{\tfrac{1}{4}(1-\sqrt{-1})} = \tfrac{1}{\sqrt[3]{4}}\left(\sqrt[3]{1+\sqrt{-1}} + \sqrt[3]{1-\sqrt{-1}}\right). \tag{1.6.3}$$

It turns out that the latter expression one uniquely interprets as a real number which is a root of the equation (1.6.2). To evaluate it consider the following expression

$$\sqrt[3]{(1+\sqrt{-1})^2} - \sqrt[3]{(1+\sqrt{-1})}\sqrt[3]{(1-\sqrt{-1})} + \sqrt[3]{(1-\sqrt{-1})^2}. \tag{1.6.4}$$

Since

$$(1+\sqrt{-1})^2 = 1^2 + 2\sqrt{-1} + \sqrt{-1}^2 = 1 + 2\sqrt{-1} - 1 = 2\sqrt{-1},$$

the left summand of (1.6.4) is equal to

$$\sqrt[3]{2\sqrt{-1}} = \sqrt[3]{2}\sqrt[3]{\sqrt{-1}} = \sqrt[3]{2}\sqrt{\sqrt[3]{-1}} = \sqrt[3]{2}\,\sqrt{-1}.$$

Similarly $(1-\sqrt{-1})^2 = -2\sqrt{-1}$, and the right summand of (1.6.4) turns into $-\sqrt[3]{2}\sqrt{-1}$. Finally $(1+\sqrt{-1})(1-\sqrt{-1}) = 1^2 - \sqrt{-1}^2 = 2$ and the central one is $-\sqrt[3]{2}$. As a result the whole expression (1.6.4) is evaluated as $-\sqrt[3]{2}$.

On the other hand one evaluates the product of (1.6.3) and (1.6.4) by the usual formula as the sum of cubes

$$\frac{1}{\sqrt[3]{4}}((1+\sqrt{-1})+(1-\sqrt{-1})) = \frac{1}{\sqrt[3]{4}}((1+1)+(\sqrt{-1})-\sqrt{-1})) = \frac{1}{\sqrt[3]{4}}(2+0) = \sqrt[3]{2}.$$

Consequently (1.6.3) is equal to $\frac{\sqrt[3]{2}}{-\sqrt[3]{2}} = -1$. And -1 is a true root of (1.6.2).

Arithmetic of complex numbers. In the sequel we use i instead of $\sqrt{-1}$. There are two basic ways to represent a complex number. The representation $z = a + ib$, where a and b are real numbers we call the *Cartesian form* of z. The numbers a and b are called respectively the *real* and the *imaginary* parts of z and are denoted by $\operatorname{Re} z$ and by $\operatorname{Im} z$ respectively. Addition and multiplication of complex numbers are defined via their real and imaginary parts as follows

$$\begin{aligned} \operatorname{Re}(z_1+z_2) &= \operatorname{Re} z_1 + \operatorname{Re} z_2, \\ \operatorname{Im}(z_1+z_2) &= \operatorname{Im} z_1 + \operatorname{Im} z_2, \\ \operatorname{Re}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2, \\ \operatorname{Im}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Im} z_1 \operatorname{Re} z_2. \end{aligned}$$

The *trigonometric form* of a complex number is $z = \rho(\cos\phi + i\sin\phi)$, where $\rho \geq 0$ is called the *module* or the *absolute value* of a complex number z and is denoted $|z|$, and ϕ is called its *argument*. The argument of a complex number is defined modulo 2π. We denote by $\operatorname{Arg} z$ the set of all arguments of z, and by $\arg z$ the element of $\operatorname{Arg} z$ which satisfies the inequalities $-\pi < \arg z \leq \pi$. So $\arg z$ is uniquely defined for all complex numbers. $\arg z$ is called the *principal argument* of z.

The number $a - bi$ is called the *conjugate* to $z = a + bi$ and denoted $\overline{z}$. One has $z\overline{z} = |z|^2$. This allows us to express z^{-1} as $\frac{\overline{z}}{|z|^2}$.

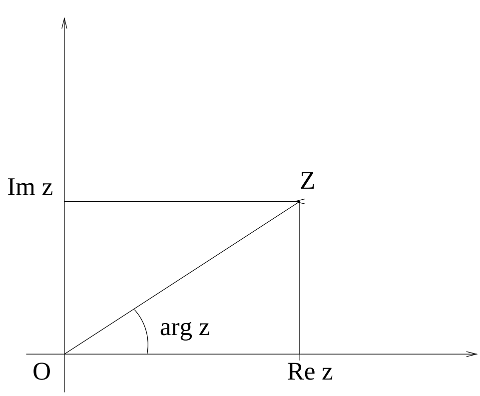

FIGURE 1.6.1. The representation of a complex number

If $z = a+ib$ then $|z| = \sqrt{a^2+b^2}$ and $\arg z = \operatorname{arctg}\frac{b}{a}$. One represents a complex number $z = a+bi$ as a point Z of the plane with coordinates (a,b). Then $|z|$ is equal to the distance from Z to the origin O. And $\arg z$ represents the angle between the axis of abscises and the ray $\overrightarrow{OZ}$. Addition of complex numbers corresponds to usual vector addition. And the usual triangle inequality turns into the *module inequality*:

$$|z+\zeta| \leq |z| + |\zeta|.$$

The multiplication formula for complex numbers in the trigonometric form is especially simple:

$$\begin{aligned} r(\cos\phi + i\sin\phi)r'(\cos\psi + i\sin\psi) \\ = rr'(\cos(\phi+\psi) + i\sin(\phi+\psi)). \end{aligned} \tag{1.6.5}$$

Indeed, the left-hand side and the right-hand side of (1.6.5) transform to

$$rr'(\cos\phi\cos\psi - \sin\phi\sin\psi) + irr'(\sin\phi\cos\psi + \sin\psi\cos\phi).$$

That is, the module of the product is equal to the product of modules and the argument of product is equal to the sum of arguments:

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 \oplus \operatorname{Arg} z_2.$$

Any complex number is uniquely defined by its module and argument.

The multiplication formula allows us to prove by induction the following:

$$(\cos\phi + i\sin\phi)^n = (\cos n\phi + i\sin n\phi). \tag{Moivre Formula}$$

Sum of a complex series. Now is the time to extend our summation theory to series made of complex numbers. We extend the whole theory without any losses to so-called absolutely convergent series. The series $\sum_{k=1}^{\infty} z_k$ with arbitrary complex terms is called *absolutely convergent*, if the series $\sum_{k=1}^{\infty} |z_k|$ of absolute values converges.

For any real number x one defines two nonnegative numbers: its *positive* x^+ *and negative* x^- *parts* as $x^+ = x[x \geq 0]$ and $x^- = -x[x < 0]$. The following identities characterize the positive and negative parts of x

$$x^+ + x^- = |x|, \qquad x^+ - x^- = x.$$

Now the sum of an absolutely convergent series of real numbers is defined as follows:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-. \tag{1.6.6}$$

That is, from the sum of all positive summands one subtracts the sum of modules of all negative summands. The two series on the right-hand side converge, because $a_k^+ \leq |a_k|$, $a_k^- \leq |a_k|$ and $\sum_{k=1}^{\infty} |a_k| < \infty$.

For an absolutely convergent complex series $\sum_{k=1}^{\infty} z_k$ we define the real and imaginary parts of its sum separately by the formulas

$$\operatorname{Re}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Re} z_k, \qquad \operatorname{Im}\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Im} z_k. \tag{1.6.7}$$

The series in the right-hand sides of these formulas are absolutely convergent, since $|\operatorname{Re} z_k| \leq |z_k|$ and $|\operatorname{Im} z_k| \leq |z_k|$.

THEOREM 1.6.1. *For any pair of absolutely convergent series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *its termwise sum* $\sum_{k=1}^{\infty}(a_k + b_k)$ *absolutely converges and*

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k. \tag{1.6.8}$$

PROOF. First, remark that the absolute convergence of the series on the left-hand side follows from the Module Inequality $|a_k+b_k| \leq |a_k|+|b_k|$ and the absolute convergence of the series on the right-hand side.

Now consider the case of real numbers. Representing all sums in (1.6.8) as differences of their positive and negative parts and separating positive and negative terms in different sides one transforms (1.6.8) into

$$\sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} b_k^+ + \sum_{k=1}^{\infty}(a_k + b_k)^- = \sum_{k=1}^{\infty} a_k^- + \sum_{k=1}^{\infty} b_k^- + \sum_{k=1}^{\infty}(a_k + b_k)^+.$$

But this equality is true due to termwise addition for positive series and the following identity,

$$x^- + y^- + (x + y)^+ = x^+ + y^+ + (x + y)^-.$$

Moving terms around turns this identity into

$$(x + y)^+ - (x + y)^- = (x^+ - x^-) + (y^+ - y^-),$$

which is true due to the identity $x^+ - +x^- = x$.

In the complex case the equality (1.6.8) splits into two equalities, one for real parts and another for imaginary parts. As for real series the termwise addition is already proved, we can write the following chain of equalities,

$$\begin{aligned}\operatorname{Re}\left(\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k\right) &= \operatorname{Re}\sum_{k=1}^{\infty} a_k + \operatorname{Re}\sum_{k=1}^{\infty} b_k \\ &= \sum_{k=1}^{\infty} \operatorname{Re} a_k + \sum_{k=1}^{\infty} \operatorname{Re} b_k \\ &= \sum_{k=1}^{\infty}(\operatorname{Re} a_k + \operatorname{Re} b_k) \\ &= \sum_{k=1}^{\infty} \operatorname{Re}(a_k + b_k) \\ &= \operatorname{Re}\sum_{k=1}^{\infty}(a_k + b_k),\end{aligned}$$

which proves the equality of real parts in (1.6.8). The same proof works for the imaginary parts. □

Sum Partition Theorem. An unordered sum of a family of complex numbers is defined by the same formulas (1.6.6) and (1.6.7). Since for positive series non-ordered sums coincide with the ordered sums, we get the same coincidence for all absolutely convergent series. Hence the commutativity law holds for all absolutely convergence series.

THEOREM 1.6.2. *If* $I = \bigsqcup_{j \in J} I_j$ *and* $\sum_{k=1}^{\infty} |a_k| < \infty$ *then* $\sum_{j \in J} \left|\sum_{i \in I_j} a_i\right| < \infty$ *and* $\sum_{j \in J} \sum_{i \in I_j} a_i = \sum_{i \in I} a_i$.

PROOF. At first consider the case of real summands. By definition $\sum_{i \in I} a_i = \sum_{i \in I} a_i^+ - \sum_{i \in I} a_i^-$. By Sum Partition Theorem positive series one transforms the original sum into

$$\sum_{j \in J} \sum_{i \in I_j} a_i^+ - \sum_{j \in J} \sum_{i \in I_j} a_i^-.$$

Now by the Termwise Addition applied at first to external and after to internal sums one gets

$$\sum_{j\in J}\left(\sum_{i\in I_j} a_i^+ - \sum_{i\in I_j} a_i^-\right) = \sum_{j\in J}\sum_{i\in I_j}(a_i^+ - a_i^-) = \sum_{j\in J}\sum_{i\in I_j} a_i.$$

So the Sum Partition Theorem is proved for all absolutely convergent real series. And it immediately extends to absolutely convergent complex series by its splitting into real and imaginary parts. □

THEOREM 1.6.3 (Termwise Multiplication). *If* $\sum_{k=1}^\infty |z_k| < \infty$ *then for any (complex)* c*,* $\sum_{k=1}^\infty |cz_k| < \infty$ *and* $\sum_{k=1}^\infty cz_k = c\sum_{k=1}^\infty z_k$.

PROOF. Termwise Multiplication for positive numbers gives the first statement of the theorem $\sum_{k=1}^\infty |cz_k| = \sum_{k=1}^\infty |c||z_k| = |c|\sum_{k=1}^\infty |z_k|$. The further proof is divided into five cases.

At first suppose c is positive and z_k real. Then $cz_k^+ = cz_k^+$ and by virtue of Termwise Multiplication for positive series we get

$$\begin{aligned}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty cz_k^+ - \sum_{k=1}^\infty cz_k^- \\ &= c\sum_{k=1}^\infty z_k^+ - c\sum_{k=1}^\infty z_k^- \\ &= c\left(\sum_{k=1}^\infty z_k^+ - \sum_{k=1}^\infty z_k^-\right) \\ &= c\sum_{k=1}^\infty z_k.\end{aligned}$$

The second case. Let $c = -1$ and z_k be real. In this case

$$\sum_{k=1}^\infty -z_k = \sum_{k=1}^\infty(-z_k)^+ - \sum_{k=1}^\infty(-z_k)^- = \sum_{k=1}^\infty z_k^- - \sum_{k=1}^\infty z_k^+ = -\sum_{k=1}^\infty z_k.$$

The third case. Let c be real and z_k complex. In this case $\operatorname{Re} cz_k = c\operatorname{Re} z_k$ and the two cases above imply the Termwise Multiplication for any real c. Hence

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty cz_k &= \sum_{k=1}^\infty \operatorname{Re} cz_k \\ &= \sum_{k=1}^\infty c\operatorname{Re} z_k \\ &= c\sum_{k=1}^\infty \operatorname{Re} z_k \\ &= c\operatorname{Re}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} c\sum_{k=1}^\infty z_k.\end{aligned}$$

The same is true for imaginary parts.

The fourth case. Let $c = i$ and z_k be complex. Then $\operatorname{Re} iz_k = -\operatorname{Im} z_k$ and $\operatorname{Im} iz_k = \operatorname{Re} z_k$. So one gets for real parts

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^\infty iz_k &= \sum_{k=1}^\infty \operatorname{Re}(iz_k) \\ &= \sum_{k=1}^\infty -\operatorname{Im} z_k \\ &= -\sum_{k=1}^\infty \operatorname{Im} z_k \\ &= -\operatorname{Im}\sum_{k=1}^\infty z_k \\ &= \operatorname{Re} i\sum_{k=1}^\infty z_k.\end{aligned}$$

The general case. Let $c = a + bi$ with real a, b. Then

$$\begin{aligned} c\sum_{k=1}^{\infty} z_k &= a\sum_{k=1}^{\infty} z_k + ib\sum_{k=1}^{\infty} z_k \\ &= \sum_{k=1}^{\infty} az_k + \sum_{k=1}^{\infty} ibz_k \\ &= \sum_{k=1}^{\infty}(az_k + ibz_k) \\ &= \sum_{k=1}^{\infty} cz_k. \end{aligned}$$

□

Multiplication of Series. For two given series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, one defines their *convolution* as a series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

THEOREM 1.6.4 (Cauchy). *For any pair of absolutely convergent series* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *their convolution* $\sum_{k=0}^{\infty} c_k$ *absolutely converges and*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k.$$

PROOF. Consider the double series $\sum_{i,j} a_i b_j$. Then by the Sum Partition Theorem its sum is equal to

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_i b_j\right) = \sum_{j=0}^{\infty} b_j \left(\sum_{i=0}^{\infty} a_i\right) = \left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right).$$

On the other hand, $\sum_{i,j} a_i b_j = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1-1} a_k b_{n-k}$. But the last sum is just the convolution.

This proof goes through for positive series. In the generalcase we have to prove absolute convergence of the double series. But this follows from

$$\left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{k=0}^{\infty} |b_k|\right) = \sum_{k=0}^{\infty} |c_k|.$$

□

Module Inequality.

$$\left|\sum_{k=1}^{\infty} z_k\right| \leq \sum_{k=1}^{\infty} |z_k|. \tag{1.6.9}$$

Let $z_k = x_k + iy_k$. Summation of the inequalities $-|x_k| \leq x_k \leq |x_k|$ gives $-\sum_{k=1}^{\infty} |x_k| \leq \sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} |x_k|$, which means $|\sum_{k=1}^{\infty} x_k| \leq \sum_{k=1}^{\infty} |x_k|$. The same inequality is true for y_k. Consider $z'_k = |x_k| + i|y_k|$. Then $|z_k| = |z'_k|$ and $|\sum_{k=1}^{\infty} z_k| \leq |\sum_{k=1}^{\infty} z'_k|$. Therefore it is sufficient to prove the inequality (1.6.9) for z'_k, that is, for numbers with non-negative real and imaginary parts. Now supposing x_k, y_k to be nonnegative one gets the following chain of equivalent transformations of (1.6.9):

$$\begin{aligned} &\left(\sum_{k=1}^{\infty} x_k\right)^2 + \left(\sum_{k=1}^{\infty} y_k\right)^2 \leq \left(\sum_{k=1}^{\infty} |z_k|\right)^2 \\ &\qquad\qquad \sum_{k=1}^{\infty} x_k \leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{\infty} y_k\right)^2} \\ &\qquad\qquad \sum_{k=1}^{n} x_k \leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{\infty} y_k\right)^2}, \quad \forall n = 1, 2, \dots \\ &\qquad\qquad \sum_{k=1}^{\infty} y_k \leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\operatorname{Re}\sum_{k=1}^{n} x_k\right)^2}, \quad \forall n = 1, 2, \dots \\ &\qquad\qquad \sum_{k=1}^{m} y_k \leq \sqrt{\left(\sum_{k=1}^{\infty} |z_k|\right)^2 - \left(\sum_{k=1}^{n} x_k\right)^2}, \quad \forall n, m = 1, 2, \dots \\ &\left(\sum_{k=1}^{n} x_k\right)^2 + \left(\sum_{k=1}^{m} y_k\right)^2 \leq \left(\sum_{k=1}^{\infty} |z_k|\right)^2, \quad \forall m, n = 1, 2, \dots \end{aligned}$$

$$\sqrt{\left(\sum_{k=1}^N x_k\right)^2 + \left(\sum_{k=1}^N y_k\right)^2} \leq \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$
$$\left|\sum_{k=1}^N z_k\right| \leq \sum_{k=1}^\infty |z_k|, \quad \forall N = 1, 2, \dots$$

The inequalities of the last system hold because $\left|\sum_{k=1}^N z_k\right| \leq \sum_{k=1}^N |z_k| \leq \sum_{k=1}^\infty |z_k|$.

Complex geometric progressions. The sum of a geometric progression with a complex ratio is given by the same formula

$$\sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z}. \tag{1.6.10}$$

And the proof is the same as in the case of real numbers. But the meaning of this formula is different. Any complex formula is in fact a pair of formulas. Any complex equation is in fact a pair of equations.

In particular, for $z = q(\sin\phi + i\cos\phi)$ the real part of the left-hand side of (1.6.10) owing to the Moivre Formula turns into $\sum_{k=0}^{n-1} q^k \sin k\phi$ and the right-hand side turns into $\sum_{k=0}^{n-1} q^k \cos k\phi$. So the formula for a geometric progression splits into two formulas which allow us to telescope some trigonometric series.

Especially interesting is the case with the ratio $\varepsilon_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. In this case the geometric progression cyclically takes the same values, because $\varepsilon_n^n = 1$. The terms of this sequence are called the *roots of unity*, because they satisfy the equation $z^n - 1 = 0$.

Lemma 1.6.5. $(z^n - 1) = \prod_{k=1}^n (z - \varepsilon_n^k)$.

Proof. Denote by $P(z)$ the right-hand side product. This polynomial has degree n, has major coefficient 1 and has all ε_n^k as its roots. Then the difference $(z^n - 1) - P(z)$ is a polynomial of degree $< n$ which has n different roots. Such a polynomial has to be 0 by virtue of the following general theorem. □

Theorem 1.6.6. *The number of roots of any nonzero complex polynomial does not exceed its degree.*

Proof. The proof is by induction on the degree of $P(z)$. A polynomial of degree 1 has the form $az+b$ and the only root is $-\frac{b}{a}$. Suppose our theorem is proved for any polynomial of degree $< n$. Consider a polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n, where the coefficients are complex numbers. Suppose it has at least n roots $z_1, \dots, z_n$. Consider the polynomial $P^*(z) = a_n \prod_{k=1}^n (z - z_k)$. The difference $P(z) - P^*(z)$ has degree $< n$ and has at least n roots (all z_k). By the induction hypothesis this difference is zero. Hence, $P(z) = P^*(z)$. But $P^*(z)$ has only n roots. Indeed, for any z different from all z_k one has $|z - z_k| > 0$. Therefore $|P^*(z)| = |a_n| \prod_{k=1}^n |z - z_k| > 0$. □

By blocking conjugated roots one gets a pure real formula:

$$z^n - 1 = (z - 1) \prod_{k=1}^{(n-1)/2} \left(z^2 - 2z\cos\frac{2k\pi}{n} + 1\right).$$

Complexification of series. Complex numbers are effectively applied to sum up so-called *trigonometric series*, i.e., series of the type $\sum_{k=0}^{\infty} a_k \cos kx$ and $\sum_{k=0}^{\infty} a_k \sin kx$. For example, to sum the series $\sum_{k=1}^{\infty} q^k \sin k\phi$ one couples it with its dual $\sum_{k=0}^{\infty} q^k \cos k\phi$ to form a complex series $\sum_{k=0}^{\infty} q^k(\cos k\phi + i \sin k\phi)$. The last is a complex geometric series. Its sum is $\frac{1}{1-z}$, where $z = \cos\phi + i\sin\phi$. Now the sum of the sine series $\sum_{k=1}^{\infty} q^k \sin k\phi$ is equal to $\operatorname{Im} \frac{1}{1-z}$, the imaginary part of the complex series, and the real part of the complex series coincides with the cosine series. In particular, for $q = 1$, one has $\frac{1}{1-z} = \frac{1}{1+\cos\phi + i\sin\phi}$. To evaluate the real and imaginary parts one multiplies both numerator and denominator by $1 + \cos\phi - i\sin\phi$. Then one gets $(1-\cos\phi)^2 + \sin^2\phi = 1 - 2\cos^2\phi + \cos^2\phi + \sin^2\phi = 2 - 2\cos\phi$ as the denominator. Hence $\frac{1}{1-z} = \frac{1-\cos\phi + i\sin\phi}{2-2\cos\phi} = \frac{1}{2} + \frac{1}{2}\cot\frac{\phi}{2}$. And we get two remarkable formulas for the sum of the divergent series

$$\sum_{k=0}^{\infty} \cos k\phi = \frac{1}{2}, \qquad \sum_{k=1}^{\infty} \sin k\phi = \frac{1}{2}\cot\frac{\phi}{2}.$$

For $\phi = 0$ the left series turns into $\sum_{k=0}^{\infty}(-1)^k$. The evaluation of the Euler series via this cosine series is remarkably short, it takes one line. But one has to know integrals and a something else to justify this evaluation.

Problems.

1. Find real and imaginary parts for $\frac{1}{1-i}$, $(\frac{1-i}{1+i})^3$, $\frac{i^5+2}{i^{19}+1}$, $\frac{(1+i)^5}{(1-i)^3}$.
2. Find trigonometric form for -1, $1+i$, $\sqrt{3}+i$.
3. Prove that $z_1 z_2 = 0$ implies either $z_1 = 0$ or $z_2 = 0$.
4. Prove the distributivity law for complex numbers.
5. Analytically prove the inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.
6. Evaluate $\sum_{k=1}^{n-1} \frac{1}{z_k(z_k+1)}$, where $z_k = 1 + kz$.
7. Evaluate $\sum_{k=1}^{n-1} z_k^2$, where $z_k = 1 + kz$.
8. Evaluate $\sum_{k=1}^{n-1} \dfrac{\sin k}{2^k}$.
9. Solve $z^2 = i$.
10. Solve $z^2 = 3 - 4i$.
11. Telescope $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
12. Prove that the conjugated to a root of polynomial with real coefficient is the root of the polynomial.
13. Prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
14. Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
***15.** Solve $8x^3 - 6x - 1 = 0$.
16. Evaluate $\sum_{k=1}^{\infty} \dfrac{\sin k}{2^k}$.
17. Evaluate $\sum_{k=1}^{\infty} \frac{\sin 2k}{3^k}$.
18. Prove absolute convergence of $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ for any z.
19. For which z the series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ absolutely converges?
20. Multiply a geometric series onto itself several times applying Cauchy formula.
21. Find series for $\sqrt{1+x}$ by method of indefinite coefficients.
22. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ absolutely converge?
23. Does series $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ absolutely converge?

CHAPTER 2

Integrals

2.1. Natural Logarithm

On the contents of the lecture.

> In the beginning of Calculus was the Word, and the Word was with Arithmetic, and the Word was *Logarithm*[1]

Logarithmic tables. Multiplication is much more difficult than addition. A logarithm reduces multiplication to addition. The invention of logarithms was one of the great achievements of our civilization.

In early times, when logarithms were unknown instead of them one used trigonometric functions. The following identity

$$2\cos x \cos y = \cos(x+y) + \cos(x-y)$$

can be applied to calculate products via tables of cosines. To multiply numbers x and y, one represents them as cosines $x = \cos a$, $y = \cos b$ using the cosine table. Then evaluate $(a+b)$ and $(a-b)$ and find their cosines in the table. Finally, the results are summed and divided by 2. That is all. A single multiplication requires four searches in the table of cosines, two additions, one subtraction and one division by 2.

A logarithmic function $l(x)$ is a function such that $l(xy) = l(x) + l(y)$ for any x and y. If one has a logarithmic table, to evaluate the product xy one has to find in the logarithmic table $l(x)$ and $l(y)$ then sum them and find the antilogarithm of the sum. This is much easier.

The idea of logarithms arose in 1544, when M. Stiefel compared geometric and arithmetic progressions. The addition of exponents corresponds to the multiplication of powers. Hence consider a number close to 1, say, 1.000001. Calculate the sequence of its powers and place them in the left column. Place in the right column the corresponding values of exponents, which are just the line numbers. The logarithmic table is ready.

Now to multiply two numbers x and y, find them (or their approximations) in the left column of the logarithmic table, and read their logarithms from the right column. Sum the logarithms and find the value of the sum in the right column. Next to this sum in the left column the product xy stands. The first tables of such logarithms were composed by John Napier in 1614.

Area of a curvilinear trapezium. Recall that a sequence is said to be monotone, if it is either increasing or decreasing. The minimal interval which contains all elements of a given sequence of points will be called *supporting interval* of the sequence. And a sequence is called *exhausting* for an interval I if I is the supporting interval of the sequence.

Let f be a non-negative function defined on $[a,b]$. The set $\{(x,y) \mid x \in [a,b] \text{ and } 0 \le y \le f(x)\}$ is called a *curvilinear trapezium* under the graph of f over the interval $[a,b]$.

To estimate the area of a curvilinear trapezium under the graph of f over $[a,b]$, choose an exhausting sequence $\{x_i\}_{i=0}^n$ for $[a,b]$ and consider the following sums:

$$(2.1.1) \qquad \sum_{k=0}^{n-1} f(x_k)|\delta x_k|, \qquad \sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k| \quad (\text{where } \delta x_k = x_{k+1} - x_k).$$

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[1] $\lambda o\gamma o\varsigma$ is Greek for "word", $\alpha\varrho\iota\theta\mu o\varsigma$ means "number".

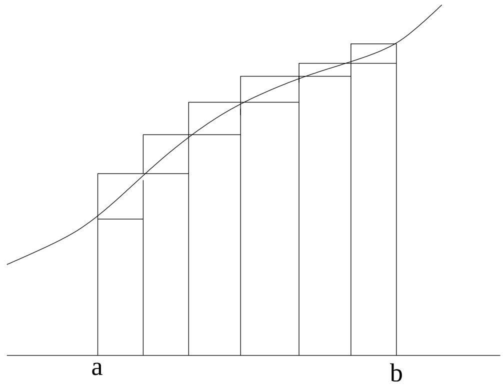

FIGURE 2.1.1. A curvilinear trapezium

We will call the first of them the *receding sum*, and the second the *advancing sum*, of the sequence $\{x_k\}$ for the function f. If the function f is monotone the area of the curvilinear trapezium is contained between these two sums. To see this, consider the following step-figures: $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_k)]$ and $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_{k+1})]$. If f and $\{x_k\}$ both increase or both decrease the first step-figure is contained in the curvilinear trapezium and the second step-figure contains the trapezium with possible exception of a vertical segment $[a \times [0, f(a)]$ or $[b \times [0, f(b)]$. If one of f and $\{x_k\}$ increases and the other decreases, then the step-figures switch the roles. The rededing sum equals the area of the first step-figure, and the advancing sum equals the area of the second one. Thus we have proved the following lemma.

LEMMA 2.1.1. *Let f be a monotone function and let S be the area of the curvilinear trapezium under the graph of f over $[a, b]$. Then for any sequence $\{x_k\}_{k=0}^n$ exhausting $[a, b]$ the area S is contained between $\sum_{k=0}^{n-1} f(x_k)|\delta x_k|$ and $\sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k|$.*

Fermat's quadratures of parabolas. In 1636 Pierre Fermat proposed an ingenious trick to determine the area below the curve $y = x^a$.

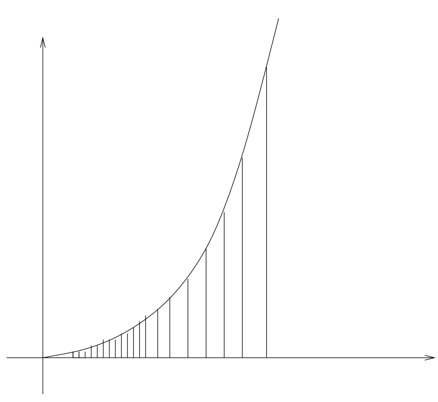

FIGURE 2.1.2. Fermat's quadratures of parabolas

If $a > -1$ then consider any interval of the form $[0, B]$. Choose a positive $q < 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[0, B]$ and the values of the function for this sequence also form a geometric progression $B^a, q^a B^a, q^{2a} B^a, q^{3a} B^a, \dots$. Then both the receding and advancing sums turn into geometric progressions:

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^k B - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(1-q)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^k B - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(1-q)q^a}{1-q^{a+1}}.\end{aligned}$$

For a natural a, one has $\frac{1-q}{1-q^{a+1}} = \frac{1}{1+q+q^2+\dots+q^a}$. As q tends to 1 both sums converge to $\frac{B^{a+1}}{a+1}$. This is the area of the curvilinear trapezium. Let us remark that for $a < 0$ this trapezium is unbounded, nevertheless it has finite area if $a > -1$.

If $a < -1$, then consider an interval in the form $[B, \infty]$. Choose a positive $q > 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[B, \infty]$ and the values of the function for this sequence also form a geometric progression $B^a, q^a B^a, q^{2a} B^a, q^{3a} B^a, \dots$. The receding and advancing sums are

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^{k+1} B - q^{k}B) &= B^{a+1}(q-1)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(q-1)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^{k+1} B - q^{k}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(q-1)q^a}{1-q^{a+1}}.\end{aligned}$$

If a is an integer set $p = q^{-1}$. Then $\frac{q-1}{1-q^{a+1}} = q\frac{1-p}{1-p^{|a|-1}} = q\frac{1}{1+p+p^2+\dots+p^{n-2}}$. As q tends to 1 both sums converge to $\frac{B^{a+1}}{|a|-1}$. This is the area of the curvilinear trapezium.

For $a > -1$ the area of the curvilinear trapezium under the graph of x^a over $[A, B]$ is equal to the difference between the areas of trapezia over $[0, B]$ and $[0, A]$. Hence this area is $\frac{B^{a+1}-A^{a+1}}{a+1}$.

For $a < -1$ one can evaluate the area of the curvilinear trapezium under the graph of x^a over $[A, B]$ as the difference between the areas of trapezia over $[A, \infty]$ and $[B, \infty]$. The result is expressed by the same formula $\frac{B^{a+1}-A^{a+1}}{a+1}$.

THEOREM 2.1.2 (Fermat). *The area below the curve $y = x^a$ over the interval $[A, B]$ is equal to $\frac{B^{a+1}-A^{a+1}}{a+1}$ for $a \neq 1$.*

We have proved this theorem for integer a, but Fermat proved it for all real $a \neq -1$.

The Natural Logarithm. In the case $a = -1$ the geometric progression for areas of step-figures turns into an arithmetic progression. This means that the area below a hyperbola is a logarithm! This discovery was made by Gregory in 1647.

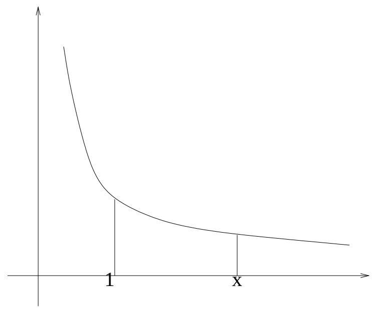

FIGURE 2.1.3. The hyperbolic trapezium over $[1, x]$

The figure bounded from above by the graph of hyperbola $y = 1/x$, from below by segment $[a, b]$ of the axis of abscissas, and on each side by vertical lines passing through the end points of the interval, is called a *hyperbolic trapezium over* $[a, b]$.

The area of hyperbolic trapezium over $[1, x]$ with $x > 1$ is called the *natural logarithm* of x, and it is denoted by $\ln x$. For a positive number $x < 1$ its logarithm is defined as the negative number whose absolute value coincides with the area of hyperbolic trapezium over $[x, 1]$. At last, $\ln 1$ is defined as 0.

THEOREM 2.1.3 (on logarithm). *The natural logarithm is an increasing function defined for all positive numbers. For each pair of positive numbers x, y*

$$\ln xy = \ln x + \ln y.$$

PROOF. Consider the case $x, y > 1$. The difference $\ln xy - \ln y$ is the area of the hyperbolic trapezium over $[y, xy]$. And we have to prove that it is equal to $\ln x$, the area of trapezium over $[1, x]$. Choose a large number n. Let $q = x^{1/n}$. Then $q^n = x$. The finite geometric progression $\{q^k\}_{k=0}^n$ exhausts $[1, x]$. Then the receding and advancing sums are

$$\sum_{k=0}^{n-1} q^{-k}(q^{k+1} - q^k) = n(q-1) \qquad \sum_{k=0}^{n-1} q^{-k-1}(q^{k+1} - q^k) = \frac{n(q-1)}{q}. \tag{2.1.2}$$

Now consider the sequence $\{xq^k\}_{k=0}^n$ exhausting $[x, xy]$. Its receding sum

$$\sum_{k=0}^{n-1} x^{-1}q^{-k}(xq^{k+1} - xq^k) = n(q-1)$$

just coincides with the receding sum (2.1.2) for $\ln x$. The same is true for the advancing sum. As a result we obtain for any natural n the following inequalities:

$$n(q-1) \geq \ln x \geq \frac{n(q-1)}{q} \qquad n(q-1) \geq \ln xy - \ln y \geq \frac{n(q-1)}{q}$$

This implies that $|\ln xy - \ln x - \ln y|$ does not exceed the difference between the the receding and advancing sums. The statement of Theorem 2.1.3 in the case $x, y > 1$ will be proved when we will prove that this difference can be made arbitrarily small by a choice of n. This will be deduced from the following general lemma.

LEMMA 2.1.4. *Let f be a monotone function over the interval $[a,b]$ and let $\{x_k\}_{k=0}^n$ be a sequence that exhausts $[a,b]$. Then*

$$\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| \le |f(b)-f(a)| \max_{k<n} |\delta x_k|$$

PROOF OF LEMMA. The proof of the lemma is a straightforward calculation. To shorten the notation, set $\delta f(x_k) = f(x_{k+1}) - f(x_k)$.

$$\begin{aligned}\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| &= \left|\sum_{k=0}^{n-1} \delta f(x_k)\delta x_k\right| \\ &\le \sum_{k=0}^{n-1} |\delta f(x_k)| \max |\delta x_k| \\ &= \max|\delta x_k| \sum_{k=0}^{n-1} |\delta f(x_k)| \\ &= \max|\delta x_k| \left|\sum_{k=0}^{n-1} \delta f(x_k)\right| \\ &= \max|\delta x_k||f(b)-f(a)|.\end{aligned}$$

The equality $\left|\sum_{k=0}^{n-1} \delta f(x_k)\right| = \sum_{k=0}^{n-1} |\delta f(x_k)|$ holds, as $\delta f(x_k)$ have the same signs due to the monotonicity of f. □

The value $\max|\delta x_k|$ is called *maximal step* of the sequence $\{x_k\}$. For the sequence $\{q^k\}$ of $[1,x]$ its maximal step is equal to $q^n - q^{n-1} = q^n(1-q^{-1}) = x(1-q)/q$. It tends to 0 as q tends to 1. In our case $|f(b)-f(a)| = 1 - \frac{1}{x} < 1$. By Lemma 2.1.4 the difference between the receding and advancing sums could be made arbitrarily small. This completes the proof in the case $x, y > 1$.

Consider the case $xy = 1$, $x > 1$. We need to prove the following

(inversion rule) $$\ln 1/x = -\ln x.$$

As above, put $q^n = x > 1$. The sequence $\{q^{-k}\}_{k=0}^n$ exhausts $[1/x, 1]$. The corresponding receding sum $\sum_{k=0}^{n-1} q^{k+1}(q^{-k} - q^{-k-1}) = \sum_{k=0}^{n-1}(q-1) = n(q-1)$ coincides with its counterpart for $\ln x$. The same is true for the advancing one. The same arguments as above prove $|\ln 1/x| = \ln x$. The sign of $\ln 1/x$ is defined as minus because $1/x < 1$. This proves the inversion rule.

Now consider the case $x < 1$, $y < 1$. Then $1/x > 1$ and $1/y > 1$ and by the first case $\ln 1/xy = (\ln 1/x + \ln 1/y)$. Replacing all terms of this equation according to the inversion rule, one gets $-\ln xy = -\ln x - \ln y$ and finally $\ln xy = \ln x + \ln y$.

The next case is $x > 1$, $y < 1$, $xy < 1$. Since both $1/x$ and xy are less then 1, then by the previous case $\ln xy + \ln 1/x = \ln \frac{xy}{x} = \ln y$. Replacing $\ln 1/x$ by $-\ln x$ one gets $\ln xy - \ln x = \ln y$ and finally $\ln xy = \ln x + \ln y$.

The last case, $x > 1$, $y < 1$, $xy > 1$ is proved by $\ln xy + \ln 1/y = \ln x$ and replacing $\ln 1/y$ by $-\ln y$. □

Base of a logarithm. Natural or hyperbolic logarithms are not the only logarithmic functions. Other popular logarithms are decimal ones. In computer science one prefers binary logarithms. Different logarithmic functions are distinguished by their *bases*. The base of a logarithmic function $l(x)$ is defined as the number b for which $l(b) = 1$. Logarithms with the base b are denoted by $\log_b x$. What is the base of the natural logarithm? This is the second most important constant in mathematics (after π). It is an irrational number denoted by e which is equal to $2.71828182845905\dots$. It was Euler who introduced this number and this notation.

Well, e is the number such that the area of hyperbolic trapezium over $[1, e]$ is 1. Consider the geometric progression q^n for $q = 1 + \frac{1}{n}$. All summands in the corresponding hyperbolic receding sum for this progression are equal to $\frac{q^{k+1}-q^k}{q^k} = q - 1 = \frac{1}{n}$. Hence the receding sum for the interval $[1, q^n]$ is equal to 1 and it is greater than $\ln q^n$. Consequently $e > q^n$. The summands of the advancing sum in this case are equal to $\frac{q^{k+1}-q^k}{q^{k+1}} = 1 - \frac{1}{q} = \frac{1}{n+1}$. Hence the advancing sum for the interval $[1, q^{n+1}]$ is equal to 1. It is less than the corresponding logarithm. Consequently, $e < q^{n+1}$. Thus we have proved the following estimates for e:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

We see that $\left(1 + \frac{1}{n}\right)^n$ rapidly tends to e as n tends to infinity.

Problems.

1. Prove that $\ln x/y = \ln x - \ln y$.
2. Prove that $\ln 2 < 1$.
3. Prove that $\ln 3 > 1$.
4. Prove that $x > y$ implies $\ln x > \ln y$.
5. Is $\ln x$ bounded?
6. Prove that $\frac{1}{n+1} < \ln(1 + 1/n) < \frac{1}{n}$.
7. Prove that $\frac{x}{1+x} < \ln(1 + x) < x$.
8. Prove the Theorem 2.1.2 (Fermat) for $a = 1/2, 1/3, 2/3$.
9. Prove the unboundedness of $\frac{n}{\ln n}$.
10. Compare $\left(1 + \frac{1}{n}\right)^n$ and $(1 + \frac{1}{n+1})^{n+1}$.
11. Prove the monotonicity of $\frac{n}{\ln n}$.
12. Prove that $\sum_{k=2}^{n-1} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$.
13. Prove that $\ln(1 + x) > x - \frac{x^2}{2}$.
14. Estimate integral part of $\ln 1000000$.
15. Prove that $\ln \frac{x+y}{2} \geq \frac{\ln x + \ln y}{2}$.
16. Prove the convergence of $\sum_{k=1}^{\infty} (\frac{1}{k} - \ln(1 + \frac{1}{k}))$.
17. Prove that $(n + \frac{1}{2})^{-1} \leq \ln(1 + \frac{1}{n}) < \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})$.
***18.** Prove that $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \ln 2$.

2.2. Definite Integral

On the contents of the lecture. Areas of curvilinear trapezia play an extraordinary important role in mathematics. They generate a key concept of Calculus — the concept of the *integral*.

Three basic rules. For a nonnegative function f its integral $\int_a^b f(x)\,dx$ along the interval $[a,b]$ is defined just as the area of the curvilinear trapezium below the graph of f over $[a,b]$. We allow a function to take infinite values. Let us remark that changing of the value of function in one point does not affect the integral, because the area of the line is zero. That is why we allow the functions under consideration to be undefined in a finite number of points of the interval.

Immediately from the definition one gets the following three *basic rules of integration*:

Rule of constant	$\int_a^b f(x)\,dx = c(b-a)$, if $f(x) = c$	for $x \in (a,b)$,
Rule of inequality	$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, if $f(x) \leq g(x)$	for $x \in (a,b)$,
Rule of partition	$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	for $b \in (a,c)$.

Partition. Let $|J|$ denote the length of an interval J. Let us say that a sequence $\{J_k\}_{k=1}^n$ of disjoint open subintervals of an interval I is a *partition* of I, if $\sum_{k=1}^n |I_k| = |I|$. The boundary of a partition $P = \{J_k\}_{k=1}^n$ is defined as the difference $I \setminus \bigcup_{k=1}^n J_k$ and is denoted ∂P.

For any finite subset S of an interval I, which contains the ends of I, there is a unique partition of I which has this set as the boundary. Such a partition is called *generated* by S. For a monotone sequence $\{x_k\}_{k=0}^n$ the generated partition is $\{(x_{k-1}, x_k)\}_{k=1}^n$.

Piecewise constant functions. A function $f(x)$ is called *partially constant* on a partition $\{J_k\}_{k=1}^n$ of $[a,b]$ if it is constant on each J_k. The Rules of Constant and Partition immediately imply:

$$\int_a^b f(x)\,dx = \sum_{k=1}^n f(J_k)|J_k|. \tag{2.2.1}$$

Proof. Indeed, the integral splits into a sum of integrals over $J_k = [x_{k-1}, x_k]$, and the function takes the value $f(J_k)$ in (x_{k-1}, x_k). □

A function is called *piecewise constant* over an interval if it is partially constant with respect to some finite partition of the interval.

Lemma 2.2.1. *Let f and g be piecewise constant functions over $[a,b]$. Then $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.*

Proof. First, suppose $f(x) = c$ is constant on the interval (a,b). Let g take the value g_k over the interval (x_k, x_{k+1}) for an exhausting $\{x_k\}_{k=0}^n$. Then $f(x) + g(x)$ takes values $(c + g_k)$ over (x_k, x_{k+1}). Hence $\int_a^b (f(x) + g(x))\,dx = \sum_{k=0}^{n-1} (c + g_k)|\delta x_k|$ due to (2.2.1). Splitting this sum and applying (2.2.1) to both summands, one gets $\sum_{k=0}^{n-1} c|\delta x_k| + \sum_{k=0}^{n-1} g_k|\delta x_k| = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. This proves the case of a constant f.

Now let f be partially constant on the partition generated by $\{x_k\}_{k=0}^n$. Then, by the partition rule, $\int_a^b (f(x)+g(x))\,dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx$. As f is constant on any (x_{k-1}, x_k), for any k one gets $\int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx = \int_{x_{k-1}}^{x_k} f(x)\,dx + \int_{x_{k-1}}^{x_k} g(x)\,dx$. Summing up these equalities one completes the proof of Lemma 2.2.1 for the sum.

The statement about differences follows from the addition formula applied to $g(x)$ and $f(x) - g(x)$. □

LEMMA 2.2.2. *For any monotone nonnegative function f on the interval $[a, b]$ and for any $\varepsilon > 0$ there is such piecewise constant function f_ε such that $f_\varepsilon \leq f(x) \leq f_\varepsilon(x) + \varepsilon$.*

PROOF. $f_\varepsilon(x) = \sum_{k=0}^\infty k\varepsilon[k\varepsilon \leq f(x) < (k+1)\varepsilon]$. □

THEOREM 2.2.3 (Addition Theorem). *Let f and g be nonnegative monotone functions defined on $[a, b]$. Then*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

PROOF. Let f_ε and g_ε be ε-approximations of f and g respectively provided by Lemma 2.2.2. Set $f^\varepsilon(x) = f_\varepsilon(x) + \varepsilon$ and $g^\varepsilon(x) = g_\varepsilon(x) + \varepsilon$. Then $f_\varepsilon(x) \leq f(x) \leq f^\varepsilon(x)$ and $g_\varepsilon(x) \leq g(x) \leq g^\varepsilon(x)$ for $x \in (a, b)$. Summing and integrating these inequalities in different order gives

$$\int_a^b (f_\varepsilon(x)+g_\varepsilon(x))\,dx \leq \int_a^b (f(x)+g(x))\,dx \leq \int_a^b (f^\varepsilon(x)+g^\varepsilon(x))\,dx$$

$$\int_a^b f_\varepsilon(x)\,dx + \int_a^b g_\varepsilon(x)\,dx \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \leq \int_a^b f^\varepsilon(x)\,dx + \int_a^b g^\varepsilon(x)\,dx.$$

Due to Lemma 2.2.1, the left-hand sides of these inequalities coincide, as well as the right-hand sides. Hence the difference between the central parts does not exceed

$$\int_a^b (f^\varepsilon(x) - f_\varepsilon(x))\,dx + \int_a^b (g^\varepsilon(x) - g_\varepsilon(x))\,dx \leq 2\varepsilon(b-a).$$

Hence, for any positive ε

$$\left| \int_a^b (f(x)+g(x))\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < 2\varepsilon(b-a).$$

This implies that the left-hand side vanishes. □

Term by term integration of a functional series.

LEMMA 2.2.4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of nonnegative nondecreasing functions and let p be a piecewise constant function. If $\sum_{k=1}^\infty f_k(x) \geq p(x)$ for all $x \in [a, b]$ then $\sum_{k=1}^\infty \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx$.*

PROOF. Let p be a piecewise constant function with respect to $\{x_i\}_{i=0}^n$. Choose any positive ε. Since $\sum_{k=1}^\infty f_k(x_i) \geq p(c)$, eventually one has $\sum_{k=1}^m f_k(x_i) > p(x_i) - \varepsilon$. Fix m such that this inequality holds simultaneously for all $\{x_i\}_{i=0}^n$. Let $[x_i, x_{i+1}]$ be an interval where $p(x)$ is constant. Then for any $x \in [x_i, x_{i+1}]$ one has these inequalities: $\sum_{k=1}^m f_k(x) \geq \sum_{k=1}^m f_k(x_k) > p(x_k) - \varepsilon = p(x) - \varepsilon$. Consequently for all $x \in [a,b]$ one has the inequality $\sum_{k=1}^{m} f_k(x) > p(x) - \varepsilon$. Taking integrals gives $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx \geq \int_a^b (p(x) - \varepsilon)\,dx = \int_a^b p(x)\,dx - \varepsilon(b-a)$. By the Addition Theorem $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx = \sum_{k=1}^{m} \int_a^b f_k(x)\,dx \leq \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Therefore $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx - \varepsilon(b-a)$ for any positive ε. This implies the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx$. □

THEOREM 2.2.5. *For any sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative nondecreasing functions on an interval $[a,b]$*

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

PROOF. Since $\sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{\infty} f_k(x)$ for all x, by integrating one gets

$$\int_a^b \sum_{k=1}^{n} f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx.$$

By the the Addition Theorem the left-hand side is equal to $\sum_{k=1}^{n} \int_a^b f_k(x)\,dx$, which is a partial sum of $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Then by All-for-One one gets the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$.

To prove the opposite inequality for any positive ε, we apply Lemma 2.2.2 to find a piecewise constant function F_ε, such that $F_\varepsilon(x) \leq \sum_{k=1}^{\infty} f_k(x)dx$ and $\int_a^b \sum_{k=1}^{\infty}(f_k(x) - F_\varepsilon(x))\,dx < \varepsilon$. On the other hand, by Lemma 2.2.4 one gets

$$\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b F_\varepsilon(x)\,dx.$$

Together these inequalities imply $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx + \varepsilon \geq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$. As the last inequality holds for all $\varepsilon > 0$, it holds also for $\varepsilon = 0$ □

THEOREM 2.2.6 (Mercator,1668). *For any $x \in (-1,1]$ one has*

$$\boxed{\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}} \tag{2.2.2}$$

PROOF. Consider $x \in [0,1)$. Since $\int_0^x t^k\,dt = \frac{t^{k+1}}{k+1}$ due to the Fermat Theorem 2.1.2, termwise integration of the geometric series $\sum_{k=0}^{\infty} t^k$ over the interval $[0,x]$ for $x < 1$ gives $\int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$.

LEMMA 2.2.7. $\int_0^x \frac{1}{1-t}\,dt = \ln(1-x)$.

PROOF OF LEMMA. Construct a translation of the plane which transforms the curvilinear trapezium below $\frac{1}{1-t}$ over $[0,x]$ into the trapezium for $\ln(1-x)$. Indeed, the reflection of the plane $((x,y) \to (2-x,y))$ along the line $x = 1$ transforms this trapezium to the curvilinear trapezium under $\frac{1}{x-1}$ over $[2-x,2]$. The parallel translation by 1 to the left of the latter trapezium $(x,y) \to (x-1,y)$ transforms it just in to the ogarithmic trapezium for $\ln(1-x)$. □

The Lemma proves the Mercator Theorem for negative x. To prove it for positive x, set $f_k(x) = x^{2k-1} - x^{2k}$. All functions f_k are nonnegative on $[0,1]$ and

$\sum_{k=1}^{\infty} f_k(x) = \frac{1}{1+x}$. Termwise integration of this equality over $[0, x]$ gives (2.2.2), modulo the equality $\int_0^x \frac{1}{1+t}\,dt = \int_1^x \frac{1}{t}\,dt$. The latter is proved by parallel translation of the plane. Let us remark, that in the case $x = 1$ the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$ is not absolutely convergent, and under its sum we mean $\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = \sum_{k=1}^{\infty}(\frac{1}{2k-1} - \frac{1}{2k})$. And the above proof proves just this fact. □

The arithmetic mean of Mercator's series evaluated at x and $-x$ gives *Gregory's Series*

$$\boxed{\frac{1}{2}\ln\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots .} \tag{2.2.3}$$

Gregory's series converges much faster than Mercator's one. For example, putting $x = \frac{1}{3}$ in (2.2.3) one gets

$$\ln 2 = \frac{2}{3} + \frac{2}{3 \cdot 3^3} + \frac{2}{5 \cdot 3^5} + \frac{2}{7 \cdot 3^7} + \dots .$$

Problems.

1. Prove that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

2. Prove the following formulas via piecewise constant approximations:

(multiplication formula) $$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

(shift formula) $$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

(reflection formula) $$\int_0^a f(x)\,dx = \int_{-a}^{0} f(-x)\,dx$$

(compression formula) $$\int_0^a f(x)\,dx = \frac{1}{k}\int_0^{ka} f\left(\frac{x}{k}\right)\,dx$$

3. Evaluate $\int_0^{2\pi}(\sin x + 1)\,dx$.

4. Prove the inequality $\int_{-2}^{2}(2 + x^3 2^x)\,dx > 8$.

5. Prove $\int_0^{2\pi} x(\sin x + 1)\,dx < 2\pi$.

6. Prove $\int_{100\pi}^{200\pi} \frac{x+\sin(x)}{x}\,dx \le 100\pi + \frac{1}{50}\pi$.

7. Denote by s_n the area of $\{(x, y) \mid 0 \le x \le 1, (1-x)\ln n + x\ln(n+1) \le y \le \ln(1+x)\}$. Prove that $\sum_{k=1}^{\infty} s_k < \infty$.

8. Prove that $\sum_{k=1}^{2n}(-1)^{k+1}\frac{x^k}{k} < \ln(1+x) < \sum_{k=1}^{2n+1}(-1)^{k+1}\frac{x^k}{k}$ for $x > 0$.

9. Compute the logarithms of the primes $2, 3, 5, 7$ with accuracy 0.01.

10. Evaluate $\int_0^1 \sqrt{x}\,dx$.

***11.** Evaluate $\int_0^{\pi} \sin x\,dx$.

2.3. Stieltjes Integral

On the contents of the lecture. The Stieltjes relativization of the integral makes the integral flexible. We learn the main transformations of integrals. They allow us to evaluate a lot of integrals.

Basic rules. A *parametric curve* is a mapping of an interval into the plane. In cartesian coordinates a parametric curve can be presented as a pair of functions $x(t), y(t)$. The first function $x(t)$ represents the value of abscises at the moment t, and the second $y(t)$ is the ordinate at the same moment. We define the integral $\int_a^b f(t)\,dg(t)$ for a nonnegative function f, called the *integrand*, and with respect to a nondecreasing *continuous* function g, called the *differand*, as the area below the curve $f(t), g(t) \mid t \in [a, b]$.

A monotone function f is called *continuous* over the interval $[a, b]$ if it takes all intermediate values, that is, the image $f[a, b]$ of $[a, b]$ coincides with $[f(a), f(b)]$. If it is not continuous for some $y \in [f(a), f(b)] \setminus f[a, b]$, there is a point $x(y) \in [a, b]$ with the following property: $f(x) < y$ if $x < x(y)$ and $f(x) > y$ if $x > x(y)$. Let us define a *generalized preimage* $f^{[-1]}(y)$ of a point $y \in [f(a), f(b)]$ either as its usual preimage $f^{-1}(y)$ if it is not empty, or as $x(y)$ in the opposite case.

Now the curvilinear trapezium below the curve $f(t), g(t)$ over $[a, b]$ is defined as $\{(x, y) \mid 0 \le y \le g(f^{[-1]}(x))\}$.

The basic rules for relative integrals transform into:

Rule of constant	$\int_a^b f(t)\,dg(t) = c(g(b) - g(a))$, if $f(t) = c$	for $t \in (a, b)$,
Rule of inequality	$\int_a^b f_1(t)\,dg(t) \le \int_a^b f_2(t)\,dg(t)$, if $f_1(t) \le f_2(t)$	for $t \in (a, b)$,
Rule of partition	$\int_a^c f(t)\,dg(t) = \int_a^b f(t)\,dg(t) + \int_b^c f(t)\,dg(t)$	for $b \in (a, c)$.

Addition theorem. The proofs of other properties of the integral are based on piecewise constant functions. For any number x, let us define its ε-integral part as $\varepsilon[x/\varepsilon]$. Immediately from the definition one gets:

LEMMA 2.3.1. *For any monotone nonnegative function f on the interval $[a, b]$ and for any $\varepsilon > 0$, the function $[f]_\varepsilon$ is piecewise constant such that $[f(x)]_\varepsilon \le f(x) \le [f(x)]_\varepsilon + \varepsilon$ for all x.*

THEOREM 2.3.2 (on multiplication). *For any nonnegative monotone f, and any continuous nondecreasing g and any positive constant c one has*

$$\int_a^b cf(x)\,dg(x) = c\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dcg(x). \tag{2.3.1}$$

PROOF. For the piecewise constant $f_\varepsilon = [f]_\varepsilon$, the proof is by a direct calculation. Hence

$$\int_a^b cf_\varepsilon(x)\,dg(x) = c\int_a^b f_\varepsilon(x)\,dg(x) = \int_a^b f_\varepsilon(x)\,dcg(x) = I_\varepsilon. \tag{2.3.2}$$

Now let us estimate the differences between integrals from (2.3.1) and their approximations from (2.3.2). For example, for the right-hand side integrals one has:

$$\int_a^b f\,dcg - \int_a^b f_\varepsilon\,dcg = \int_a^b (f - f_\varepsilon)\,dcg \le \int_a^b \varepsilon\,dcg = \varepsilon(cg(b) - cg(a)). \tag{2.3.3}$$

Hence $\int_a^b f\,dcg = I_\varepsilon + \varepsilon_1$, where $\varepsilon_1 \leq c\varepsilon(g(b)-g(a))$. The same argument proves $c\int_a^b f\,dg = I_\varepsilon + \varepsilon_2$ and $\int_a^b cf\,dg = I_\varepsilon + \varepsilon_3$, where $\varepsilon_2, \varepsilon_3 \leq c\varepsilon(g(b)-g(a))$. Then the pairwise differences between the integrals of (2.3.1) do not exceed $2c\varepsilon(g(b)-g(a))$. Consequently they are less than any positive number, that is, they are zero. □

THEOREM 2.3.3 (Addition Theorem). *Let f_1, f_2 be nonnegative monotone functions and g_1, g_2 be nondecreasing continuous functions over $[a,b]$, then*

$$\int_a^b (f_1(t)+f_2(t))\,dg_1(t) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_2(t)\,dg_1(t), \tag{2.3.4}$$

$$\int_a^b f_1(t)\,d(g_1(t)+g_2(t)) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_1(t)\,dg_2(t). \tag{2.3.5}$$

PROOF. For piecewise constant integrands both the equalities follow from the Rule of Constant and the Rule of Partition. To prove (2.3.4) replace f_1 and f_2 in both parts by $[f_1]_\varepsilon$ and $[f_2]_\varepsilon$. We get equality and denote by I_ε the common value of both sides of this equality. Then by (2.3.3) both integrals on the right-hand side differ from they approximation at most by $\varepsilon(g_1(b)-g_1(a))$, therefore the right-hand side of (2.3.4) differs from I_ε at most by $2\varepsilon(g_1(b)-g_1(a))$. The same is true for the left-hand side of (2.3.4). This follows immediately from (2.3.3) in case $f = f_1 + f_2$, $f_\varepsilon = [f_1]_\varepsilon + [f_2]_\varepsilon$ and $g = g_1$. Consequently, the difference between left-hand and right-hand sides of (2.3.4) does not exceed $4\varepsilon(g_1(b)-g_1(a))$. As ε can be chosen arbitrarily small this difference has to be zero.

The proof of (2.3.5) is even simpler. Denote by I_ε the common value of both parts of (2.3.5) where f_1 is changed by $[f_1]_\varepsilon$. By (2.3.3) one can estimate the differences between the integrals of (2.3.5) and their approximations as being $\leq \varepsilon(g_1(b)+g_2(b)-g_1(a)-g_2(a))$ for the left-hand side, and as $\leq \varepsilon(g_1(b)-g_1(a))$ and $\leq \varepsilon(g_2(b)-g_2(a))$ for the corresponding integrals of the right-hand side of (2.3.5). So both sides differ from I_ε by at most $\leq \varepsilon(g_1(b)-g_1(a)+g_2(b)-g_2(a))$. Hence the difference vanishes. □

Differential forms. An expression of the type $f_1dg_1 + f_2dg_2 + \cdots + f_ndg_n$ is called a *differential form*. One can add differential forms and multiply them by functions. The integral of a differential form $\int_a^b (f_1\,dg_1 + f_2dg_2 + \cdots + f_ndg_n)$ is defined as the sum of the integrals $\sum_{k=1}^n \int_a^b f_k\,dg_k$. Two differential forms are called equivalent on the interval $[a,b]$ if their integrals are equal for all subintervals of $[a,b]$. For the sake of brevity we denote the differential form $f_1dg_1 + f_2dg_2 + \cdots + f_ndg_n$ by FdG, where $F = \{f_1,\dots,f_n\}$ is a collection of integrands and $G = \{g_1,\dots,g_n\}$ is a collection of differands.

THEOREM 2.3.4 (on multiplication). *Let FdG and $F'dG'$ be two differential forms, with positive increasing integrands and continuous increasing differands, which are equivalent on $[a,b]$. Then their products by any increasing function f on $[a,b]$ are equivalent on $[a,b]$ too.*

PROOF. If f is constant then the statement follows from the multiplication formula. If f is piecewise constant, then divide $[a,b]$ into intervals where it is constant and prove the equality for parts and after collect the results by the Partition Rule. In the general case, $0 \leq \int_a^b fF\,dG - \int_a^b [f]_\varepsilon F\,dG \leq \int_a^b \varepsilon F\,dG = \varepsilon\int_a^b F\,dG$. Since $\int_a^b [f]_\varepsilon F'\,dG' = \int_a^b [f]_\varepsilon F\,dG$, one concludes that $\left|\int_a^b fF'\,dG' - \int_a^b fF\,dG\right| \leq$

$\varepsilon \int_a^b F\,dG + \varepsilon \int_a^b F'\,dG'$. The right-hand side of this inequality can be made arbitrarily small. Hence the left-hand side is 0. □

Integration by parts.

THEOREM 2.3.5. *If f and g are continuous nondecreasing nonnegative functions on $[a,b]$ then $d(fg)$ is equivalent to $fdg + gdf$.*

PROOF. Consider $[c,d] \subset [a,b]$. The integral $\int_c^d f\,dg$ represents the area below the curve $(f(t), g(t))_{t\in[c,d]}$. And the integral $\int_c^d g\,df$ represents the area on the left of the same curve. Its union is equal to $[0,f(d)]\times[0,g(d)] \setminus [0,f(c)]\times[0,g(c)]$. The area of this union is equal to $(f(d)g(d) - f(c)g(c) = \int_c^d dfg$. On the other hand the area of this union is the sum of the areas of curvilinear trapezia representing the integrals $\int_c^d f\,dg$ and $\int_c^d g\,df$. □

Change of variable. Consider a Stieltjes integral $\int_a^b f(\tau)\,dg(\tau)$ and suppose there is a continuous nondecreasing mapping $\tau\colon [t_0,t_1] \to [a,b]$, such that $\tau(t_0) = a$ and $\tau(t_1) = b$. The composition $g(\tau(t))$ is a continuous nondecreasing function and the curve $\{(f(\tau(t), g(\tau(t))) \mid t \in [t_0,t_1]\}$ just coincides with the curve $\{(f(\tau), g(\tau)) \mid \tau \in [a,b]$. Hence, the following equality holds; it is known as the *Change of Variable* formula:

$$\int_{t_0}^{t_1} f(\tau(t))\,dg(\tau(t)) = \int_{\tau(t_0)}^{\tau(t_1)} f(\tau)\,dg(\tau).$$

For differentials this means that the equality $F(x)dG(x) = F'(x)dG'(x)$ conserves if one substitutes instead of an independent variable x a function.

Differential Transformations.

Case dx^n. Integration by parts for $f(t) = g(t) = t$ gives $dt^2 = tdt + tdt$. Hence $tdt = d\frac{t^2}{2}$. If we already know that $dx^n = ndx^{n-1}$, then $dx^{n+1} = d(xx^n) = xdx^n + x^n dx = nxx^{n-1}dx + x^n dx = (n+1)x^n dx$. This proves the Fermat Theorem for natural n.

Case $d\sqrt[n]{x}$. To evaluate $d\sqrt[n]{x}$ substitute $x = y^n$ into the equality $dy^n = ny^{n-1}dy$. One gets $dx = \frac{nx}{\sqrt[n]{x}}d\sqrt[n]{x}$, hence $d\sqrt[n]{x} = \frac{\sqrt[n]{x}}{nx}dx$.

Case $\ln x dx$. We know $d\ln x = \frac{1}{x}dx$. Integration by parts gives $\ln x dx = d(x\ln x) - xd\ln x = d(x\ln x) - dx = d(x\ln x - x)$.

Problems.

1. Evaluate $dx^{2/3}$.
2. Evaluate dx^{-1}.
3. Evaluate $x\ln x\,dx$.
4. Evaluate $d\ln^2 x$.
5. Evaluate $\ln^2 x\,dx$.
6. Evaluate de^x.
7. Investigate the convergence of $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$.

2.4. Asymptotics of Sums

On the contents of the lecture. We become at last acquainted with the fundamental concept of a *limit*. We extend the notion of the sum of a series and discover that a change of order of summands can affect the ultimate sum. Finally we derive the famous Stirling formula for $n!$.

Asymptotic formulas. The Mercator series shows how useful series can be for evaluating integrals. In this lecture we will use integrals to evaluate both partial and ultimate sums of series. Rarely one has an explicit formula for partial sums of a series. There are lots of important cases where such a formula does not exist. For example, it is known that partial sums of the Euler series cannot be expressed as a finite combination of elementary functions. When an explicit formula is not available, one tries to find a so-called *asymptotic formula*. An asymptotic formula for a partial sum S_n of a series is a formula of the type $S_n = f(n) + R(n)$ where f is a known function called the *principal part* and $R(n)$ is a *remainder*, which is small, in some sense, with respect to the principal part. Today we will get an asymptotic formula for partial sums of the harmonic series.

Infinitesimally small sequences. The simplest asymptotic formula has a constant as its principal part and an infinitesimally small remainder. One says that a sequence $\{z_k\}$ is *infinitesimally small* and writes $\lim z_k = 0$, if z_k tends to 0 as n tends to infinity. That is for any positive ε eventually (i.e., beginning with some n) $|z_k| < \varepsilon$. With Iverson notation, this definition can be expressed in the following clear form:

$$[\{z_k\}_{k=1}^{\infty} \text{ is infinitesimally small}] = \prod_{m=1}^{\infty} 2\left|\sum_{n=1}^{\infty}(-1)^n \prod_{k=1}^{\infty}[m[k>n]|z_k|<1]\right|.$$

Three basic properties of infinitesimally small sequences immediately follow from the definition:

- if $\lim a_k = \lim b_k = 0$ then $\lim(a_k + b_k) = 0$;
- if $\lim a_k = 0$ then $\lim a_k b_k = 0$ for any bounded sequence $\{b_k\}$;
- if $a_k \leq b_k \leq c_k$ for all k and $\lim a_k = \lim c_k = 0$, then $\lim b_k = 0$.

The third property is called the *squeeze rule*.

Today we need just one property of infinitesimally small sequences:

THEOREM 2.4.1 (Addition theorem). *If the sequences $\{a_k\}$ and $\{b_k\}$ are infinitesimally small, than their sum and their difference are infinitesimally small too.*

PROOF. Let ε be a positive number. Then $\varepsilon/2$ also is positive number. And by definition of infinitesimally small, the inequalities $|a_k| < \varepsilon/2$ and $|b_k| < \varepsilon/2$ hold eventually beginning with some n. Then for $k > n$ one has $|a_k \pm b_k| \leq |a_k| + |b_k| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Limit of sequence.

DEFINITION. *A sequence $\{z_k\}$ of (complex) numbers* converges *to a number z if* $\lim z - z_k = 0$. *The number z is called the* limit *of the sequence $\{z_k\}$ and denoted by* $\lim z_k$.

An infinite sum represents a particular case of a limit as demonstrated by the following.

THEOREM 2.4.2. *The partial sums of an absolutely convergent series* $\sum_{k=1}^{\infty} z_k$ *converge to its sum.*

PROOF. $|\sum_{k=1}^{n-1} z_k - \sum_{k=1}^{\infty} z_k| = |\sum_{k=n}^{\infty} z_k| \leq \sum_{k=n}^{\infty} |z_k|$. Since $\sum_{k=1}^{\infty} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$, there is a partial sum such that $\sum_{k=1}^{n-1} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$. Then for all $m \geq n$ one has $\sum_{k=m}^{\infty} |z_k| \leq \sum_{k=n}^{\infty} |z_k| < \varepsilon$. □

Conditional convergence. The concept of the limit of sequence leads to a notion of convergence generalizing absolute convergence.

A series $\sum_{k=1}^{\infty} a_k$ is called (conditionally) *convergent* if $\lim \sum_{k=1}^{n} a_k = A + \alpha_n$, where $\lim \alpha_n = 0$. The number A is called its ultimate sum.

The following theorem gives a lot of examples of conditionally convergent series which are not absolutely convergent. By $[[n]]$ we denote the even part of the number n, i.e., $[[n]] = 2[n/2]$.

THEOREM 2.4.3 (Leibniz). *For any of positive decreasing infinitesimally small sequence* $\{a_n\}$, *the series* $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *converges.*

PROOF. Denote the difference $a_k - a_{k+1}$ by δa_k. The series $\sum_{k=1}^{\infty} \delta a_{2k-1}$ and $\sum_{k=1}^{\infty} \delta a_{2k}$ are positive and convergent, because their termwise sum is $\sum_{k=1}^{\infty} \delta a_k = a_1$. Hence $S = \sum_{k=1}^{\infty} \delta a_{2k-1} \leq a_1$. Denote by S_n the partial sum $\sum_{k=1}^{n-1}(-1)^{k+1}a_k$. Then $S_{2n} = \sum_{k=1}^{n-1} \delta a_{2n-1} = S + \alpha_n$, where $\lim \alpha_n = 0$. Then $S_n = S_{[[n]]} + a_n[n \text{ is odd}] + \alpha_{[[n]]}$. As $a_n[n \text{ is odd}] + \alpha_{[[n]]}$ is infinitesimally small, this implies the theorem. □

LEMMA 2.4.4. *Let* f *be a non-increasing nonnegative function. Then the series* $\sum_{k=1}^{\infty} (f(k) - \int_k^{k+1} f(x)\,dx)$ *is positive and convergent and has sum* $c_f \leq f(1)$.

PROOF. Integration of the inequalities $f(k) \geq f(x) \geq f(k+1)$ over $[k, k+1]$ gives $f(k) \geq \int_k^{k+1} f(x)\,dx \geq f(n+1)$. This proves the positivity of the series and allows us to majorize it by the telescopic series $\sum_{k=1}^{\infty}(f(k) - f(k+1)) = f(1)$. □

THEOREM 2.4.5 (Integral Test on Convergence). *If a nonnegative function* $f(x)$ *decreases monotonically on* $[1, +\infty)$, *then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if* $\int_1^{\infty} f(x)\,dx < \infty$.

PROOF. Since $\int_1^{\infty} f(x)\,dx = \sum_{k=1}^{\infty} \int_k^{k+1} f(x)\,dx$, one has $\sum_{k=1}^{\infty} f(k) = c_f + \int_1^{\infty} f(x)\,dx$. □

Euler constant. The sum $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right)$, which is c_f for $f(x) = \frac{1}{x}$, is called *Euler's constant* and denoted by γ. Its first ten digits are $0.5772156649\ldots$.

Harmonic numbers. The sum $\sum_{k=1}^{n} \frac{1}{k}$ is denoted H_n and is called the n-th *harmonic number.*

THEOREM 2.4.6. $H_n = \ln n + \gamma + o_n$ *where* $\lim o_n = 0$.

PROOF. Since $\ln n = \sum_{k=1}^{n-1}(\ln(k+1) - \ln k) = \sum_{k=1}^{n-1} \ln(1 + \frac{1}{k})$, one has $\ln n + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = H_{n-1}$. But $\sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = \gamma + \alpha_n$, where $\lim \alpha_n = 0$. Therefore $H_n = \ln n + \gamma + (\frac{1}{n} + \alpha_n)$. □

Alternating harmonic series. The *alternating harmonic series* $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is a conditionally convergent series due to the Leibniz Theorem 2.4.3, and it is not absolutely convergent. To find its sum we apply our Theorem 2.4.6 on asymptotics of harmonic numbers.

Denote by $S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$ the partial sum. Then $S_n = H'_n - H''_n$, where $H'_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is odd}]$ and $H''_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is even}]$. Since $H''_{2n} = \frac{1}{2}H_n$ and $H'_{2n} = H_{2n} - H''_{2n} = H_{2n} - \frac{1}{2}H_n$ one gets

$$\begin{aligned} S_{2n} &= H_{2n} - \tfrac{1}{2}H_n - \tfrac{1}{2}H_n \\ &= H_{2n} - H_n \\ &= \ln 2n + \gamma + o_{2n} - \ln n - \gamma - o_n \\ &= \ln 2 + (o_{2n} - o_n). \end{aligned}$$

Consequently $S_n = \ln 2 + (o_{[[n]]} - o_{[n/2]} + \frac{(-1)^{n+1}}{n}[n \text{ is odd}])$. As the sum in brackets is infinitesimally small, one gets

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \ln 2.$$

The same arguments for a permutated alternating harmonic series give

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots = \tfrac{3}{2}\ln 2. \tag{2.4.1}$$

Indeed, in this case its $3n$-th partial sum is

$$\begin{aligned} S_{3n} &= H'_{4n} - H''_{2n} \\ &= H_{4n} - \tfrac{1}{2}H_{2n} - \tfrac{1}{2}H_n \\ &= \ln 4n + \gamma + o_{4n} - \tfrac{1}{2}(\ln 2n + \gamma + o_{2n} + \ln n + \gamma + o_n) \\ &= \ln 4 - \tfrac{1}{2}\ln 2 + o'_n \\ &= \tfrac{3}{2}\ln 2 + o'_n, \end{aligned}$$

where $\lim o'_n = 0$. Since the difference between S_n and S_{3m} where $m = [n/3]$ is infinitesimally small, this proves (2.4.1).

Stirling's Formula. We will try to estimate $\ln n!$. Integration of the inequalities $\ln[x] \leq \ln x \leq \ln[x+1]$ over $[1,n]$ gives $\ln(n-1)! \leq \int_1^n \ln x\,dx \leq \ln n!$. Let us estimate the difference D between $\int_1^n \ln x\,dx$ and $\frac{1}{2}(\ln n! + \ln(n-1)!)$.

$$\begin{aligned} D &= \int_1^n (\ln x - \tfrac{1}{2}(\ln[x] + \ln[x+1]))\,dx \\ &= \sum_{k=1}^{n-1}\int_0^1 \left(\ln(k+x) - \ln\sqrt{k(k+1)}\right)\,dx. \end{aligned} \tag{2.4.2}$$

To prove that all summands on the left-hand side are nonnegative, we apply the following general lemma.

LEMMA 2.4.7. *$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ for any function.*

PROOF. The reflection of the plane across the line $y = \frac{1}{2}$ transforms the curvilinear trapezium of $f(x)$ over $[0,1]$ into curvilinear trapezium of $f(1-x)$ over $[0,1]$. □

LEMMA 2.4.8. $\int_0^1 \ln(k+x)\,dx \geq \ln\sqrt{k(k+1)}$.

PROOF. Due to Lemma 2.4.7 one has

$$\begin{aligned}\int_0^1 \ln(k+x)\,dx &= \int_0^1 \ln(k+1-x)\,dx \\ &= \int_0^1 \tfrac{1}{2}(\ln(k+x)+\ln(k+1-x))\,dx \\ &= \int_0^1 \ln\sqrt{(k+x)(k+1-x)}\,dx \\ &= \int_0^1 \ln\sqrt{k(k+1)+x-x^2}\,dx \\ &\geq \int_0^1 \ln\sqrt{k(k+1)}\,dx \\ &= \ln\sqrt{k(k+1)}.\end{aligned}$$

□

Integration of the inequality $\ln(1+x/k) \leq x/k$ over $[0,1]$ gives

$$\int_0^1 \ln(1+x/k)\,dx \leq \int_0^1 \frac{x}{k}\,dx = \frac{1}{2k}.$$

This estimate together with the inequality $\ln(1+1/k) \geq 1/(k+1)$ allows us to estimate the summands from the right-hand side of (2.4.2) in the following way:

$$\begin{aligned}\int_0^1 \ln(k+x) - \ln\sqrt{k(k+1)}\,dx &= \int_0^1 \ln(k+x) - \ln k - \tfrac{1}{2}(\ln(k+1)-\ln k)\,dx \\ &= \int_0^1 \ln\left(1+\tfrac{x}{k}\right) - \tfrac{1}{2}\ln\left(1+\tfrac{1}{k}\right)\,dx \\ &\leq \tfrac{1}{2k} - \tfrac{1}{2(k+1)}.\end{aligned}$$

We see that $D_n \leq \sum_{k=1}^{\infty} \frac{1}{2k} - \frac{1}{2(k+1)} = \frac{1}{2}$ for all n. Denote by D_∞ the sum (2.4.2) for infinite n. Then $R_n = D_\infty - D_n = \frac{\theta}{2n}$ for some nonnegative $\theta < 1$, and we get

$$\begin{aligned}D_\infty - \tfrac{\theta}{2n} &= \int_1^n \ln x\,dx - \tfrac{1}{2}\left(\ln n! + \ln(n-1)!\right) \\ &= \int_1^n \ln x\,dx - \ln n! + \tfrac{1}{2}\ln n.\end{aligned} \tag{2.4.3}$$

Substituting in (2.4.3) the value of the integral $\int_1^n \ln x\,dx = \int_1^n d(x\ln x - x) = (n\ln n - n) - (1\ln 1 - 1) = n\ln n - n + 1$, one gets

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + (1-D_\infty) + \tfrac{\theta}{2n}.$$

Now we know that $1 \geq (1-D_\infty) \geq \frac{1}{2}$, but it is possible to evaluate the value of D_∞ with more accuracy. Later we will prove that $1 - D_\infty = \sqrt{2\pi}$.

Problems.

1. Does $\sum_{k=1}^{\infty} \sin k$ converge?
2. Does $\sum_{k=1}^{\infty} \sin k^2$ converge?
3. Evaluate $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \dots - \frac{2}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} - \dots$.
4. Prove: If $\lim \frac{a_{n+1}}{a_n} < 1$, then $\sum_{k=1}^{\infty} a_k$ converge.
5. Prove: If $\sum_{k=1}^{\infty} |a_k - a_{k-1}| < \infty$, then $\{a_k\}$ converges.
6. Prove the convergence of $\sum_{k=1}^{\infty} \dfrac{(-1)^{[\sqrt{k}]}}{k}$.
7. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{\ln^3 k}$.
8. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k \sqrt{\ln \ln k}}$.
9. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k (\ln \ln k)^2}$.
10. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ and find its asymptotic formula.
11. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$.
12. Which partial sum of the above series is 0.01 close to its ultimate sum?
13. Evaluate $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$ with precision 0.01.
14. Evaluate $\int_1^3 \ln x \, d[x]$.
15. Express the Stirling constant via the Wallis product $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1}$.

2.5. Quadrature of Circle

On the contents of the lecture. We extend the concept of the integral to complex functions. We evaluate a very important integral $\oint \frac{1}{z}dz$ by applying Archimedes' theorem on the area of circular sector. As a consequence, we evaluate the Wallis product and the Stirling constant.

Definition of a complex integral. To specify an integral of a complex function one has to indicate not only its limits, but also the *path of integration*. A path of integration is a mapping $p\colon [a,b] \to \mathbb{C}$, of an interval $[a,b]$ of the real line into complex plane. The integral of a complex differential form fdg (here f and g are complex functions of complex variable) along the path p is defined via separate integration of different combinations of real and imaginary parts in the following way:

$$\int_a^b \operatorname{Re} f(p(t))\, d\operatorname{Re} g(p(t)) - \int_a^b \operatorname{Im} f(p(t))\, d\operatorname{Im} g(p(t)) \\ + i\int_a^b \operatorname{Re} f(p(t))\, d\operatorname{Im} g(p(t)) + i\int_a^b \operatorname{Im} f(p(t))\, d\operatorname{Re} g(p(t))$$

Two complex differential forms are called equal if their integrals coincide for all paths. So, the definition above can be written shortly as $fdg = \operatorname{Re} f d\operatorname{Re} g - \operatorname{Im} f d\operatorname{Im} g + i\operatorname{Re} f d\operatorname{Im} g + i\operatorname{Im} f d\operatorname{Re} g$.

The integral $\int \frac{1}{z}dz$. The Integral is the principal concept of Calculus and $\int \frac{1}{z}dz$ is the principal integral. Let us evaluate it along the path $p(t) = \cos t + i\sin t$, $t \in [0,\phi]$, which goes along the arc of the circle of the length $\phi \leq \pi/2$. Since $\frac{1}{\cos t + i\sin t} = \cos t - i\sin t$, one has

$$\int_p \frac{1}{z}\, dz = \int_0^\phi \cos t\, d\cos t + \int_0^\phi \sin t\, d\sin t \\ - i\int_0^\phi \sin t\, d\cos t + i\int_0^\phi \cos t\, d\sin t. \tag{2.5.1}$$

Its real part transforms into $\int_0^\phi \frac{1}{2}\, d\cos^2 t + \int_0^\phi \frac{1}{2}\, d\sin^2 t = \int_0^\phi \frac{1}{2}\, d(\cos^2 t + \sin^2 t) = \int_0^\phi \frac{1}{2}\, d1 = 0$. An attentive reader has to object: integrals were defined only for differential forms with non-decreasing differands, while $\cos t$ decreases.

Sign rule. Let us define the integral for any differential form fdg with any continuous monotone differand g and any integrand f of a constant sign (i.e, non-positive or non-negative). The definition relies on the following *Sign Rule.*

$$\int_a^b -f\, dg = -\int_a^b f\, dg = \int_a^b f\, d(-g) \tag{2.5.2}$$

If f is of constant sign, and g is monotone, then among the forms fdg, $-fdg$, $fd(-g)$ and $-fd(-g)$ there is just one with non-negative integrand and non-decreasing differand. For this form, the integral was defined earlier, for the other cases it is defined by the Sign Rule.

Thus the integral of a negative function against an increasing differand and the integral of a positive function against a decreasing differand are negative. And the integral of a negative function against a decreasing differand is positive.

The Sign Rule agrees with the Constant Rule: the formula $\int_a^b c\,dg = c(g(b) - g(a))$ remains true either for negative c or decreasing g.

The Partition Rule also is not affected by this extension of the integral.

The Inequality Rule takes the following form: if $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$ then $\int_a^b f_1(x)\,dg(x) \leq \int_a^b f_2(x)\,dg(x)$ for non-decreasing g and $\int_a^b f_1(x)\,dg(x) \geq \int_a^b f_2(x)\,dg(x)$ for non-increasing g.

Change of variable. Now all integrals in (2.5.1) are defined. The next objection concerns transformation $\cos t d \cos t = \frac{1}{2} d\cos^2 t$. This transformation is based on a decreasing change of variable $x = \cos t$ in $dx^2/2 = xdx$. But what happens with an integral when one applies a decreasing change of variable? The curvilinear trapezium, which represents the integral, does not change at all under any change of variable, even for a non-monotone one. Hence the only thing that may happen is a change of sign. And the sign changes by the Sign Rule, simultaneously on both sides of equality $dx^2/2 = xdx$. If the integrals of xdx and dx^2 are positive, both integrals of $\cos t d\cos t$ and $\cos^2 t$ are negative and have the same absolute value. These arguments work in the general case:

A decreasing change of variable reverses the sign of the integral.

Addition Formula. The next question concerns the legitimacy of addition of differentials, which appeared in the calculation $d\cos^2 t + d\sin^2 t = d(\cos^2 t + \sin^2 t) = 0$, where differands are not *comonotone*: $\cos t$ decreases, while $\sin t$ increases. The addition formula in its full generality will be proved in the next lecture, but this special case is not difficult to prove. Our equality is equivalent to $d\sin^2 t = -d\cos^2 t$. By the Sign Rule $-d\cos^2 t = d(-\cos^2 t)$, but $-\cos^2 t$ is increasing. And by the Addition Theorem $d(-\cos^2 t + 1) = d(-\cos^2 t) + d1 = d(-\cos^2 t)$. But $-\cos^2 t + 1 = \sin^2 t$. Hence our evaluation of the real part of (2.5.1) is justified.

Trigonometric integrals. We proceed to the evaluation of the imaginary part of (2.5.1), which is $\cos t\, d\sin t - \sin t\, d\cos t$. This is a simple geometric problem.

The integral of $\sin t\, d\cos t$ is negative as $\cos t$ is decreasing on $[0, \frac{\pi}{2}]$, and its absolute value is equal to the area of the curvilinear triangle $A'BA$, which is obtained from the circular sector OBA with area $\phi/2$ by deletion of the triangle $OA'B$, which has area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \sin t\, d\cos t$ is $\phi/2 - \frac{1}{2}\cos\phi\sin\phi$.

The integral of $\cos t\, d\sin t$ is equal to the area of curvilinear trapezium $OB'BA$. The latter consists of a circular sector OBA with area $\phi/2$ and a triangle $OB'B$ with area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \cos t\, d\sin t = \phi/2 + \frac{1}{2}\cos\phi\sin\phi$.

As a result we get $\int_p \frac{1}{z}\,dz = i\phi$. This result has a lot of consequences. But today we restrict our attention to the integrals of $\sin t$ and $\cos t$.

Multiplication of differentials. We have proved

$$\cos t\, d\sin t - \sin t\, d\cos t = dt. \tag{2.5.3}$$

Multiplying this equality by $\cos t$, one gets

$$\cos^2 t\, d\sin t - \sin t\cos t\, d\cos t = \cos t\, dt.$$

Replacing $\cos^2 t$ by $(1 - \sin^2 t)$ and moving $\cos t$ into the differential, one transforms the left-hand side as

$$d\sin t - \sin^2 t\, d\sin t - \tfrac{1}{2}\sin t\, d\cos^2 t = d\sin t - \tfrac{1}{2}\sin t\, d\sin^2 t - \tfrac{1}{2}\sin t\, d\cos^2 t.$$

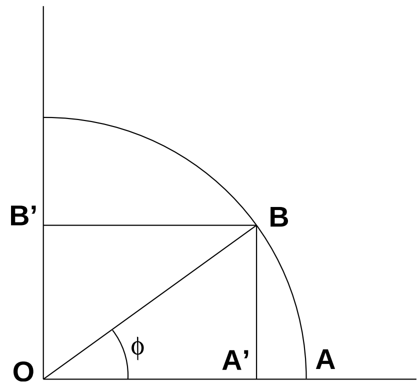

FIGURE 2.5.1. Trigonometric integrals

We already know that $d\sin^2 t + d\cos^2 t$ is zero. Now we have to prove the same for the product of this form by $\frac{1}{2}\sin t$. The arguments are the same: we multiply by $\frac{1}{2}\sin t$ the equivalent equality $d\sin^2 t = d(-\cos^2 t)$ whose differands are increasing. This is a general way to extend the theorem on multiplication of differentials to the case of any monotone functions. We will do it later. Now we get just $d\sin t = \cos t\, dt$.

Further, multiplication of the left-hand side of (2.5.3) by $\sin t$ gives

$$\sin t\cos t\, d\sin t - \sin^2 t\, d\cos t = \tfrac{1}{2}\cos t\, d\sin^2 t - d\cos t + \tfrac{1}{2}\cos t\, d\cos^2 t = -d\cos t.$$

So we get $d\cos t = -\sin t dt$.

THEOREM 2.5.1. *$d\sin t = \cos t\, dt$ and $d\cos t = -\sin t\, dt$.*

We have proved this equality only for $[0, \pi/2]$. But due to well-known symmetries this suffices.

Application of trigonometric integrals.

LEMMA 2.5.2. *For any convergent infinite product of factors ≥ 1 one has*

$$\lim \prod_{k=1}^{n} p_k = \prod_{k=1}^{\infty} p_k. \tag{2.5.4}$$

PROOF. Let ε be a positive number. Then $\prod_{k=1}^{\infty} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$, and by All-for-One there is n such that $\prod_{k=1}^{n} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Then for any $m > n$ one has the inequalities $\prod_{k=1}^{\infty} p_k \geq \prod_{k=1}^{m} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Therefore $|\prod_{k=1}^{m} p_k - \prod_{k=1}^{\infty} p_k| < \varepsilon$. □

Wallis product. Set $I_n = \int_0^\pi \sin^n x\, dx$. Then $I_0 = \int_0^\pi 1\, dx = \pi$ and $I_1 = \int_0^\pi \sin x\, dx = -\cos\pi + \cos 0 = 2$. For $n \geq 2$, let us replace the integrand $\sin^n x$ by

$\sin^{n-2} x(1-\cos^2 x)$ and obtain

$$\begin{aligned} I_n &= \int_0^\pi \sin^{n-2} x(1-\cos^2 x)\,dx \\ &= \int_0^\pi \sin^{n-2} x\,dx - \int_0^\pi \sin^{n-2} x\cos x\,d\sin x \\ &= I_{n-2} - \tfrac{1}{n-1}\int_0^\pi \cos x\,d\sin^{n-1}(x) \\ &= I_{n-2} - \int_0^\pi d(\cos x\sin^{n-1} x) + \int_0^\pi \sin^{n-1} x\,d\cos x \\ &= I_{n-2} - \tfrac{1}{n-1} I_n. \end{aligned}$$

We get the recurrence relation $I_n = \frac{n-1}{n} I_{n-2}$, which gives the formula

$$I_{2n} = \pi\frac{(2n-1)!!}{2n!!}, \quad I_{2n-1} = 2\frac{(2n-2)!!}{(2n-1)!!} \tag{2.5.5}$$

where $n!!$ denotes the product $n(n-2)(n-4)\cdots(n \mod 2+1)$. Since $\sin^n x \leq \sin^{n-1} x$ for all $x \in [0,\pi]$, the sequence $\{I_n\}$ decreases. Since $I_n \leq I_{n-1} \leq I_{n-2}$, one gets $\frac{n-1}{n} = \frac{I_n}{I_{n-2}} \leq \frac{I_{n-1}}{I_{n-2}} \leq 1$. Hence $\frac{I_{n-1}}{I_{n-2}}$ differs from 1 less than $\frac{1}{n}$. Consequently, $\lim \frac{I_{n-1}}{I_{n-2}} = 1$. In particular, $\lim \frac{I_{2n+1}}{I_{2n}} = 1$. Substituting in this last formula the expressions of I_n from (2.5.5) one gets

$$\lim \frac{\pi}{2}\frac{(2n+1)!!(2n-1)!!}{2n!!2n!!} = 1.$$

Therefore this is the famous Wallis Product

$$\frac{\pi}{2} = \lim \frac{2n!!2n!!}{(2n-1)!!(2n+1)!!} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}.$$

Stirling constant. In Lecture 2.4 we have proved that

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n, \tag{2.5.6}$$

where o_n is infinitesimally small and σ is a constant. Now we are ready to determine this constant. Consider the difference $\ln 2n! - 2\ln n!$. By (2.5.6) it expands into

$$\begin{aligned} (2n\ln 2n - 2n + \tfrac{1}{2}\ln 2n + \sigma + o_{2n}) - 2(n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n) \\ = 2n\ln 2 + \tfrac{1}{2}\ln 2n - \ln n - \sigma + o'_n, \end{aligned}$$

where $o'_n = o_{2n} - 2o_n$ is infinitesimally small. Then σ can be presented as

$$\sigma = 2\ln n! - \ln 2n! + 2n\ln 2 + \tfrac{1}{2}\ln n + \tfrac{1}{2}\ln 2 - \ln n + o'_n.$$

Multiplying by 2 one gets

$$2\sigma = 4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2 + 2o'_n.$$

Hence $2\sigma = \lim(4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2)$. Switching to product and keeping in mind the identities $n! = n!!(n-1)!!$ and $n!2^n = 2n!!$ one gets

$$\sigma^2 = \lim \frac{n!^4 2^{4n+1}}{(2n!)^2 n} = \lim \frac{2\cdot(2n!!)^4}{(2n!!)^2(2n-1)!!^2 n} \lim \frac{2\cdot(2n!!)^2(2n+1)}{(2n-1)!!(2n+1)!!n} = 2\pi.$$

Problems.

1. Evaluate $\int \sqrt{1-x^2}\,dx$.
2. Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx$.
3. Evaluate $\int \sqrt{5-x^2}\,dx$.
4. Evaluate $\int \cos^2 x\,dx$.
5. Evaluate $\int \tan x\,dx$.
6. Evaluate $\int \sin^4 x\,dx$.
7. Evaluate $\int \sin x^2\,dx$.
8. Evaluate $\int \tan x\,dx$.
9. Evaluate $\int x^2 \sin x\,dx$.
10. Evaluate $d \arcsin x$.
11. Evaluate $\int \arcsin x\,dx$.
12. Evaluate $\int e^x \cos x\,dx$.

2.6. Virtually monotone functions

Monotonization of the integrand. Let us say that a pair of functions f_1, f_2 *monotonize* a function f, if f_1 is non-negative and non-decreasing, f_2 is non-positive and non-increasing and $f = f_1 + f_2$.

LEMMA 2.6.1. *Let $f = f_1 + f_2$ and $f = f_1' + f_2'$ be two monotonizations of f. Then for any monotone h one has $f_1dh + f_2dh = f_1dh + f_2'dh$.*

PROOF. Our equality is equivalent to $f_1dh - f_2'dh = f_1'dh - f_2dh$. By the sign rule this turns into $f_1dh + (-f_2')dh = f_1'dh + (-f_2)dh$. Now all integrands are nonnegative and for non-decreasing h we can apply the Addition Theorem and transform the inequality into $(f_1 - f_2')dh = (f_1' - f_2)dh$. This is true because $(f_1 - f_2') = (f_1' - f_2)$.

The case of a non-increasing differand is reduced to the case of a non-decreasing one by the transformation $f_1d(-h)+f_2d(-h) = f_1'd(-h)+f_2'd(-h)$, which is based on the Sign Rule. □

A function which has a monotonization is called *virtually monotone*.

We define the integral $\int_a^b f\,dg$ for any virtually monotone integrand f and any continuous monotone differand g via a monotonization $f = f_1 + f_2$ by

$$\int_a^b f\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$$

Lemma 2.6.1 demonstrates that this definition does not depend on the choice of a monotonization.

LEMMA 2.6.2. *Let f and g be virtually monotone functions; then $f + g$ is virtually monotone and $fdh + gdh = (f + g)dh$ for any continuous monotone h.*

PROOF. Let h be nondecreasing. Consider monotonizations $f = f_1 + f_2$ and $g = g_1 + g_2$. Then $fdh + gdh = f_1dh + f_2dh + g_1dh + g_2dh$ by definition via monotonization of the integrand. By virtue of the Addition Theorem 2.3.3 this turns into $(f_1 + g_1)dh + (f_2 + g_2)dh$. But the pair of brackets monotonize $f + g$. Hence $f+g$ is proved to be virtually monotone and the latter expression is $(f+g)dh$ by definition, via monotonization of the integrand. The case of non-increasing h is reduced to the previous case via $-fd(-h) - gd(-h) = -(f + g)d(-h)$. □

Lemma on locally constant functions. Let us say that a function $f(x)$ is *locally constant* at a point x if $f(y) = f(x)$ for all y sufficiently close to x, i.e., for all y from an interval $(x - \varepsilon, x + \varepsilon)$.

LEMMA 2.6.3. *A function f which is locally constant at each point of an interval is constant.*

PROOF. Suppose $f(x)$ is not constant on $[a, b]$. We will construct by induction a sequence of intervals $I_k = [a_k, b_k]$, such that $I_0 = [a, b]$, $I_{k+1} \subset I_k$, $|b_k - a_k| \geq 2|b_{k+1} - a_{k+1}|$ and the function f is not constant on each I_k. First step: Let $c = (a + b)/2$, as f is not constant $f(x) \neq f(c)$ for some x. Then choose $[x, c]$ or $[c, x]$ as for $[a_1, b_1]$. On this interval f is not constant. The same are all further steps. The intersection of the sequence is a point such that any of its neighborhoods contains some interval of the sequence. Hence f is not locally constant at this point. □

LEMMA 2.6.4. *If $f(x)$ is a continuous monotone function and $a < f(x) < b$ then $a < f(y) < b$ for all y sufficiently close to x.*

PROOF. If f takes values greater than b, then it takes value b and if $f(x)$ takes values less than a then it takes value a due to continuity. Then $[f^{-1}(a), f^{-1}(b)]$ is the interval where inequalities hold. □

LEMMA 2.6.5. *Let g_1, g_2 be continuous* comonotone *functions. Then $g_1 + g_2$ is continuous and monotone, and for any virtually monotone f one has*

$$fdg_1 + fdg_2 = fd(g_1 + g_2). \tag{2.6.1}$$

PROOF. Suppose $g_1(x) + g_2(x) < p$, let $\varepsilon = p - g_1(x) - g_2(x)$. Then $g_1(y) < g_1(y) + \varepsilon/2$ and $g_2(y) < g_2(y) + \varepsilon/2$ for all y sufficiently close to x. Hence $g_(y) + g_2(y) < p$ for all y sufficiently close to x. The same is true for the opposite inequality. Hence $\operatorname{sgn}(g_1(x)+g_2(x)-p)$ is locally constant at all points where it is not 0. But it is not constant if p is an intermediate value, hence it is not locally constant, hence it takes value 0. At this point $g_1(x) + g_2(x) = p$ and the continuity of $g_1 + g_2$ is proved.

Consider a monotonization $f = f_1 + f_2$. Let g_i be nondecreasing. By definition via monotonization of the integrand, the left-hand side of (2.6.1) turns into $(f_1dg_1 + f_2dg_1) + (f_1dg_2 + f_2dg_2) = (f_1dg_1 + f_1dg_2) + (f_2dg_1 + f_2dg_2)$. By the Addition Theorem 2.3.3 $f_1dg_1 + f_1dg_2 = f_1d(g_1 + g_2)$. And the equality $f_2dg_1 + f_2dg_2 = f_2d(g_1 + g_2)$ follows from $(-f_2)dg_1 + (-f_2)dg_2 = (-f_2)d(g_1 + g_2)$ by the Sign Rule. Hence the left-hand side is equal to $f_1d(g_1 + g_2) + f_2d(g_1 + g_2)$, which coincides with the right-hand side of (2.6.1) by definition via monotonization of integrand. The case of non-increasing differands is taken care of via transformation of (2.6.1) by the Sign Rule into $fd(-g_1) + fd(-g_2) = fd(-g_1 - g_2)$. □

LEMMA 2.6.6. *Let $g_1 + g_2 = g_3 + g_4$ where all $(-1)^k g_k$ are non-increasing continuous functions. Then $fdg_1 + fdg_2 = fdg_3 + fdg_4$ for any virtually monotone f.*

PROOF. Our equality is equivalent to $fdg_1 - fdg_4 = fdg_3 - fdg_2$. By the Sign Rule it turns into $fdg_1 + fd(-g_4) = fdg_3 + fd(-g_2)$. Now all differands are nondecreasing and by Lemma 2.6.5 it transforms into $fd(g_1 - g_4) = fd(g_3 - g_2)$. This is true because $g_1 - g_4 = g_3 - g_2$. □

Monotonization of the differand. A monotonization by continuous functions is called continuous. A virtually monotone function which has a continuous monotonization is called *continuous*. The integral for any virtually monotone integrand f against a virtually monotone continuous differand g is defined via a continuous virtualization $g = g_1 + g_2$ of the differand

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

The integral is well-defined because of Lemma 2.6.6.

THEOREM 2.6.7 (Addition Theorem). *For any virtually monotone functions f, f' and any virtually monotone continuous g, g', $fdg + f'dg = (f + f')dg$ and $fdg + fdg' = fd(g + g')$*

PROOF. To prove $fdg + f'dg = (f + f')dg$, consider a continuous monotonization $g = g_1 + g_2$. Then by definition of the integral for virtually monotone differands this equality turns into $(fdg_1 + fdg_2) + (f'dg_1 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. After rearranging it turns into $(fdg_1 + f'dg_1) + (fdg_2 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. But this is true due to Lemma 2.6.2.

To prove $fdg + fdg' = fd(g + g')$, consider monotonizations $g = g_1 + g_2$, $g' = g'_1 + g'_2$. Then $(g_1 + g'_1) + (g_2 + g'_2)$ is a monotonization for $g + g'$. And by the definition of the integral for virtually monotone differands our equality turns into $fdg_1 + fdg_2 + fdg'_1 + fdg'_2$ □

Change of variable.

LEMMA 2.6.8. *If f is virtually monotone and g is monotone, then $f(g(x))$ is virtually monotone.*

PROOF. Let $f_1 + f_2$ be a monotonization of f. If h is non-decreasing then $f_1(h(x)) + f_2(h(x))$ gives a monotonization of $f(g(x))$. If h is decreasing then the monotonization is given by $(f_2(h(x)) + c) + (f_1(h(x)) - c)$ where c is a sufficiently large constant to provide positivity of the first brackets and negativity of the second one. □

The following natural convention is applied to define an integral with reversed limits: $\int_a^b f(x)\,dg(x) = -\int_b^a f(x)\,dg(x)$.

THEOREM 2.6.9 (on change of variable). *If $h\colon [a, b] \to [h(a), h(b)]$ is monotone, $f(x)$ is virtually monotone and $g(x)$ is virtually monotone continuous then*

$$\int_a^b f(h(t))\,dg(h(t)) = \int_{h(a)}^{h(b)} f(x)\,dg(x).$$

PROOF. Let $f = f_1 + f_2$ and $g = g_1 + g_2$ be a monotonization and a continuous monotonization of f and g respectively. The $\int_a^b f(h(t))\,dg(h(t))$ splits into sum of four integrals: $\int_a^b f_i(h(t))\,dg_j(h(t))$ where f_i are of constant sign and g_j are monotone continuous. These integrals coincide with the corresponding integrals $\int_{h(a)}^{h(b)} f_i(x)\,dg_i(x)$. Indeed their absolute values are the areas of the same curvilinear trapezia. And their signs determined by the Sign Rule are the same. □

Integration by parts. We have established the Integration by Parts formula for non-negative and non-decreasing differential forms. Now we extend it to the case of continuous monotone forms. In the first case f and g are non-decreasing. In this case choose a positive constant c sufficiently large to provide positivity of $f + c$ and $g + c$ on the interval of integration. Then $d(f + c)(g + c) = (f + c)d(g + c) + (g + c)d(f + c)$. On the other hand $d(f + c)(g + c) = dfg + cdf + cdg$ and $(f + c)d(g + c) + (g + c)d(f + c) = fdg + cdg + cdf$. Compare these results to get $dfg = fdg + gdf$. Now if f is increasing and g is decreasing then $-g$ is increasing and we get $-dfg = df(-g) = fd(-g) + (-g)df = -fdg - gdf$, which leads to $dfg = fdg + gdf$. The other cases: f decreasing, g increasing and both decreasing are proved by the same arguments. The extension of the Integration by Parts formula to piecewise monotone forms immediately follows by the Partition Rule.

Variation. Define the *variation of a sequence* of numbers $\{x_k\}_{k=1}^n$ as the sum $\sum_{k=1}^{\infty} |x_{k+1} - x_k|$. Define the variation of a function f along a sequence $\{x_k\}_{k=0}^n$

as the variation of sequence $\{f(x_k)\}_{k=0}^n$. Define a *chain* on an interval $[a,b]$ as a nondecreasing sequence $\{x_k\}_{k=0}^n$ such that $x_0 = a$ and $x_n = b$. Define the *partial variation* of f on an interval $[a,b]$ as its variation along a chain on the interval.

The least number surpassing all partial variations function f over $[a,b]$ is called *the (ultimate) variation of a function* $f(x)$ on an interval $[a,b]$ and is denoted by $\operatorname{var}_f[a,b]$.

LEMMA 2.6.10. *For any function f one has the inequality* $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$. *If f is a monotone function on $[a,b]$, then* $\operatorname{var}_f[a,b] = |f(b) - f(a)|$.

PROOF. The inequality $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$ follows immediately from the definition because $\{a,b\}$ is a chain. For monotone f, all partial variations are telescopic sums equal to $|f(b) - f(a)|$ □

THEOREM 2.6.11 (additivity of variation). $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] = \operatorname{var}_f[a,c]$.

PROOF. Consider a chain $\{x_k\}_{k=0}^n$ of $[a,c]$, which contains b. In this case the variation of f along $\{x_k\}_{k=0}^n$ splits into sums of partial variations of f along $[a,b]$ and along $[b,c]$. As a partial variations does not exceed an ultimate, we get that in this case the variation of f along $\{x_k\}_{k=0}^n$ does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$.

If $\{x_k\}_{k=0}^n$ does not contain b, let us add b to the chain. Then in the sum expressing the partial variation of f, the summand $|f(x_{i+1}) - f(x_i)|$ changes by the sum $|f(b) - f(x_i)| + |f(x_{i+1} - f(b)|$ which is greater or equal. Hence the variation does not decrease after such modification. But the variation along the modified chain does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$ as was proved above. As all partial variations of f over $[a,c]$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$, the same is true for the ultimate variation.

To prove the opposite inequality we consider a *relaxed* inequality $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] \leq \operatorname{var}_f[a,c] + \varepsilon$ where ε is an positive number. Choose chains $\{x_k\}_{k=0}^n$ on $[a,b]$ and $\{y_k\}_{k=0}^m$ on $[b,c]$ such that corresponding partial variations of f are $\geq \operatorname{var}_f[a,b] + \varepsilon/2$ and $\geq \operatorname{var}_f[b,c] + \varepsilon/2$ respectively. As the union of these chains is a chain on $[a,c]$ the sum of these partial variations is a partial variation of f on $[a,c]$. Consequently this sum is less or equal to $\operatorname{var}_f[a,c]$. On the other hand it is greater or equal to $\operatorname{var}_f[a,b] + \varepsilon/2 + \operatorname{var}_f[b,c] + \varepsilon/2$. Comparing these results gives just the relaxed inequality. As the relaxed inequality is proved for all $\varepsilon > 0$ it also holds for $\varepsilon = 0$. □

LEMMA 2.6.12. *For any functions f, g one has the inequality* $\operatorname{var}_{f+g}[a,b] \leq \operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$.

PROOF. Since $|f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \leq |f(x_{k+1}) - f(x_k)| + |g(x_{k+1}) - g(x_k)|$, the variation of $f + g$ along any sequence does not exceed the sum of the variations of f and g along the sequence. Hence all partial variations of $f + g$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$, and so the same is true for the ultimate variation. □

LEMMA 2.6.13. *For any function of finite variation on $[a,b]$, the functions* $\operatorname{var}_f[a,x]$ *and* $\operatorname{var}_f[a,x] - f(x)$ *are both nondecreasing functions of x.*

PROOF. That $\operatorname{var}_f[a,x]$ is nondecreasing follows from nonnegativity and additivity of variation. If $x > y$ then the inequality $\operatorname{var}_f[a,x] - f(x) \geq \operatorname{var}_f[a,y] - f(y)$

is equivalent to $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] \geq f(x)-f(y)$. This is true because $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] = \operatorname{var}_f[x,y] \geq |f(x)-f(y)|$. □

LEMMA 2.6.14. $\operatorname{var}_{f^2}[a,b] \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b]$.

PROOF. For all $x, y \in [a,b]$ one has

$$\begin{aligned} |f(x)+f(y)| &= |2f(a)+f(x)-f(a)+f(y)-f(a)| \\ &\leq 2|f(a)| + \operatorname{var}_f[a,x] + \operatorname{var}_f[a,y] \\ &\leq 2|f(a)| + 2\operatorname{var}_f[a,b]. \end{aligned}$$

Hence

$$\begin{aligned} \textstyle\sum_{k=0}^{n-1} |f^2(x_{k+1}) - f^2(x_k)| &= \textstyle\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)||f(x_{k+1}) + f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \textstyle\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b] \end{aligned}$$

□

LEMMA 2.6.15. *If* $\operatorname{var}_f[a,b] < \infty$ *and* $\operatorname{var}_g[a,b] < \infty$, *then* $\operatorname{var}_{fg}[a,b] < \infty$.

PROOF. $4fg = (f+g)^2 - (f-g)^2$. □

THEOREM 2.6.16. *The function f is virtually monotone on $[a,b]$ if and only if it has a finite variation.*

PROOF. Since monotone functions have finite variation on finite intervals, and the variation of a sum does not exceed the sum of variations, one gets that all virtually monotone functions have finite variation. On the other hand, if f has finite variation then $f = (\operatorname{var}_f[a,x] + c) + (f(x) - \operatorname{var}_f[a,x] - c)$, the functions in the brackets are monotone due to Lemma 2.6.13, and by choosing a constant c sufficiently large, one obtains that the second bracket is negative. □

Problems.

1. Evaluate $\int_1^i z^2\,dz$.
2. Prove that $1/f(x)$ has finite variation if it is bounded.
3. Prove $\int_a^b f(x)\,dg(x) \leq \max_{[a,b]} f \operatorname{var}_g[a,b]$.

CHAPTER 3

Derivatives

3.1. Newton-Leibniz Formula

On the contents of the lecture. In this lecture appears the celebrated Newton-Leibniz formula — the main tool in the evaluation of integrals. It is accompanied with such fundamental concepts as the derivative, the limit of a function and continuity.

Motivation. Consider the following problem: for a given function F find a function f such that $dF(x) = f(x)\,dx$, over $[a,b]$, that is, $\int_c^d f(t)\,dt = F(d) - F(c)$ for any subinterval $[c,d]$ of $[a,b]$.

Suppose that such an f exists. Since the value of f at a single point does not affects the integral, we cannot say anything about the value of f at any given point. But if f is continuous at a point x_0, its value is uniquely defined by F.

To be precise, the difference quotient $\frac{F(x)-F(x_0)}{x-x_0}$ tends to $f(x_0)$ as x tends to x_0. Indeed, $F(x) = F(x_0) + \int_{x_0}^x f(t)\,dt$. Furthermore, $\int_{x_0}^x f(t)\,dt = f(x_0)(x - x_0) + \int_{x_0}^x (f(t) - f(x_0))\,dt$. Also, $|\int_{x_0}^x (f(t) - f(x_0)\,dt| \le \operatorname{var}_f[x_0, x]|x - x_0|$. Consequently

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \operatorname{var}_f[x, x_0]. \tag{3.1.1}$$

However, $\operatorname{var}_f[x, x_0]$ can be made arbitrarily small by choosing x sufficiently close to x_0, since $\operatorname{var}_f x_0 = 0$.

Infinitesimally small functions. A set is called a *neighborhood* of a point x if it contains all points *sufficiently close* to x, that is, all points y such that $|y - x|$ is less then a positive number ε.

We will say that a function f is *locally bounded* (above) by a constant C at a point x, if $f(x) \le C$ for all y sufficiently close to x.

A function $o(x)$ is called *infinitesimally small* at x_0, if $|o(x)|$ is locally bounded at x_0 by any $\varepsilon > 0$.

LEMMA 3.1.1. *If the functions o and ω are infinitesimally small at x_0 then $o \pm \omega$ are infinitesimally small at x_0.*

PROOF. Let $\varepsilon > 0$. Let O_1 be a neighborhood of x_0 where $|o(x)| < \varepsilon/2$, and let O_2 be a neighborhood of x_0 where $|\omega(x)| < \varepsilon/2$. Then $O_1 \cap O_2$ is a neighborhood where both inequalities hold. Hence for all $x \in O_1 \cap O_2$ one has $|o(x) \pm \omega(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

LEMMA 3.1.2. *If $o(x)$ is infinitesimally small at x_0 and $f(x)$ is locally bounded at x_0, then $f(x)o(x)$ is infinitesimally small at x_0.*

PROOF. The neighborhood where $|f(x)o(x)|$ is bounded by a given $\varepsilon > 0$ can be constructed as the intersection of a neighborhood U, where $|f(x)|$ is bounded by a constant C, and a neighborhood V, where $|o(x)|$ is bounded by ε/C. □

DEFINITION. *One says that a function $f(x)$* tends to A *as x tends to x_0 and writes* $\lim_{x\to x_0} f(x) = A$, *if $f(x) = A + o(x)$ on the complement of x_0, where $o(x)$ is infinitesimally small at x_0.*

COROLLARY 3.1.3. *If both the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then the limit* $\lim_{x\to x_0}(f(x) + g(x))$ *also exists and* $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$.

PROOF. This follows immediately from Lemma 3.1.1. □

LEMMA 3.1.4. *If the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then also* $\lim_{x\to x_0} f(x)g(x)$ *exists and* $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x)\lim_{x\to x_0} g(x)$.

PROOF. If $f(x) = A+o(x)$ and $g(x) = B+\omega(x)$, then $f(x)g(x) = AB+A\omega(x)+Bo(x)+\omega(x)o(x)$, where $A\omega(x)$, $Bo(x)$ and $\omega(x)o(x)$ all are infinitesimally small at x_0 by Lemma 3.1.2, and their sum is infinitesimally small by Lemma 3.1.1. □

DEFINITION. *A function f is called* continuous *at x_0, if* $\lim_{x\to x_0} f(x) = f(x_0)$.

A function is said to be continuous (without mentioning a point), if it is continuous at all points under consideration.

The following lemma gives a lot of examples of continuous functions.

LEMMA 3.1.5. *If f is a monotone function on $[a,b]$ such that $f[a,b] = [f(a), f(b)]$ then f is continuous.*

PROOF. Suppose f is nondecreasing. Suppose a positive ε is given. For a given point x denote by $x^\varepsilon = f^{-1}(f(x)+\varepsilon)$ and $x_\varepsilon = f^{-1}(f(x)-\varepsilon)$. Then $[x_\varepsilon, x^\varepsilon]$ contains a neighborhood of x, and for any $y \in [x_\varepsilon, x^\varepsilon]$ one has $f(x)+\varepsilon = f(x_\varepsilon) \le f(y) \le f(x^\varepsilon) = f(x)+\varepsilon$. Hence the inequality $|f(y)-f(x)| < \varepsilon$ holds locally at x for any ε. □

The following theorem immediately follows from Corollary 3.1.3 and Lemma 3.1.4.

THEOREM 3.1.6. *If the functions f and g are continuous at x_0, then $f+g$ and fg are continuous at x_0.*

The following property of continuous functions is very important.

THEOREM 3.1.7. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g(f(x))$ is continuous at x_0.*

PROOF. Given $\varepsilon > 0$, we have to find a neighborhood U of x_0, such that $|g(f(x)) - g(f(x_0))| < \varepsilon$ for $x \in U$. As $\lim_{y\to f(x_0)} g(y) = g(f(x_0))$, there exists a neighborhood V of $f(x_0)$ such that $|g(y)-g(y_0)| < \varepsilon$ for $y \in V$. Thus it is sufficient to find a U such that $f(U) \subset V$. And we can do this. Indeed, by the definition of neighborhood there is $\delta > 0$ such that V contains $V_\delta = \{y \mid |y - f(x_0)| < \delta\}$. Since $\lim_{x\to x_0} f(x) = f(x_0)$, there is a neighborhood U of x_0 such that $|f(x)-f(x_0)| < \delta$ for all $x \in U$. Then $f(U) \subset V_\delta \subset V$. □

DEFINITION. *A function f is called* differentiable *at a point x_0 if the* difference quotient $\frac{f(x)-(f_0)}{x-x_0}$ *has a limit as x tends to x_0. This limit is called the* derivative *of the function F at the point x_0, and denoted* $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

Immediately from the definition one evaluates the derivative of linear function

$$(ax+b)' = a \tag{3.1.2}$$

The following lemma is a direct consequence of Lemma 3.1.3.

LEMMA 3.1.8. *If f and g are differentiable at x_0, then $f+g$ is differentiable at x_0 and* $(f+g)'(x_0) = f'(x_0)+g'(x_0)$.

Linearization. Let f be differentiable at x_0. Denote by $o(x)$ the difference $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0), \tag{3.1.3}$$

where $o(x)$ is infinitesimally small at x_0. We will call such a representation a *linearization* of $f(x)$.

LEMMA 3.1.9. *If f is differentiable at x_0, then it is continuous at x_0.*

PROOF. All summands but $f(x_0)$ on the right-hand side of (3.1.3) are infinitesimally small at x_0; hence $\lim_{x\to x_0} f(x) = f(x_0)$. □

LEMMA 3.1.10 (on uniqueness of linearization). *If $f(x) = a + b(x - x_0) + o(x)(x - x_0)$, where $\lim_{x\to x_0} o(x) = 0$, then f is differentiable at x_0 and $a = f(x_0)$, $b = f'(x_0)$.*

PROOF. The difference $f(x) - f(x_0)$ is infinitesimally small at x_0 because f is continuous at x_0 and the difference $f(x) - a = b(x - x_0) + o(x)(x - x_0)$ is infinitesimally small by the definition of linearization. Hence $f(x_0) - a$ is infinitesimally small. But it is constant, hence $f(x_0) - a = 0$. Thus we established $a = f(x_0)$.

The difference $\frac{f(x)-a}{x-x_0} - b = o(x)$ is infinitesimally small as well as $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. But $\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-a}{x-x_0}$. Therefore $b - f'(x_0)$ is infinitesimally small. That is $b = f'(x_0)$. □

LEMMA 3.1.11. *If f and g are differentiable at x_0, then fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$.*

PROOF. Consider lineariations $f(x_0)+f'(x_0)(x-x_0)+o(x)(x-x_0)$ and $g(x_0)+g'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Their product is $f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) + (f(x)\omega(x) + f(x_0)o(x))(x - x_0)$. This is the linearization of $f(x)g(x)$ at x_0, because $f\omega$ and go are infinitesimally small at x_0. □

THEOREM 3.1.12. *If f is differentiable at x_0, and g is differentiable at $f(x_0)$ then $g(f(x))$ is differentiable at x_0 and $(g(f(x_0)))' = g'(f(x_0))f'(x_0)$.*

PROOF. Denote $f(x_0)$ by y_0 and substitute into the linearization $g(y) = g(y_0)+g'(y_0)(y - y_0) + o(y)(y - y_0)$ another linearization $y = f(x_0) + f'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Since $y - y_0 = f'(x_0)(x - x_0) + \omega(x)(x - x_0)$, we get $g(y) = g(y_0) + g'(y_0)f'(x_0)(x - x_0) + g'(y_0)(x - x_0)\omega(x) + o(f(x))(x - x_0)$. Due to Lemma 3.1.10, it is sufficient to prove that $g'(y_0)\omega(x) + o(f(x))$ is infinitesimally small at x_0. The first summand is obviously infinitesimally small. To prove that the second one also is infinitesimally small, we remark that $o(f(x_0) = 0$ and $o(y)$ is continuous at $f(x_0)$ and that $f(x)$ is continuous at x_0 due to Lemma 3.1.9. Hence by Theorem 3.1.6 the composition is continuous at x_0 and infinitesimally small. □

THEOREM 3.1.13. *Let f be a virtually monotone function on $[a, b]$. Then $F(x) = \int_a^x f(t)\,dt$ is virtually monotone and continuous on $[a, b]$. It is differentiable at any point x_0 where f is continuous, and $F'(x_0) = f(x_0)$.*

PROOF. If f has a constant sign, then F is monotone. So, if $f = f_1 + f_2$ is a monotonization of f, then $\int_a^x f_1(x)\,dx + \int_a^x f_1(x)\,dx$ is a monotonization of $F(x)$. This proves that $F(x)$ is virtually monotone.

To prove continuity of $F(x)$ at x_0, fix a constant C which bounds f in some neighborhood U of x_0. Then for $x \in U$ one proves that $|F(x) - F(x_0)|$ is infinitesimally small via the inequalities $|F(x) - F(x_0)| = |\int_{x_0}^{x} f(x)\,dx| \leq |\int_{x_0}^{x} C\,dx| = C|x - x_0|$.

Now suppose f is continuous at x_0. Then $o(x) = f(x_0) - f(x)$ is infinitesimally small at x_0. Therefore $\lim_{x\to x_0} \frac{1}{x-x_0}\int_{x_0}^{x} o(x)\,dx = 0$. Indeed for any $\varepsilon > 0$ the inequality $|o(x)| \leq \varepsilon$ holds over $[x_\varepsilon, x_0]$ for some x_ε. Hence $|\int_{x_0}^{x} o(x)\,dx| \leq |\int_{x_0}^{x} \varepsilon\,dx| = \varepsilon|x - x_0|$ for any $x \in [x_0, x_\varepsilon]$.

Then $F(x) = F(x_0) + f(x_0)(x-x_0) + (\frac{1}{x-x_0}\int_{x_0}^{x} o(t)\,dt)(x-x_0)$ is a linearization of $F(x)$ at x_0. □

COROLLARY 3.1.14. *The functions* ln, sin, cos *are differentiable and* $\ln'(x) = \frac{1}{x}$, $\sin' = \cos$, $\cos' = -\sin$.

PROOF. Since $d\sin x = \cos x\,dx$, $d\cos x = -\sin x\,dx$, due to Theorem 3.1.13 both $\sin x$ and $\cos x$ are continuous, and, as they are continuous, the result follows from Theorem 3.1.13. And $\ln' x = \frac{1}{x}$, by Theorem 3.1.13, follows from the continuity of $\frac{1}{x}$. The continuity follows from Lemma 3.1.5. □

Since $\sin'(0) = \cos 0 = 1$ and $\sin 0 = 0$, the linearization of $\sin x$ at 0 is $x + xo(x)$. This implies the following very important equality

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{3.1.4}$$

LEMMA 3.1.15. *If* $f'(x) > 0$ *for all* $x \in [a, b]$, *then* $f(b) > f(a)|$.

PROOF. Suppose $f(a) \geq f(b)$. We construct a sequence of intervals $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ such that their lengths tend to 0 and $f(a_k) \geq f(b_k)$. All steps of construction are the same. The general step is: let m be the middle point of $[a_k, b_k]$. If $f(m) \leq f(a_k)$ we set $[a_{k+1}, b_{k+1}] = [a_k, m]$, otherwise $f(m) > f(a_k) \geq f(b_k)$ and we set $[a_{k+1}, b_{k+1}] = [m, b_k]$.

Now consider a point x belonging to all $[a_k, b_k]$. Let $f(y) = f(x) + (f'(x) + o(x))(y - x)$ be the linearization of f at x. Let U be neighborhood where $|o(x)| < f'(x)$. Then $\operatorname{sgn}(f(y) - f(x)) = \operatorname{sgn}(y - x)$ for all $y \in U$. However for some n we get $[a_n, b_n] \subset U$. If $a_n \leq x < b_n$ we get $f(a_n) \leq f(x) < f(b_n)$ else $a_n < x$ and $f(a_n) < f(x) \leq f(b_n)$. In the both cases we get $f(a_n) < f(b_n)$. This is a contradiction with our construction of the sequence of intervals. □

THEOREM 3.1.16. *If* $f'(x) = 0$ *for all* $x \in [a, b]$, *then* $f(x)$ *is constant.*

PROOF. Set $k = \frac{f(b)-f(a)}{b-a}$. If $k < 0$ then $g(x) = f(x) - kx/2$ has derivative $g'(x) = f'(x) - k/2 > 0$ for all x. Hence by Lemma 3.1.15 $g(b) > g(a)$ and further $f(b) - f(a) > k(b - a)/2$. This contradicts the definition of k. If $k > 0$ then one gets the same contradiction considering $g(x) = -f(x) + kx/2$. □

THEOREM 3.1.17 (Newton-Leibniz). *If* $f'(x)$ *is a continuous virtually monotone function on an interval* $[a, b]$, *then* $\int_a^b f'(x)\,dx = f(b) - f(a)$.

PROOF. Due to Theorem 3.1.13, the derivative of the difference $\int_a^x f'(t)\,dt - f(x)$ is zero. Hence the difference is constant by Theorem 3.1.16. Substituting

$x = a$ we find the constant which is $f(a)$. Consequently, $\int_a^x f'(t)\,dt - f(x) = f(a)$ for all x. In particular, for $x = b$ we get the Newton-Leibniz formula. □

Problems.

1. Evaluate $(1/x)'$, $\sqrt{x}'$, $(\sqrt{\sin x^2})'$.
2. Evaluate $\exp' x$.
3. Evaluate $\operatorname{arctg}' x$, $\tan' x$.
4. Evaluate $|x|'$, $\operatorname{Re} z'$.
5. Prove: $f'(x) \equiv 1$ if and only if $f(x) = x + \text{const}$.
6. Evaluate $\left(\int_x^{x^2} \frac{\sin t}{t}\,dt\right)'$ as a function of x.
7. Evaluate $\sqrt{1 - x^2}'$.
8. Evaluate $(\int_0^1 \frac{\sin kt}{t}\,dt)'$ as a function of k.
9. Prove: If f is continuous at a and $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} f(x_n) = f(a)$.
10. Evaluate $\left(\int_0^y [x]\,dx\right)_y'$.
11. Evaluate $\arcsin' x$.
12. Evaluate $\int \frac{dx}{2+3x^2}$.
13. Prove: If $f'(x) < 0$ for all $x < m$ and $f'(x) > 0$ for all $x > m$ then $f'(m) = 0$.
14. Prove: If $f'(x)$ is bounded on $[a, b]$ then f is virtually monotone on $[a, b]$.

3.2. Exponential Functions

On the contents of the lecture. We solve the principal differential equation $y' = y$. Its solution, the exponential function, is expanded into a power series. We become acquainted with hyperbolic functions. And, finally, we prove the irrationality of e.

Debeaune's problem. In 1638 F. Debeaune posed Descartes the following geometrical problem: find a curve $y(x)$ such that for each point P the distances between V and T, the points where the vertical and the tangent lines cut the x-axis, are always equal to a given constant a. Despite the efforts of Descartes and Fermat, this problem remained unsolved for nearly 50 years. In 1684 Leibniz solved the problem via infinitesimal analysis of this curve: let x, y be a given point P (see the picture). Then increase x by a small increment of b, so that y increases almost by yb/a. Indeed, in small the curve is considered as the line. Hence the point P' of the curve with vertical projection V', one considers as lying on the line TP. Hence the triangle $TP'V'$ is similar to TPV. As $TV = a$, $TV' = b+a$ this similarity gives the equality $\frac{a+b}{y+\delta y} = \frac{a}{y}$ which gives $\delta y = yb/a$.

Repeating we obtain a sequence of values

$$y,\ y(1+\tfrac{b}{a}),\ y(1+\tfrac{b}{a})^2,\ y(1+\tfrac{b}{a})^3,\dots.$$

We see that "in small" $y(x)$ transforms an arithmetic progression into a geometric one. This is the inverse to what the logarithm does. And the solution is a function which is the inverse to a logarithmic function. Such functions are called *exponential.*

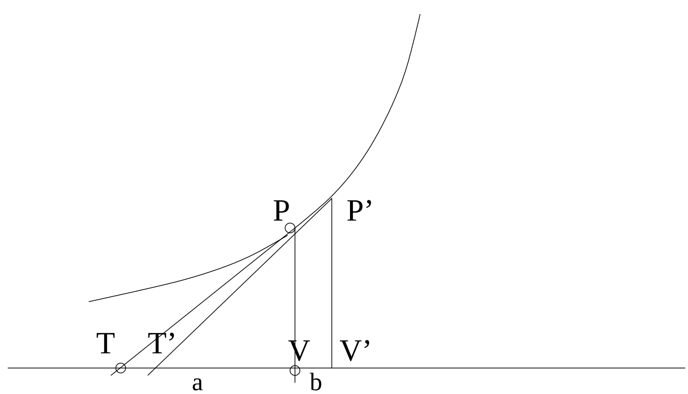

FIGURE 3.2.1. Debeaune's problem

Tangent line and derivative. A tangent line to a smooth convex curve at a point x is defined as a straight line such that the line intersects the curve just at x and the whole curve lies on one side of the line.

We state that the equation of the tangent line to the graph of function f at a point x_0 is just the principal part of linearization of $f(x)$ at x_0. In other words, the equation is $y = f(x_0) + (x - x_0)f'(x_0)$.

First, consider the case of a horizontal tangent line. In this case $f(x_0)$ is either maximal or minimal value of $f(x)$.

LEMMA 3.2.1. *If a function $f(x)$ is differentiable at an extremal point x_0, then $f'(x_0) = 0$.*

PROOF. Consider the linearization $f(x) = f(x_0)+f'(x_0)(x-x_0)+o(x))(x-x_0)$. Denote $x-x_0$ by δx, and $f(x)-f(x_0)$ by $\delta f(x)$. If we suppose that $f'(x_0) \neq 0$, then, for sufficiently small δx, we get $|o(x\pm\delta x)| < |f'(x)|$, hence $\operatorname{sgn}(f'(x_0)+o(x_0+\delta x)) = \operatorname{sgn}(f'(x_0)+o(x_0-\delta x))$, and $\operatorname{sgn}\delta f(x) = \operatorname{sgn}\delta x$. Therefore the sign of $\delta f(x)$ changes whenever the sign of δx changes. The sign of $\delta f(x)$ cannot be positive if $f(x_0)$ is the maximal value of $f(x)$, and it cannot be negative if $f(x_0)$ is the minimal value. This is the contradiction. □

THEOREM 3.2.2. *If a function $f(x)$ is differentiable at x_0 and its graph is convex, then the tangent line to the graph of $f(x)$ at x_0 is $y = f(x_0)+f'(x_0)(x-x_0)$.*

PROOF. Let $y = ax+b$ be the equation of a tangent line to the graph $y = f(x)$ at the point x_0. Since $ax+b$ passes through x_0, one has $ax_0+b = f(x_0)$, therefore $b = f(x_0) - ax_0$, and it remains to prove that $a = f'(x_0)$. If the tangent line $ax+b$ is not horizontal, consider the function $g(x) = f(x) - ax$. At x_0 it takes either a maximal or a minimal value and $g'(x_0) = 0$ by Lemma 3.2.1. On the other hand, $g'(x_0) = f'(x_0) - a$. □

Differential equation. The Debeaune problem leads to a so-called differential equation on $y(x)$. To be precise, the equation of the tangent line to $y(x)$ at x_0 is $y = y(x_0) + y'(x_0)(x - x_0)$. So the x-coordinate of the point T can be found from the equation $0 = y(x_0) + y'(x_0)(x - x_0)$. The solution is $x = x_0 - \frac{y(x_0)}{y'(x_0)}$. The x-coordinate of V is just x_0. Hence TV is equal to $\frac{y(x_0)}{y'(x_0)}$. And Debeaune's requirement is $\frac{y(x_0)}{y'(x_0)} = a$. Or $ay' = y$. Equations that include derivatives of functions are called *differential equations*. The equation above is the simplest differential equation. Its solution takes one line. Indeed passing to differentials one gets $ay'\,dx = y\,dx$, further $ady = y\,dx$, then $a\frac{dy}{y} = dx$ and $a\,d\ln y = dx$. Hence $a\ln y = x+c$ and finally $y(x) = \exp(c+\frac{x}{a})$, where $\exp x$ denotes the function inverse to the natural logarithm and c is an arbitrary constant.

Exponenta. The function inverse to the natural logarithm is called the *exponential function*. We shall call it the *exponenta* to distinguish it from other exponential functions.

THEOREM 3.2.3. *The exponenta is the unique solution of the differential equation $y' = y$ such that $y'(0) = 1$.*

PROOF. Differentiation of the equality $\ln\exp x = x$ gives $\frac{\exp' x}{\exp x} = 1$. Hence $\exp x$ satisfies the differential equation $y' = y$. For $x = 0$ this equation gives $\exp'(0) = \exp 0$. But $\exp 0 = 1$ as $\ln 1 = 0$.

For the converse, let $y(x)$ be a solution of $y' = y$. The derivative of $\ln y$ is $\frac{y'}{y} = 1$. Hence the derivative of $\ln y(x) - x$ is zero. By Theorem 3.1.16 from the previous lecture, this implies $\ln y(x) - x = c$ for some constant c. If $y'(0) = 1$, then $y(0) = 1$ and $c = \ln 1 - 0 = 0$. Therefore $\ln y(x) = x$ and $y(x) = \exp\ln y(x) = \exp x$. □

Exponential series. Our next goal is to prove that

$$\exp x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \dots + \frac{x^k}{k!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{3.2.1}$$

where $0! = 1$. This series is absolutely convergent for any x. Indeed, the ratio of its subsequent terms is $\frac{x}{n}$ and tends to 0, hence it is eventually majorized by any geometric series.

Hyperbolic functions. To prove that the function presented by series (3.2.1) is virtually monotone, consider its odd and even parts. These parts represent the so-called *hyperbolic functions*: hyperbolic sine $\operatorname{sh} x$, and hyperbolic cosine $\operatorname{ch} x$.

$$\operatorname{sh}(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \operatorname{ch}(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

The hyperbolic sine is an increasing function, as all odd powers are increasing over the whole line. The hyperbolic cosine is increasing for positive x and decreasing for negative. Hence both are virtually monotone; and so is their sum.

Consider the integral $\int_0^x \operatorname{sh} t\,dt$. As all terms of the series representing sh are increasing, we can integrate the series termwise. This integration gives $\operatorname{ch} x$. As $\operatorname{sh} x$ is locally bounded, $\operatorname{ch} x$ is continuous by Theorem 3.1.13. Consider the integral $\int_0^x \operatorname{ch} t\,dt$; here we also can integrate the series representing ch termwise, because for positive x all the terms are increasing, and for negative x, decreasing. Integration gives $\operatorname{sh} x$, since the continuity of $\operatorname{ch} x$ was already proved. Further, by Theorem 3.1.13 we get that $\operatorname{sh} x$ is differentiable and $\operatorname{sh}' x = \operatorname{ch} x$. Now returning to the equality $\operatorname{ch} x = \int_0^x \operatorname{sh} t\,dt$ we get $\operatorname{ch}' x = \operatorname{sh} x$, as $\operatorname{sh} x$ is continuous.

Therefore $(\operatorname{sh} x + \operatorname{ch} x)' = \operatorname{ch} x + \operatorname{sh} x$. And $\operatorname{sh} 0 + \operatorname{ch} 0 = 0 + 1 = 1$. Now by the above Theorem 3.2.3 one gets $\exp x = \operatorname{ch} x + \operatorname{sh} x$.

Other exponential functions. The exponenta as a function inverse to the logarithm transforms sums into products. That is, for all x and y one has

$$\exp(x + y) = \exp x \exp y.$$

A function which has this property (i.e., transform sums into products) is called *exponential.*

THEOREM 3.2.4. *For any positive a there is a unique differentiable function denoted by a^x called the exponential function to base a, such that $a^1 = a$ and $a^{x+y} = a^x a^y$ for any x, y. This function is defined by the formula* $\exp a \ln x$.

PROOF. Consider $l(x) = \ln a^x$. This function has the property $l(x+y) = l(x) + l(y)$. Therefore its derivative at any point is the same: it is equal to $k = \lim_{x\to 0} \frac{l(x)}{x}$. Hence the function $l(x) - kx$ is constant, because its derivative is 0. This constant is equal to $l(0)$, which is 0. Indeed $l(0) = l(0+0) = l(0) + l(0)$. Thus $\ln a^x = kx$. Substituting $x = 1$ one gets $k = \ln a$. Hence $a^x = \exp(x \ln a)$. So if a differentiable exponential function with base a exists, it coincides with $\exp(x \ln a)$. On the other hand it is easy to see that $\exp(x \ln a)$ satisfies all the requirements for an exponential function to base a, that is $\exp(1 \ln a) = a$, $\exp((x+y)\ln a) = \exp(x \ln a)\exp(y \ln a)$; and it is differentiable as composition of differentiable functions. ☐

Powers. Hence for any positive a and any real b, one defines the number a^b as

$$a^b = \exp(b \ln a)$$

a is called the base, and b is called the exponent. For rational b this definition agrees with the old definition. Indeed if $b = \frac{p}{q}$ then the properties of the exponenta and the logarithm imply $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Earlier, we have defined logarithms to base b as the number c, and called the *logarithm of* b *to base* a, if $a^c = b$ and denoted $c = \log_a b$.

The basic properties of powers are collected here.

THEOREM 3.2.5.

$$\left(a^b\right)^c = a^{(bc)}, \quad a^{b+c} = a^b a^c, \quad (ab)^c = a^c b^c, \quad \log_a b = \frac{\log b}{\log a}.$$

Power functions. The power operation allows us to define the power function x^α for any real degree α. Now we can prove the equality $(x^\alpha)' = \alpha x^{\alpha-1}$ in its full value. Indeed, $(x^\alpha)' = (\exp(\alpha \ln x))' = \exp'(\alpha \ln x)(\alpha \ln x)' = \exp(\alpha \ln x)\frac{\alpha}{x} = \alpha x^{\alpha-1}$.

Infinite products via the Logarithm.

LEMMA 3.2.6. *Let $f(x)$ be a function continuous at x_0. Then for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ one has $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

PROOF. For any given $\varepsilon > 0$ there is a neighborhood U of x_0 such that $|f(x) - f(x_0)| \leq \varepsilon$ for $x \in U$. As $\lim_{n\to\infty} x_n = x_0$, eventually $x_n \in U$. Hence eventually $|f(x_n) - f(x_0)| < \varepsilon$. □

As we already have remarked, infinite sums and infinite products are limits of partial products.

THEOREM 3.2.7. $\ln \prod_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \ln p_k$.

PROOF.

$$\begin{aligned}\exp(\textstyle\sum_{k=1}^{\infty} \ln p_k) &= \exp(\lim_{n\to\infty} \textstyle\sum_{k=1}^{n} \ln p_k) \\ &= \lim_{n\to\infty} \exp(\textstyle\sum_{k=1}^{n} \ln p_k) \\ &= \lim_{n\to\infty} \textstyle\prod_{k=1}^{n} p_k \\ &= \textstyle\prod_{k=1}^{\infty} p_k.\end{aligned}$$

Now take logarithms of both sides of the equation. □

Symmetric arguments prove the following: $\exp \sum_{k=1}^{\infty} a_k = \prod_{k=1}^{\infty} \exp a_k$.

Irrationality of e. The expansion of the exponenta into a power series gives an expansion into a series for e which is $\exp 1$.

LEMMA 3.2.8. *For any natural n one has $\frac{1}{n+1} < en! - [en!] < \frac{1}{n}$.*

PROOF. $en! = \sum_{k=0}^{\infty} \frac{n!}{k!}$. The partial sum $\sum_{k=0}^{n} \frac{n!}{k!}$ is an integer. The tail $\sum_{k=n+1}^{\infty} \frac{n!}{k!}$ is termwise majorized by the geometric series $\sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n}$. On the other hand the first summand of the tail is $\frac{1}{n+1}$. Consequently the tail has its sum between $\frac{1}{n+1}$ and $\frac{1}{n}$. □

THEOREM 3.2.9. *The number e is irrational.*

PROOF. Suppose $e = \frac{p}{q}$ where p and q are natural. Then $eq!$ is a natural number. But it is not an integer by Lemma 3.2.8. □

Problems.

1. Prove the inequalities $1 + x \leq \exp x \leq \frac{1}{1-x}$.
2. Prove the inequalities $\frac{x}{1+x} \leq \ln(1 + x) \leq x$.
3. Evaluate $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.
4. Evaluate $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$.
5. Evaluate $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$.
6. Find the derivative of x^x.
7. Prove: $x > y$ implies $\exp x > \exp y$.
8. Express via e: $\exp 2$, $\exp(1/2)$, $\exp(2/3)$, $\exp(-1)$.
9. Prove that $\exp(m/n) = e^{\frac{m}{n}}$.
10. Prove that $\exp x > 0$ for any x.
11. Prove the addition formulas $\operatorname{ch}(x + y) = \operatorname{ch}(x)\operatorname{ch}(y) + \operatorname{sh}(x)\operatorname{sh}(y)$, $\operatorname{sh}(x + y) = \operatorname{sh}(x)\operatorname{ch}(y) + \operatorname{sh}(y)\operatorname{ch}(x)$.
12. Prove that $\Delta \operatorname{sh}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{ch}(x)$, $\Delta \operatorname{ch}(x - 0.5) = \operatorname{sh} 0.5 \operatorname{sh}(x)$.
13. Prove $\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$.
14. Prove $\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = 1$.
15. Solve the equation $\operatorname{sh} x = 4/5$.
16. Express via e the sum $\sum_{k=1}^{\infty} k/k!$.
17. Express via e the sum $\sum_{k=1}^{\infty} k^2/k!$.
18. Prove that $\{\frac{\exp k}{k^n}\}$ is unbounded.
19. Prove: The product $\prod(1 + p_n)$ converges if and only if the sum $\sum p_n$ ($p_n \geq 0$) converges.
20. Determine the convergence of $\prod \frac{e^{1/n}}{1+\frac{1}{n}}$.
21. Does $\prod n(e^{1/n} - 1)$ converges?
22. Prove the divergence of $\sum_{k=1}^{\infty} \frac{[k-prime]}{k}$.
23. Expand a^x into a power series.
24. Determine the geometrical sense of $\operatorname{sh} x$ and $\operatorname{ch} x$.
25. Evaluate $\lim_{n\to\infty} \sin \pi e n!$.
26. Does the series $\sum_{k=1}^{\infty} \sin \pi e k!$ converge?
***27.** Prove the irrationality of e^2.

3.3. Euler Formula

On the contents of the lecture. The reader becomes acquainted with the most famous Euler formula. Its special case $e^{i\pi} = -1$ symbolizes the unity of mathematics: here e represents Analysis, i represents Algebra, and π represents Geometry. As a direct consequence of the Euler formula we get power series for sin and cos, which we need to sum up the Euler series.

Complex Newton-Leibniz. For a function of a complex variable $f(z)$ the derivative is defined by the same formula $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We will denote it also by $\frac{df(z)}{dz}$, to distinguish from derivatives of paths: complex valued functions of real variable. For a path $p(t)$ its derivative will be denoted either $p'(t)$ or $\frac{dp(t)}{dt}$. The Newton-Leibniz formula for real functions can be expressed by the equality $\frac{df(t)}{dt}dt = df(t)$. Now we extend this formula to complex functions.

The linearization of a complex function $f(z)$ at z_0 has the same form $f(z_0) + f'(z_0)(z - z_0) + o(z)(z - z_0)$, where $o(z)$ is an infinitesimally small function of complex variable. The same arguments as for real numbers prove the basic rules of differentiation: the derivative of sums, products and compositions.

THEOREM 3.3.1. $\frac{dz^n}{dz} = nz^{n-1}$.

PROOF. $\frac{dz}{dz} = 1$ one gets immediately from the definition of the derivative. Suppose the equality $\frac{dz^n}{dz} = nz^{n-1}$ is proved for n. Then $\frac{dz^{n+1}}{dz} = \frac{dzz^n}{dz} = z\frac{dz^n}{dz} + z^n\frac{dz}{dz} = znz^{n-1} + z^n = (n+1)z^n dz$. And the theorem is proved by induction. □

A *smooth path* is a differentiable mapping $p\colon [a,b] \to \mathbb{C}$ with a continuous bounded derivative. A function $f(z)$ of a complex variable is called *virtually monotone* if for any smooth path $p(t)$ the functions $\operatorname{Re} f(p(t))$ and $\operatorname{Im} f(p(t))$ are virtually monotone.

LEMMA 3.3.2. *If $f'(z)$ is bounded, then $f(z)$ is virtually monotone.*

PROOF. Consider a smooth path p. Then $\frac{df(p(t))}{dt} = f'(p(t))p'(t)$ is bounded by some K. Due to Lemma 3.1.15 one has $|f(p(t)) - f(p(t_0))| \leq K|t - t_0|$. Hence any partial variation of $f(p(t))$ does not exceed $K(b-a)$. Therefore $\operatorname{var}_{f(p(t))}[a,b] \leq K$. □

THEOREM 3.3.3. *If a complex function $f(z)$ has a bounded virtually monotone continuous complex derivative over the image of a smooth path $p\colon [a,b] \to \mathbb{C}$, then $\int_p f'(z)\,dz = f(p(b)) - f(p(a))$.*

PROOF. $\frac{df(p(t))}{dt} = f'(p(t))p'(t) = \frac{d\operatorname{Re} f(p(t))}{dt} + i\frac{d\operatorname{Im} f(p(t))}{dt}$. All functions here are continuous and virtually monotone by hypothesis. Passing to differential forms one gets

$$\begin{aligned}\frac{df(p(t))}{dt}\,dt &= \frac{d\operatorname{Re} f(p(t))}{dt}\,dt + i\,\frac{d\operatorname{Im} f(p(t))}{dt}\,dt \\ &= d(\operatorname{Re} f(p(t))) + i\,d(\operatorname{Im} f(p(t))) \\ &= d(\operatorname{Re} f(p(t)) + i\operatorname{Im} f(p(t))) \\ &= d(f(p(t)).\end{aligned}$$

Hence $\int_p f'(z)\,dz = \int_p df(z)$. □

COROLLARY 3.3.4. *If $f'(z) = 0$ then $f(z)$ is constant.*

PROOF. Consider $p(t) = z_0 + (z - z_0)t$, then $f(z) - f(z_0) = \int_p f'(\zeta)\, d\zeta = 0$. □

Differentiation of series. Let us say that a complex series $\sum_{k=1}^{\infty} a_k$ majorizes (eventually) another such series $\sum_{k=1}^{\infty} b_k$ if $|b_k| \leq |a_k|$ for all k (resp. for k beyond some n).

The series $\sum_{k=1}^{\infty} kc_k(z - z_0)^{k-1}$ is called a *formal derivative* of $\sum_{k=0}^{\infty} c_k(z - z_0)^k$.

LEMMA 3.3.5. *Any power series $\sum_{k=0}^{\infty} c_k(z - z_0)^k$ eventually majorizes its formal derivative $\sum_{k=0}^{\infty} kc_k(z_1 - z_0)^{k-1}$ if $|z_1 - z_0| < |z - z_0|$.*

PROOF. The ratio of the n-th term of the derivative to the n-th term of the series tends to 0 as n tends to infinity. Indeed, this ratio is $\frac{k(z_1 - z_0)^k}{(z - z_0)^k} = kq^k$, where $|q| < 1$ since $|z_1 - z_0| < |z - z_0|$. The fact that $\lim_{n\to\infty} nq^n = 0$ follows from the convergence of $\sum_{k=1}^{\infty} kq^k$ which we already have proved before. This series is eventually majorized by any geometric series $\sum_{k=0}^{\infty} AQ^k$ with $Q > q$. □

A path $p(t)$ is called *monotone* if both $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$ are monotone.

LEMMA 3.3.6. *Let $p\colon [a, b] \to \mathbb{C}$ be a smooth monotone path, and let $f(z)$ be virtually monotone. If $|f(p(t))| \leq c$ for $t \in [a, b]$ then $\left|\int_p f(z)\, dz\right| \leq 4c|p(b) - p(a)|$.*

PROOF. Integration of the inequalities $-c \leq \operatorname{Re} f(p(t)) \leq c$ against $d \operatorname{Re} z$ along the path gives $|\int_p \operatorname{Re} f(z)\, d \operatorname{Re} z| \leq c|\operatorname{Re} p(b) - \operatorname{Re} p(a)| \leq c|p(b) - p(a)|$. The same arguments prove $|\int_p \operatorname{Im} f(z)\, d \operatorname{Im} z| \leq c|\operatorname{Im} p(b) - \operatorname{Im} p(a)| \leq c|p(b) - p(a)|$. The sum of these inequalities gives $|\operatorname{Re} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. The same arguments yields $|\operatorname{Im} \int_p f(z)\, dz| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. And the addition of the two last inequalities allows us to accomplish the proof of the Lemma because $|\int_p f(z)\, dz| \leq |\operatorname{Re} \int_p f(z)\, dz| + |\int_p f(z)\, dz|$. □

LEMMA 3.3.7. $|z^n - \zeta^n| \leq n|z - \zeta| \max\{|z^{n-1}|, |\zeta^{n-1}|\}$.

PROOF. $(z^n - \zeta^n) = (z - \zeta) \sum_{k=0}^{n-1} z^k \zeta^{n-k-1}$ and $|z^k \zeta^{n-k-1}| \leq \max\{|z^{n-1}|, |\zeta^{n-1}|\}$. □

A *linear path* from z_0 to z_1 is defined as a linear mapping $p\colon [a, b] \to \mathbb{C}$, such that $p(a) = z_0$ and $p(b) = z_1$, that is $p(t) = z_0(t - a) + (z_1 - z_0)(t - a)/(b - a)$.

We denote by $\int_a^b f(z)\, dz$ the integral along the linear path from a to b.

LEMMA 3.3.8. *For any complex z, ζ and natural $n > 0$ one has*

$$|z^n - z_0^n - nz_0^{n-1}(z - z_0)| \leq 2n(n - 1)|z - z_0|^2 \max\{|z|^{n-2}, |z_0|^{n-2}\}. \tag{3.3.1}$$

PROOF. By the Newton-Leibniz formula, $z^n - z_0^n = \int_{z_0}^z n\zeta^{n-1}\, d\zeta$. Further,

$$\begin{aligned}\int_{z_0}^z n\zeta^{n-1}\, d\zeta &= \int_{z_0}^z nz_0^{n-1}\, d\zeta + \int_{z_0}^z n(\zeta^{n-1} - z_0^{n-1})\, d\zeta \\ &= nz_0^{n-1} + \int_{z_0}^z n(\zeta^{n-1} - z_0^{n-1})\, d\zeta.\end{aligned}$$

Consequently, the left-hand side of (3.3.1) is equal to $\left|\int_{z_0}^z n(\zeta^{n-1} - z_0^{n-1})\, d\zeta\right|$. Due to Lemma 3.3.7 the absolute value of the integrand along the linear path does not exceed $(n-1)|z-z_0|\max\{|z^{n-2}|,|z_0^{n-2}|\}$. Now the estimation of the integral by Lemma 3.3.6 gives just the inequality (3.3.1). $\square$

THEOREM 3.3.9. *If $\sum_{k=0}^\infty c_k(z_1-z_0)^k$ converges absolutely, then $\sum_{k=0}^\infty c_k(z-z_0)^k$ and $\sum_{k=1}^\infty kc_k(z-z_0)^{k-1}$ absolutely converge provided by $|z-z_0|<|z_1-z_0|$, and the function $\sum_{k=1}^\infty kc_k(z-z_0)^{k-1}$ is the complex derivative of $\sum_{k=0}^\infty c_k(z-z_0)^k$.*

PROOF. The series $\sum_{k=0}^\infty c_k(z-z_0)^k$ and its formal derivative are eventually majorized by $\sum_{k=0}^\infty c_k(z_1-z_0)^k$ if $|z-z_0|\le|z_1-z_0|$ by the Lemma 3.3.5. Hence they absolutely converge in the circle $|z-z_0|\le|z_1-z_0|$. Consider

$$R(z)=\sum_{k=0}^\infty c_k(z-z_0)^k-\sum_{k=0}^\infty c_k(\zeta-z_0)^k-(z-\zeta)\sum_{k=1}^\infty kc_k(\zeta-z_0)^{k-1}.$$

To prove that the formal derivative is the derivative of $\sum_{k=0}^\infty c_k(z-z_0)^k$ at ζ it is sufficient to prove that $R(z)=o(z)(z-\zeta)$, where $o(z)$ is infinitesimally small at ζ. One has $R(z)=\sum_{k=1}^\infty c_k\left((z-z_0)^k-(\zeta-z_0)^k-k(\zeta-z_0)^{k-1}\right)$. By Lemma 3.3.8 one gets the following estimate: $|R(z)|\le\sum_{k=1}^\infty 2|c_k|k(k-1)|z-\zeta|^2|z_2-z_0|^{n-2}$, where $|z_2-z_0|=\max\{|z-z_0|,|\zeta-z_0|\}$. Hence all we need now is to prove that $\sum_{k=1}^\infty 2k(k-1)|c_k||z_2-z_0|^{k-2}|z-\zeta|$ is infinitesimally small at ζ. And this in its turn follows from the convergence of $\sum_{k=1}^\infty 2k(k-1)|c_k||z_2-z_0|^{k-2}$. The latter may be deduced from Lemma 3.3.5. Indeed, consider z_3, such that $|z_2-z_0|<|z_3-z_0|<|z_1-z_0|$. The convergence of $\sum_{k=1}^\infty k|c_k||z_3-z_0|^{k-1}$ follows from the convergence of $\sum_{k=0}^\infty|c_k||z_1-z_0|^k$ by Lemma 3.3.5. And the convergence of $\sum_{k=2}^\infty k(k-1)|c_k||z_2-z_0|^{k-2}$ follows from the convergence of $\sum_{k=1}^\infty k|c_k||z_3-z_0|^{k-1}$ by the same lemma. $\square$

COROLLARY 3.3.10. *Let $f(z)=\sum_{k=0}^\infty c_kz^k$ converge absolutely for $|z|<r$, and let a,b have absolute values less then r. Then $\int_a^b f(z)\,dz=\sum_{k=0}^\infty\frac{c_k}{k+1}(b^{k+1}-a^{k+1})$.*

PROOF. Consider $F(z)=\sum_{k=0}^\infty\frac{c_kz^{k+1}}{k+1}$. This series is termwise majorized by the series of $f(z)$, hence it converges absolutely for $|z|<r$. By Theorem 3.3.9 $f(z)$ is its derivative for $|z|<r$. In our case $f(z)$ is differentiable and its derivative is bounded by $\sum_{k=0}^\infty k|c_k|r_0^k$, where $r_0=\max\{|a|,|b|\}$. Hence $f(z)$ is continuous and virtually monotone and our result now follows from Theorem 3.3.3. $\square$

Exponenta in $\mathbb{C}$. The exponenta for any complex number z is defined as $\exp z=\sum_{k=0}^\infty\frac{z^k}{k!}$. The definition works because the series $\sum_{k=0}^\infty\frac{z^k}{k!}$ absolutely converges for any $z\in\mathbb{C}$.

THEOREM 3.3.11. *The exponenta is a differentiable function of a complex variable with derivative* $\exp' z=\exp z$, *such that for all complex z, ζ the following* addition formula *holds:* $\exp(z+\zeta)=\exp z\exp\zeta$.

PROOF. The derivative of the exponenta can be evaluated termwise by Theorem 3.3.9. And this evaluation gives $\exp' z=\exp z$. To prove the addition formula consider the following function $r(z)=\frac{\exp(z+\zeta)}{\exp z}$. Differentiation of the equality $r(z)\exp z=\exp(z+\zeta)$ gives $r'(z)\exp z+r(z)\exp z=\exp(z+\zeta)$. Division by $\exp z$ gives $r'(z)+r(z)=r(z)$. Hence $r(z)$ is constant. This constant is determined by substitution $z=0$ as $r(z)=\exp\zeta$. This proves the addition formula. $\square$

LEMMA 3.3.12. *Let* $p\colon [a,b] \to \mathbb{C}$ *be a smooth path contained in the complement of a neighborhood of* 0. *Then* $\exp \int_p \frac{1}{\zeta}\, d\zeta = \frac{p(b)}{p(a)}$.

PROOF. First consider the case when p is contained in a circle $|z - z_0| < |z_0|$ with center $z_0 \neq 0$. In this circle, $\frac{1}{z}$ expands in a power series:

$$\frac{1}{\zeta} = \frac{1}{z_0 - (z_0 - \zeta)} = \frac{1}{z_0}\frac{1}{1 - \frac{z_0-\zeta}{z_0}} = \sum_{k=0}^{\infty} \frac{(z_0-\zeta)^k}{z_0^{k+1}}.$$

Integration of this series is possible to do termwise due to Corollary 3.3.10. Hence the result of the integration does not depend on the path. And Theorem 3.3.9 provides differentiability of the termwise integral and the possibility of its termwise differentiation. Such differentiation simply gives the original series, which represents $\frac{1}{z}$ in this circle.

Consider the function $l(z) = \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta$. Then $l'(z) = \frac{1}{z}$. The derivative of the composition $\exp l(z)$ is $\frac{\exp l(z)}{z}$. Hence the composition satisfies the differential equation $y'z = y$. We search for a solution of this equation in the form $y = zw$. Then $y' = w + w'z$ and our equation turns into $wz + w'z^2 = wz$. Therefore $w' = 0$ and w is constant. To find this constant substitute $z = z_0$ and get $1 = \exp 0 = \exp l(z_0) = wz_0$. Hence $w = \frac{1}{z_0}$ and $\exp l(z) = \frac{z}{z_0}$.

To prove the general case consider a partition $\{x_k\}_{k=0}^n$ of $[a,b]$. Denote by p_k the restriction of p over $[x_k, x_{k+1}]$. Choose the partition so small that $|p(x) - p(x_k)| < |p(x_k)|$ for all $x \in [x_k, x_{k+1}]$. Then any p_k satisfies the requirement of the above considered case. Hence $\exp \int_{p_k} \frac{1}{\zeta}\, d\zeta = \frac{p(x_{k+1})}{p(x_k)}$. Further $\exp \int_p \frac{1}{\zeta}\, d\zeta = \exp \sum_{k=0}^{n-1} \int_{p_k} \frac{1}{\zeta}\, d\zeta = \prod_{k=0}^{n-1} \frac{p(x_{k+1})}{p(x_k)} = p(b)/p(a)$. □

THEOREM 3.3.13 (Euler Formula). *For any real* ϕ *one has*

$$\boxed{\exp i\phi = \cos\phi + i \sin\phi}$$

PROOF. In Lecture 2.5 we have evaluated $\int_p \frac{1}{z}\, dz = i\phi$ for $p(t) = \cos t + i \sin t$, $t \in [0, \phi]$. Hence Lemma 3.3.12 applied to $p(t)$ immediately gives the Euler formula. □

Trigonometric functions in $\mathbb{C}$. The Euler formula gives power series expansions for $\sin x$ and $\cos x$:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

These expansions are used to define trigonometric functions for complex variable. On the other hand the Euler formula allows us to express trigonometric functions via the exponenta:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \qquad \cos z = \frac{\exp(iz) + \exp(-iz)}{2}.$$

The other trigonometric functions tan, cot, sec, cosec are defined for complex variables by the usual formulas via sin and cos.

Problems.

1. Evaluate $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

2. Prove the formula of Joh. Bernoulli $\int_0^1 x^x\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$.

3. Find $\ln(-1)$.

4. Solve the equation $\exp z = i$.

5. Evaluate i^i.

6. Prove $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cos z = \frac{e^{iz}+e^{-iz}}{2}$.

7. Prove the identity $\sin^2 z + \cos^2 z = 1$.

8. Solve the equation $\sin z = 5/3$.

9. Solve the equation $\cos z = 2$.

10. Evaluate $\sum_{k=0}^{\infty} \frac{\cos k}{k!}$.

11. Evaluate $\oint_{|z|=1} \dfrac{dz}{z^2}$.

12. Evaluate $\sum_{k=1}^{\infty} q^k \dfrac{\sin kx}{k}$.

13. Expand into a power series $e^x \cos x$.

3.4. Abel's Theorem

On the contents of the lecture. The expansion of the logarithm into power series will be extended to the complex case. We learn the very important Abel's transformation of sum. This transformation is a discrete analogue of integrations by parts. Abel's theorem on the limit of power series will be applied to the evaluation of trigonometric series related to the logarithm. The concept of Abel's sum of a divergent series will be introduced.

Principal branch of the Logarithm. Since $\exp(x+iy) = e^x(\cos y + i\sin y)$, one gets the following formula for the logarithm: $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$, where $\operatorname{Arg} z = \arg z + 2\pi k$. We see that the logarithm is a multi-valued function, that is why one usually chooses a *branch* of the logarithm to work. For our purposes it is sufficient to consider the *principal* branch of the logarithm:

$$\ln z = \ln|z| + i\arg z, \quad -\pi < \arg z \leq \pi.$$

The principal branch of the logarithm is a differentiable function of a complex variable with derivative $\frac{1}{z}$, inverse to $\exp z$. This branch is not continuous at negative numbers. However its restriction on the upper half-plane is continuous and even differentiable at negative numbers.

LEMMA 3.4.1. *For any nonnegative z one has $\int_1^z \frac{1}{\zeta}\, d\zeta = \ln z$.*

PROOF. If $\operatorname{Im} z \neq 0$, the segment $[0, z]$ is contained in the circle $|\zeta - z_0| < |z_0|$ for $z_0 = \frac{|z|^2}{\operatorname{Im} z}$. In this circle $\frac{1}{\zeta}$ expands into a power series, which one can integrate termwise. Since for z^k the result of integration depends only on the ends of path of integration, the same is true for power series. Hence, we can change the path of integration without changing the result. Consider the following path: $p(t) = \cos t + i\sin t$, $t \in [0, \arg z]$. We know the integral $\int_p \frac{1}{\zeta}\, d\zeta = i\arg z$. This path terminates at $\frac{z}{|z|}$. Continue this path by the linear path to z. The integral satisfies $\int_{z/|z|}^z \frac{1}{\zeta}\, d\zeta = \int_1^{|z|} \frac{1}{z/|z|t}\, dtz/|z| = \int_1^{|z|} \frac{1}{t}\, dt = \ln|z|$. Therefore $\int_1^z \frac{1}{\zeta}\, d\zeta = \int_p \frac{1}{\zeta}\, d\zeta + \int_{z/|z|}^z \frac{1}{\zeta}\, d\zeta = i\arg z + \ln|z|$. □

Logarithmic series. In particular for $|1-z| < 1$ termwise integration of the series $\frac{1}{\zeta} = \sum_{k=0}^\infty (1-\zeta)^k$ gives the complex Mercator series:

$$\ln(1+z) = \sum_{k=1}^\infty (-1)^{k+1} \frac{z^k}{k}. \tag{3.4.1}$$

Substitute in this series $-z$ for z and subtract the obtained series from (3.4.1) to get the complex Gregory series:

$$\frac{1}{2}\ln\frac{1+z}{1-z} = \sum_{k=0}^\infty (-1)^k \frac{z^{2k+1}}{2k+1}.$$

In particular for $z = ix$, one has $\left|\frac{1+ix}{1-ix}\right| = 1$ and $\arg\frac{1+ix}{1-ix} = 2\operatorname{arctg} x$. Therefore one gets

$$\operatorname{arctg} x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Since $\arg(1+e^{i\phi}) = \operatorname{arctg}\frac{\sin\phi}{1+\cos\phi} = \operatorname{arctg}\tan(\phi/2) = \frac{\phi}{2}$, the substitution of $\exp(i\phi)$ for z in the Mercator series $\ln(1+e^{i\phi}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^{ik\phi}}{k}$ gives for the imaginary parts:

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\sin k\phi}{k} = \frac{\phi}{2}. \tag{3.4.2}$$

However the last substitution is not correct, because $|e^{i\phi}| = 1$ and (3.4.1) is proved only for $|z| < 1$. To justify it we will prove a general theorem, due to Abel.

Summation by parts. Consider two sequences $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$. The difference of their product $\delta a_k b_k = a_{k+1}b_{k+1} - a_k b_k$ can be presented as

$$\delta(a_k b_k) = a_{k+1}\delta b_k + b_k\delta a_k.$$

Summation of these equalities gives

$$a_n b_n - a_1 b_1 = \sum_{k=1}^{n-1} a_{k+1}\delta b_k + \sum_{k=1}^{n-1} b_k \delta a_k.$$

A permutation of the latter equality gives the so-called *Abel's transformation* of sums

$$\sum_{k=1}^{n-1} b_k\Delta a_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}\Delta b_k.$$

Abel's theorem. One writes $x \to a-0$ instead of $x \to a$ and $x < a$, and $x \to a+0$ means $x > a$ and $x \to a$.

THEOREM 3.4.2 (Abel).

If $\sum_{k=0}^{\infty} a_k$ *converges, then* $\lim_{x\to 1-0}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k$.

PROOF. $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < 1$, because of the boundedness of $\{a_k\}$.

Suppose $\varepsilon > 0$. Set $A(n,m) = \sum_{k=n}^{m} a_k$, $A(n,m)(x) = \sum_{k=n}^{m} a_k x^k$. Choose N so large that

$$|A(0,n) - A(0,\infty)| < \frac{\varepsilon}{9}, \quad \forall n > N. \tag{3.4.3}$$

Applying the Abel transformation for any $m > n$ one gets

$$\begin{aligned}
A(n,m) - A(n,m)(x) &= \sum_{k=n}^{m} a_k(1-x^k)\\
&= (1-x)\sum_{k=n}^{m}\delta A(n-1,k-1)\sum_{j=0}^{k-1}x^j\\
&= (1-x)\Big[A(n-1,m)\sum_{j=0}^{m}x^j - A(n-1,n)\sum_{j=0}^{n}x^j - \sum_{k=n}^{m}A(n-1,k)x^k\Big].
\end{aligned}$$

By (3.4.3) for $n > N$, one gets $|A(n-1,m)| = |(A(0,m) - A) + (A - A(0,n))| \leq \varepsilon/9 + \varepsilon/9 = 2\varepsilon/9$. Hence, we can estimate from above by $\frac{2\varepsilon/3}{1-x}$ the absolute value of the expression in the brackets of the previous equation for $A(n,m) - A(n,m)(x)$. As a result we get

$$|A(n,m) - A(n,m)(x)| \le \frac{2\varepsilon}{3}, \quad \forall m \ge n > N, \forall x. \tag{3.4.4}$$

Since $\lim_{x\to 1-0} A(0,N)(x) = A(0,N)$ one chooses δ so small that for $x > 1-\delta$ the following inequality holds:

$$|A(0,N) - A(0,N)(x)| < \frac{\varepsilon}{3}.$$

Summing up this inequality with (3.4.4) for $n = N+1$ one gets:

$$|A(0,m) - A(0,m)(x)| < \varepsilon, \quad \forall m > N, |1-x| < \delta.$$

Passing to limits as m tends to infinity the latter inequality gives

$$|A(0,\infty) - A(0,\infty)(x)| \le \varepsilon, \quad \text{for } |1-x| < \delta.$$

□

Leibniz series. As the first application of the Abel Theorem we evaluate the Leibniz series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. This series converges by the Leibniz Theorem 2.4.3. By the Abel Theorem its sum is

$$\lim_{x\to 1-0} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2k+1} = \lim_{x\to 1-0} \operatorname{arctg} x = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

We get the following remarkable equality:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \dots.$$

Abel sum of a series. One defines the *Abel sum* of a series $\sum_{k=0}^{\infty} a_k$ as the limit $\lim_{x\to 1-0} \sum_{k=0}^{\infty} a_k x^k$. The series which have an Abel sum are called *Abel summable*. The Abel Theorem shows that all convergent series have Abel sums coinciding with their usual sums. However there are a lot of series, which have an Abel sum, but do not converge.

Abel's inequality. Consider a series $\sum_{k=1}^{\infty} a_k b_k$, where the partial sums $A_n = \sum_{k=1}^{n-1} a_k$ are bounded by some constant A and the sequence $\{b_k\}$ is monotone. Then $\sum_{k=1}^{n-1} a_k b_k = \sum_{k=1}^{n-1} b_k \delta A_k = A_n b_n - A_1 b_1 + \sum_{k=1}^{n-1} A_{k+1} \delta b_k$. Since $\sum_{k=1}^{n-1} |\delta b_k| = |b_n - b_1|$, one gets the following inequality:

$$\left|\sum_{k=1}^{n-1} a_k b_k\right| \le 3A \max\{|b_k|\}.$$

Convergence test.

THEOREM 3.4.3. *Let the sequence of partial sums $\sum_{k=1}^{n-1} a_k$ be bounded, and let $\{b_k\}$ be non-increasing and infinitesimally small. Then $\sum_{k=1}^{\infty} a_k b_k$ converges to its Abel sum, if the latter exists.*

PROOF. The difference between a partial sum $\sum_{k=1}^{n-1} a_k b_k$ and the Abel sum is equal to

$$\lim_{x\to 1-0} \sum_{k=1}^{n-1} a_k b_k (1 - x^k) + \lim_{x\to 1-0} \sum_{k=n}^{\infty} a_k b_k x^k.$$

The first limit is zero, the second limit can be estimated by Abel's inequality from above by $3Ab_n$. It tends to 0 as n tends to infinity. □

Application. Now we are ready to prove the equality (3.4.2). The series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$ has an Abel sum. Indeed,

$$\begin{aligned}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{q^k\sin kx}{k} &= \operatorname{Im}\lim_{q\to 1-0}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(qe^{ix})^k}{k}\\ &= \operatorname{Im}\lim_{q\to 1-0}\ln(1+qe^{ix})\\ &= \operatorname{Im}\ln(1+e^{ix}).\end{aligned}$$

The sums $\sum_{k=1}^{n-1}\sin kx = \operatorname{Im}\sum_{k=1}^{n-1}e^{ikx} = \operatorname{Im}\frac{1-e^{inx}}{1-e^{ix}}$ are bounded. And $\frac{1}{k}$ is decreasing and infinitesimally small. Hence we can apply Theorem 3.4.3.

Problems.

1. Evaluate $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\dots$.
2. Evaluate $\sum_{k=1}^{\infty}\frac{\sin 2k}{k}$.
3. $\sum_{k=1}^{\infty}\frac{\cos k\phi}{k} = -\ln|2\sin\frac{\phi}{2}|$, $(0<|\phi|\le\pi)$.
4. $\sum_{k=1}^{\infty}\frac{\sin k\phi}{k} = \frac{\pi-\phi}{2}$, $(0<\phi<2\pi)$.
5. $\sum_{k=0}^{\infty}\frac{\cos(2k+1)\phi}{2k+1} = \frac{1}{2}\ln|2\cot\frac{\phi}{2}|$, $(0<|\phi|<\pi)$
6. $\sum_{k=0}^{\infty}\frac{\sin(2k+1)\phi}{2k+1} = \frac{\pi}{4}$, $(0<\phi<\pi)$
7. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos k\phi}{k} = \ln\left(2\cos\frac{\phi}{2}\right)$, $(-\pi<\phi<\pi)$
8. Find the Abel sum of $1-1+1-1+\dots$.
9. Find the Abel sum of $1-1+0+1-1+0+\dots$.
10. Prove: A periodic series, such that the sum of the period is zero, has an Abel sum.
11. Telescope $\sum_{k=1}^{\infty}\dfrac{k^2}{2^k}$.
12. Evaluate $\sum_{k=0}^{n-1}k\cos kx$.
13. Estimate from above $\sum_{k=n}^{\infty}\frac{\sin kx}{k^2}$.
***14.** Prove: If $\sum_{k=0}^{\infty}a_k$, $\sum_{k=0}^{\infty}b_k$ and their convolution $\sum_{k=0}^{\infty}c_k$ converge, then $\sum_{k=0}^{\infty}c_k = \sum_{k=0}^{\infty}a_k\sum_{k=0}^{\infty}b_k$.

3.5. Residue Theory

On the contents of the lecture. At last, the reader learns something, which Euler did not know, and which he would highly appreciate. Residue theory allows one to evaluate a lot of integrals which were not accessible by the Newton-Leibniz formula.

Monotone curve. A *monotone curve* Γ is defined as a subset of the complex plane which is the image of a monotone path. Nonempty intersections of vertical and horizontal lines with a monotone curve are either points or closed intervals.

The points of the monotone curve which have an extremal sum of real and imaginary parts are called its *endpoints*, the other points of the curve are called its *interior* points.

A continuous injective monotone path p whose image coincides with Γ is called a *parametrization* of Γ.

LEMMA 3.5.1. *Let $p_1\colon [a,b] \to \mathbb{C}$ and $p_2\colon [c,d] \to \mathbb{C}$ be two parametrizations of the same monotone curve Γ. Then $p_1^{-1}p_2\colon [c,d] \to [a,b]$ is a continuous monotone bijection.*

PROOF. Set $P_i(t) = \operatorname{Re} p_i(t) + \operatorname{Im} p_i(t)$. Then P_1 and P_2 are continuous and strictly monotone. And $p_1(t) = p_2(\tau)$ if and only if $P_1(t) = P_2(\tau)$. Hence $p_1^{-1}p_2 = P_1^{-1}P_2$. Since P_1^{-1} and P_2 are monotone continuous, the composition $P_1^{-1}P_2$ is monotone continuous. □

Orientation. One says that two parametrizations p_1 and p_2 of a monotone curve Γ have the *same orientation*, if $p_1^{-1}p_2$ is increasing, and one says that they have *opposite orientations*, if $p_1^{-1}p_2$ is decreasing.

Orientation divides all parametrizations of a curve into two classes. All elements of one orientation class have the same orientation and all elements of the other class have the opposite orientation.

An oriented curve is a curve with fixed *circulation direction.* A choice of orientation means distinguishing one of the orientation classes as positive, corresponding to the oriented curve. For a monotone curve, to specify its orientation, it is sufficient to indicate which of its endpoints is its beginning and which is the end. Then all positively oriented parametrizations start with its beginning and finish at its end, and negatively oriented parametrizations do the opposite.

If an oriented curve is denoted by Γ, then its *body*, the curve without orientation, is denoted $|\Gamma|$ and the curve with the same body but with opposite orientation is denoted $-\Gamma$.

If Γ' is a monotone curve which is contained in an oriented curve Γ, then one defines the *induced orientation* on Γ' by Γ as the orientation of a parametrization of Γ' which extends to a positive parametrization of Γ.

Line integral. One defines the integral $\int_\Gamma f(z)\,dg(z)$ along a oriented monotone curve Γ as the integral $\int_p f(z)\,dg(z)$, where p is a positively oriented parametrization of Γ. This definition does not depend on the choice of p, because different parametrizations are obtained from each other by an increasing change of variable (Lemma 3.5.1).

One defines a *partition of a curve* Γ by a point x as a pair of monotone curves Γ_1, Γ_2, such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = x$. And we write in this case $\Gamma = \Gamma_1 + \Gamma_2$.

The Partition Rule for the line integral is

$$\int_{\Gamma_1+\Gamma_2} f(z)\,dg(z) = \int_{\Gamma_1} f(z)\,dg(z) + \int_{\Gamma_2} f(z)\,dg(z), \tag{3.5.1}$$

where the orientations on Γ_i are induced by an orientation of Γ. To prove the Partition Rule consider a positive parametrization $p\colon [a,b] \to \Gamma$. Then the restrictions of p over $[a, p^{-1}(x)]$ and $[p^{-1}(x), b]$ give positive parametrizations of Γ_1 and Γ_2. Hence the equality (3.5.1) follows from $\int_a^{p^{-1}(x)} f(z)\,dg(z) + \int_{p^{-1}(x)}^b f(z)\,dg(z) = \int_a^b f(z)\,dg(z)$.

A sequence of oriented monotone curves $\{\Gamma_k\}_{k=1}^n$ with disjoint interiors is called a *chain* of monotone curves and denoted by $\sum_{k=1}^n \Gamma_k$. The body of a chain $C = \sum_{k=1}^n \Gamma_k$ is defined as $\bigcup_{k=1}^n |\Gamma_k|$ and denoted by $|C|$. The interior of the chain is defined as the union of interiors of its elements.

The integral of a form $f\,dg$ along the chain is defined as $\int_{\sum_{k=1}^n \Gamma_k} f\,dg = \sum_{k=1}^n \int_{\Gamma_k} f\,dg$.

One says that two chains $\sum_{k=1}^n \Gamma_k$ and $\sum_{k=1}^m \Gamma'_k$ have the same orientation, if the orientations induced by Γ_k and Γ'_j on $\Gamma_k \cap \Gamma'_j$ coincide in the case when $\Gamma_k \cap \Gamma'_j$ has a nonempty interior. Two chains with the same body and orientation are called *equivalent*.

LEMMA 3.5.2. *If two chains $C = \sum_{k=1}^n \Gamma_k$ and $C' = \sum_{k=1}^m \Gamma'_k$ are equivalent then the integrals along these chains coincide for any form $f dg$.*

PROOF. For any interior point x of the chain C, one defines the subdivision of C by x as $\Gamma_j^+ + \Gamma_j^- + \sum_{k=1}^n \Gamma_k[k \neq j]$, where Γ_j is the curve containing x and $\Gamma_j^+ + \Gamma_j^-$ is the partition of Γ by x. The subdivision does not change the integral along the chain due to the Partition Rule.

Hence we can subdivide C step by step by endpoints of C' to construct a chain Q whose endpoints include all endpoints of P'. And the integral along Q is the same as along P. Another possibility to construct Q is to subdivide C' by endpoints of C. This construction shows that the integral along Q coincides with the integral along C'. Hence the integrals along C and C' coincide. □

Due to this lemma, one can introduce the integral of a differential form along any oriented piecewise monotone curve Γ. To do this one considers a *monotone partition* of Γ into a sequence $\{\Gamma_k\}_{k=1}^n$ of monotone curves with disjoint interiors and equip all Γ_k with the induced orientation. One gets a chain and the integral along this chain does not depend on the partition.

Contour integral. A *domain* D is defined as a connected bounded part of the plane with piecewise monotone boundary. The boundary of D denoted ∂D is the union of finitely many monotone curves. And we suppose that $\partial D \subset D$, that is we consider a closed domain.

For a monotone curve Γ, which is contained in the boundary of a domain D, one defines the *induced orientation* of Γ by D as the orientation of a parametrization which leaves D on the left during the movement along Γ around D.

One introduces the integral $\oint_{\partial D} f(z)dg(z)$ as the integral along any chain whose body coincides with ∂D and whose orientations of curves are induced by D.

Due to Lemma 3.5.2 the choice of chain does not affect the integral.

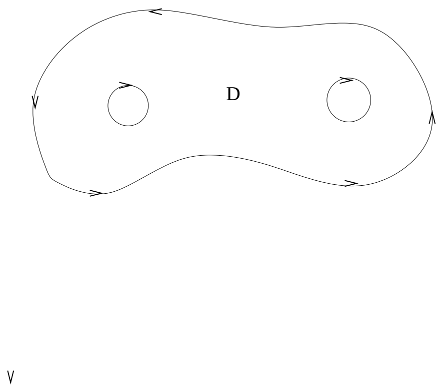

FIGURE 3.5.1. Contour integral

LEMMA 3.5.3. *Let D be a domain and l be either a vertical or a horizontal line, which bisects D into two parts: D' and D'' lying on the different sides of l. Then $\oint_{\partial D} f(z)dz = \oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz$.*

PROOF. The line l intersects the boundary of D in a finite sequence of points and intervals $\{J_k\}_{k=1}^m$.

Set $\partial' D = \partial D \cap \partial D'$ and $\partial'' D = \partial D \cap \partial D''$. The intersection $\partial' D \cap \partial'' D$ consists of finitely many points. Indeed, the interior points of J_k do not belong to this intersection, because their small neighborhoods have points of D only from one side of l. Hence

$$\int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

The boundary of D' consists of $\partial' D$ and some number of intervals. And the boundary of D'' consists of $\partial'' D$ and the same intervals, but with opposite orientation. Therefore

$$L = \int_{l\cap\partial D'} f(z)\,dz = -\int_{l\cap\partial D''} f(z)\,dz.$$

On the other hand

$$\oint_{\partial D'} f(z)dz = \int_{\partial' D} f(z)\,dz + L \text{ and } \oint_{\partial D''} f(z)dz = \int_{\partial'' D} f(z)\,dz - L,$$

hence

$$\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \int_{\partial' D} f(z)\,dz + \int_{\partial'' D} f(z)\,dz = \oint_{\partial D} f(z)dz.$$

□

LEMMA 3.5.4 (Estimation). *If $|f(z)| \leq B$ for any z from a body of a chain $C = \sum_{k=1}^n \Gamma_k$, then $\left|\int_C f(z)\,dz\right| \leq 4Bn \operatorname{diam}|C|$.*

PROOF. By Lemma 3.3.6 for any k one has $\left|\int_{\Gamma_k} f(z)\,dz\right| \leq 4B|A_k - B_k| \leq 4B \operatorname{diam}|C|$ where A_k and B_k are endpoints of Γ_k. The summation of these inequalities proves the lemma. □

THEOREM 3.5.5 (Cauchy). *If a function f is complex differentiable in a domain D then $\oint_{\partial D} f(z)dz = 0$.*

PROOF. Fix a rectangle R with sides parallel to the coordinate axis which contains D and denote by $|R|$ its area and by P its perimeter.

The proof is by contradiction. Suppose $\left|\oint_{\partial D} f(z)\,dz\right| \neq 0$. Denote by c the ratio of $\left|\oint_{\partial D} f(z)\,dz\right|/|R|$. We will construct a nested sequence of rectangles $\{R_k\}_{k=0}^{\infty}$ such that

- $R_0 = R$, $R_{k+1} \subset R_k$;
- R_{2k} is similar to R;
- $|\oint_{\partial(R_k\cap D)} f(z)\,dz| \geq c|R_k|$, where $|R_k|$ is the area of R_k.

The induction step: Suppose R_k is already constructed. Divide R_k in two equal rectanges R_k' and R_k'' by drawing either a vertical, if k is even, or a horizontal, if k is odd, interval joining the middles of the opposite sides of R_k. Set $D_k = D \cap R_k$, $D' = D \cap R_k'$, $D'' = D \cap R_k''$. We state that at least one of the following inequalities holds:

$$\left|\oint_{\partial D'} f(z)dz\right| \geq c|R_k'|, \qquad \left|\oint_{\partial D''} f(z)dz\right| \geq c|R_k''|. \tag{3.5.2}$$

Indeed, in the opposite case one gets

$$\left|\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz\right| < c|R_k'| + c|R_k'| = c|R_k|.$$

Since $\oint_{\partial D'} f(z)dz + \oint_{\partial D''} f(z)dz = \oint_{\partial D_k} f(z)dz$ by Lemma 3.5.3 we get a contradiction with the hypothesis $|\int_{p_k} f(z)\,dz| \geq c|R_k|$. Hence, one of the inequalities (3.5.2) holds. If the first inequality holds we set $R_{k+1} = R_k'$ else we set $R_{k+1} = R_k''$.

After constructing the sequence $\{R_k\}$, consider a point z_0 belonging to $\bigcap_{k=1}^{\infty} R_k$. This point belongs to D, because all its neighborhoods contain points of D. Consider the linearization $f(z) = f(z_0)+f'(z_0)(z-z_0)+o(z)(z-z_0)$. Since $\oint_{\partial D_k}(f(z_0)+f'(z_0)(z-z_0))dz = 0$ one gets

$$\left|\oint_{\partial D_k} o(z)(z-z_0)dz\right| = \left|\oint_{\partial D_k} f(z)dz\right| \geq c|R_k|. \tag{3.5.3}$$

The boundary of D_k is contained in the union $\partial R_k \cup R_k \cap \partial D$. Consider a monotone partition $\partial D = \sum_{k=1}^{n} \Gamma_k$. Since the intersection of R_k with a monotone curve is a monotone curve, one concludes that $\partial D \cap R_k$ is a union of at most n monotone curves. As ∂R_k consists of 4 monotone curves we get that ∂D_k is as a body of a chain with at most $4+n$ monotone curves.

Denote by P_k the perimeter of R_k. And suppose that $o(x)$ is bounded in R_k by a constant o_k. Then $|o(x)(z-z_0)| \leq P_k o_k$ for all $z \in R_k$.

Since $\operatorname{diam} \partial D_k \leq \frac{P_k}{2}$ by the Estimation Lemma 3.5.4, we get the following inequality:

$$\left|\oint_{\partial D_k} o(z)(z-z_0)dz\right| \leq 4(4+n)P_k o_k \frac{P_k}{2} = 2(4+n)o_k P_k^2. \tag{3.5.4}$$

The ratio $P_k^2/|R_k|$ is constant for even k. Therefore the inequalities (3.5.3) and (3.5.4) contradict each other for $o_k < \frac{c|R_k|}{2(4+n)P_k^2} = \frac{c|R|}{2(4+n)P^2}$. However the inequality $|o(x)| < \frac{c|R|}{2(4+n)P^2}$ holds for some neighborhood V of z_0 as $o(x)$ is infinitesimally small at z_0. This is a contradiction, because V contains some R_{2k}. □

Residues. By $\oint_{z_0}^r f(z)\,dz$ we denote the integral along the boundary of the disk $\{|z-z_0|\le r\}$.

LEMMA 3.5.6. *Suppose a function $f(z)$ is complex differentiable in the domain D with the exception of a finite set of points $\{z_k\}_{k=1}^n$. Then*

$$\oint_{\partial D} f(z)dz = \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz,$$

where r is so small that all disks $|z-z_k|<r$ are contained in D and disjoint.

PROOF. Denote by D' the complement of the union of the disks in D. Then $\partial D'$ is the union of ∂D and the boundary circles of the disks. By the Cauchy Theorem 3.5.5, $\oint_{\partial D'} f(z)dz = 0$. On the other hand this integral is equal to the sum $\oint_{\partial D} f(z)dz$ and the sum of integrals along boundaries of the circles. The orientation induced by D' onto the boundaries of these circles is opposite to the orientation induced from the circles. Hence

$$0 = \oint_{\partial D'} f(z)dz = \oint_{\partial D} f(z)dz - \sum_{k=1}^{n} \oint_{z_k}^{r} f(z)\,dz.$$

□

A *singular point* of a complex function is defined as a point where either the function or its derivative are not defined. A singular point is called isolated, if it has a neighborhood, where it is the only singular point. A point is called a *regular point* if it not a singular point.

One defines the *residue* of f at a point z_0 and denotes it as $\operatorname{res}_{z_0} f$ as the limit $\lim_{r\to 0} \frac{1}{2\pi i} \oint_{z_0}^r f(z)dz$. The above lemma shows that this limit exists for any isolated singular point and moreover, that all integrals along sufficiently small circumferences in this case are the same.

Since in all regular points the residues are 0 the conclusion of Lemma 3.5.6 for a function with finitely many singular points can be presented in the form:

$$\boxed{\oint_{\partial D} f(z)dz = 2\pi i \sum_{z\in D} \operatorname{res}_z f.} \tag{3.5.5}$$

An isolated singular point z_0 is called a *simple pole* of a function $f(z)$ if there exists a nonzero limit $\lim_{z\to z_0} f(z)(z-z_0)$.

LEMMA 3.5.7. *If z_0 is a simple pole of $f(z)$ then* $\operatorname{res}_{z_0} f = \lim_{z\to z_0}(z-z_0)f(z)$.

PROOF. Set $L = \lim_{z\to z_0}(z-z_0)f(z)$. Then $f(z) = L + \frac{o(z)}{(z-z_0)}$, where $o(z)$ is infinitesimally small at z_0. Hence

$$\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0} = \oint_{z_0}^{r} f(z)\,dz - \oint_{z_0}^{r} \frac{L}{z-z_0}\,dz. \tag{3.5.6}$$

Since the second integral from the right-hand side of (3.5.6) is equal to $2L\pi i$ and the other one is equal to $2\pi i \operatorname{res}_{z_0} f$ for sufficiently small r, we conclude that the integral from the left-hand side also is constant for sufficiently small r. To prove that $L = \operatorname{res}_{z_0} f$ we have to prove that this constant $c = \lim_{r\to 0} \oint_{z_0}^r \frac{o(z)}{z-z_0}\,dz$ is 0. Indeed, assume that $|c|>0$. Then there is a neighborhood U of z_0 such that $|o(z)| \le \frac{|c|}{32}$

for all $z \in U$. Then one gets a contradiction by estimation of $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right|$ (which is equal to $|c|$ for sufficiently small r) from above by $\frac{|c|}{\sqrt{2}}$ for r less than the radius of U. Indeed, the integrand is bounded by $\frac{|c|}{32r}$ and the path of integration (the circle) can be divided into four monotone curves of diameter $r\sqrt{2}$: quarters of the circle. Hence by the Estimation Lemma 3.5.4 one gets $\left|\oint_{z_0}^{r} \frac{o(z)\,dz}{z-z_0}\right| \leq 16\sqrt{2}\frac{|c|}{32} = \frac{|c|}{\sqrt{2}}$. □

REMARK 3.5.8. *Denote by* $\Gamma(r, \phi, z_0)$ *an arc of the circle* $|z - z_0| = r$*, whose angle measure is* ϕ*. Under the hypothesis of Lemma 3.5.7 the same arguments prove the following*

$$\lim_{r\to 0} \int_{\Gamma(\phi,r,0z)} f(z)\,dz = i\phi \lim_{z\to z_0} f(z)(z - z_0).$$

Problems.

1. Evaluate $\oint_1^1 \frac{dz}{1+z^4}$.
2. Evaluate $\oint_0^1 \frac{dz}{\sin z}$.
3. Evaluate $\oint_0^1 \frac{dz}{e^z-1}$.
4. Evaluate $\oint_0^1 \frac{dz}{z^2}$.
5. Evaluate $\oint_0^1 \sin\frac{1}{z}\,dz$.
6. Evaluate $\oint_0^1 ze^{\frac{1}{z}}\,dz$.
7. Evaluate $\oint_0^{5/2} z^2 \cot \pi z\,dz$.
8. Evaluate $\oint_2^{\frac{1}{2}} \frac{z\,dz}{(z-1)(z-2)^2}$.
9. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{5+3\cos\phi}$.
10. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{(1+\cos^2\phi)^2}$.
11. Evaluate $\int_0^{2\pi} \frac{d\phi}{(1+\cos\phi)^2}$.
12. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$.
13. Evaluate $\int_0^{+\infty} \frac{dx}{(1+x^2)(4+x^2)}$.
14. Evaluate $\int_{-\infty}^{+\infty} \frac{1+x^2}{1+x^4}$.
15. Evaluate $\int_{-\infty}^{+\infty} \frac{x^3}{1+x^6}\,dx$.

3.6. Analytic Functions

On the contents of the lecture. This lecture introduces the reader into the phantastically beautiful world of analytic functions. Integral Cauchy formula, Taylor series, Fundamental Theorem of Algebra. The reader will see all of these treasures in a single lecture.

THEOREM 3.6.1 (Integral Cauchy Formula). *If function f is complex differentiable in the domain D, then for any interior point $z \in D$ one has:*

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\partial D}\frac{f(\zeta)\,dz}{\zeta - z}}$$

PROOF. The function $\frac{f(z)}{z-z_0}$ has its only singular point inside the circle. This is z_0, which is a simple pole. The residue of $\frac{f(z)}{z-z_0}$ by Lemma 3.5.7 is equal to $\lim_{z\to z_0}(z-z_0)\frac{f(z)}{z-z_0} = \lim_{z\to z_0} f(z) = f(z_0)$. And by the formula (3.5.5) the integral is equal to $2\pi i f(z_0)$. □

LEMMA 3.6.2. *Let $\sum_{k=1}^{\infty} f_k$ be a series of virtually monotone complex functions, which is termwise majorized by a convergent positive series $\sum_{k=1}^{\infty} c_k$ on a monotone curve Γ (that is $|f_k(z)| \le c_k$ for natural k and $z \in \Gamma$) and such that $F(z) = \sum_{k=1}^{\infty} f_k(z)$ is virtually monotone. Then*

$$\sum_{k=1}^{\infty}\int_{\Gamma} f_k(z)\,dz = \int_{\Gamma}\sum_{k=1}^{\infty} f_k(z)\,dz. \tag{3.6.1}$$

PROOF. By the Estimation Lemma 3.5.4 one has the following inequalities:

$$\left|\int_{\Gamma} f_k(z)\,dz\right| \le 4c_k \operatorname{diam}\Gamma, \qquad \left|\int_{\Gamma}\sum_{k=n}^{\infty} f_k(z)\,dz\right| \le 4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k. \tag{3.6.2}$$

Set $F_n(z) = \sum_{k=1}^{n-1} f_k(z)$. By the left inequality of (3.6.2), the module of difference between $\int_\Gamma F_n(z)\,dz = \sum_{k=1}^{n-1}\int_\Gamma f_k(z)\,dz$ and the left-hand side of (3.6.1) does not exceed $4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k$. Hence this module is infinitesimally small as n tends to infinity. On the other hand, by the right inequality of (3.6.2) one gets $\left|\int_\Gamma F_n(z)\,dz - \int_\Gamma F(z)\,dz\right| \le 4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k$. This implies that the difference between the left-hand and right-hand sides of (3.6.1) is infinitesimally small as n tends to infinity. But this difference does not depend on n. Hence it is zero. □

LEMMA 3.6.3. *If a real function f defined over an interval $[a, b]$ is locally bounded, then it is bounded.*

PROOF. The proof is by contradiction. Suppose that f is unbounded. Divide the interval $[a, b]$ in half. Then the function has to be unbounded at least on one of the halves. Consider this half and divide it in half. Choose the half where the function is unbounded. So we construct a nested infinite sequence of intervals converging to a point, which coincides with the intersection of all the intervals. And f is obviously not locally bounded at this point. □

COROLLARY 3.6.4. *A complex function $f(z)$ continuous on the boundary of a domain D is bounded on ∂D.*

PROOF. Consider a path $p\colon [a, b] \to \partial D$. Then $|f(p(t))|$ is continuous on $[a, b]$, hence it is locally bounded, hence it is bounded. Since ∂D can be covered by images of finitely many paths this implies boundedness of f over ∂D. □

THEOREM 3.6.5. *If a function $f(z)$ is complex differentiable in the disk $|z-z_0| \leq R$, then for $|z - z_0| < R$*

$$f(z) = \sum_{k=0}^{\infty}(z - z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta,$$

where the series on the right-hand side absolutely converges for $|z - z_0| < R$.

PROOF. Fix a point z such that $|z - z_0| < R$ and consider ζ as a variable. For $|\zeta - z_0| > |z - z_0|$ one has

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}. \tag{3.6.3}$$

On the circle $|\zeta - z_0| = R$ the series on the right-hand side is majorized by the convergent series $\sum_{k=0}^{\infty} \frac{|z-z_0|^k}{R^{k+1}}$ for $r > |z - z_0|$. The function $f(\zeta)$ is bounded on $|\zeta - z_0| = R$ by Corollary 3.6.4. Therefore after multiplication of (3.6.3) by $f(\zeta)$ all the conditions of Lemma 3.6.2 are satisfied. Termwise integration gives:

$$f(z) = \oint_{z_0}^{R} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=0}^{\infty}(z - z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{k+1}}.$$

□

Analytic functions. A function $f(z)$ of complex variable is called an *analytic function* in a point z_0 if there is a positive ε such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ for all z from a disk $|z - z_0| \leq \varepsilon$ and the series absolutely converges. Since one can differentiate power series termwise (Theorem 3.3.9), any function which is analytic at z is also complex differentiable at z. Theorem 3.6.5 gives a converse. Thus, we get the following:

COROLLARY 3.6.6. *A function $f(z)$ is analytic at z if and only if it is complex differentiable in some neighborhood of z.*

THEOREM 3.6.7. *If f is analytic at z then f' is analytic at z. If f and g are analytic at z then $f + g$, $f - g$, fg are analytic at z. If f is analytic at z and g is analytic at $f(z)$ then $g(f(z))$ is analytic at z.*

PROOF. Termwise differentiation of the power series representing f in a neighborhood of z gives the power series for its derivative. Hence f' is analytic. The differentiability of $f \pm g$, fg and $g(f(z))$ follow from corresponding differentiation rules. □

LEMMA 3.6.8 (Isolated Zeroes). *If $f(z)$ is analytic and is not identically equal to 0 in some neighborhood of z_0, then $f(z) \neq 0$ for all $z \neq z_0$ sufficiently close to z_0.*

PROOF. Let $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ in a neighborhood U of z_0. Let c_m be the first nonzero coefficient. Then $\sum_{k=m}^{\infty} c_k(z - z_0)^{k-m}$ converges in U to a differentiable function $g(z)$ by Theorem 3.3.9. Since $g(z_0) = c_m \neq 0$ and $g(z)$ is continuous at z_0, the inequality $g(z) \neq 0$ holds for all z sufficiently close to z_0. As $f(z) = g(z)(z - z_0)^m$, the same is true for $f(z)$. □

THEOREM 3.6.9 (Uniqueness Theorem). *If two power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z - z_0)^k$ converge in a neighborhood of z_0 and their sums coincide for some infinite sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \neq z_0$ for all k and $\lim_{k\to\infty} z_k = z_0$, then $a_k = b_k$ for all k.*

PROOF. Set $c_k = a_k - b_k$. Then $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ has a non-isolated zero at z_0. Hence $f(z) = 0$ in a neighborhood of z_0. We get a contradiction by considering the function $g(z) = \sum_{k=m}^{\infty} c_k(z - z_0)^{k-m}$, which is nonzero for all z sufficiently close to z_0 (cf. the proof of the Isolated Zeroes Lemma 3.6.8), and satisfies the equation $f(z) = g(z)(z - z_0)^m$. □

Taylor series. Set $f^{(0)} = f$ and by induction define the $(k + 1)$-th derivative $f^{(k+1)}$ of f as the derivative of its k-th derivative $f^{(k)}$. For the first and the second derivatives one prefers the notation f' and f''. For example, the k-th derivative of z^n is $n^{\underline{k}} z^{n-k}$. (Recall that $n^{\underline{k}} = n(n - 1)\dots(n - k + 1)$.)

The following series is called the *Taylor series* of a function f at point z_0:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

The Taylor series is defined for any analytic function, because an analytic function has derivative of any order due to Theorem 3.6.7.

THEOREM 3.6.10. *If a function f is analytic in the disk $|z - z_0| < r$ then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ for any z from the disk.*

PROOF. By Theorem 3.6.5, $f(z)$ is presented in the disk by a convergent power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$. To prove our theorem we prove that

$$a_k = \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)}{k!}. \tag{3.6.4}$$

Indeed, $a_0 = f(z_0)$ and termwise differentiatiion of $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ applied n times gives $f^{(n)}(z) = \sum_{k=n}^{\infty} k^{\underline{n}} a_k(z - z_0)^k$. Putting $z = z_0$, one gets $f^{(n)}(z_0) = n^{\underline{n}} a_n = a_n n!$. □

THEOREM 3.6.11 (Liouville). *If a function f is analytic and bounded on the whole complex plane, then f is constant.*

PROOF. If f is analytic on the whole plane then $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where a_k is defined by (3.6.4). If $|f(z)| \leq B$ by the Estimation Lemma 3.5.4 one gets

$$|a_k| = \left| \oint_0^R \frac{f(\zeta)}{z^{k+1}}\, d\zeta \right| \leq 4 \cdot 4 \frac{B}{R^{k+1}} \frac{R}{\sqrt{2}} = \frac{C}{R^k}. \tag{3.6.5}$$

Consequently a_k for $k > 0$ is infinitesimally small as R tends to infinity. But a_k does not depend on R, hence it is 0. Therefore $f(z) = a_0$. □

THEOREM 3.6.12 (Fundamental Theorem of Algebra). *Any nonconstant polynomial $P(z)$ has a complex root.*

PROOF. If $P(z)$ has no roots the function $f(z) = \frac{1}{P(z)}$ is analytic on the whole plane. Since $\lim_{z\to\infty} f(z) = 0$ the inequality $|f(z)| < 1$ holds for $|z| = R$ if R is sufficiently large. Therefore the estimation (3.6.5) for the k-th coefficient of f holds with $B = 1$ for sufficiently large R. Hence the same arguments as in proof of the Liouville Theorem 3.6.11 show that $f(z)$ is constant. This is a contradiction. □

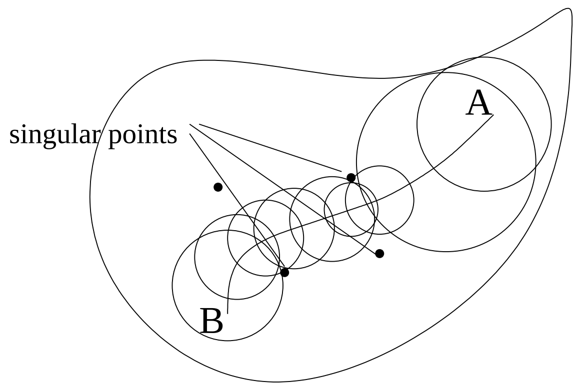

FIGURE 3.6.1. Analytic continuation

Analytic continuation.

LEMMA 3.6.13. *If an analytic function $f(z)$ has finitely many singular points in a domain D and a non isolated zero at a point $z_0 \in D$ then $f(z) = 0$ for all regular $z \in D$.*

PROOF. For any nonsingular point $z \in D$, we construct a sequence of sufficiently small disks $D_0, D_1, D_2, \dots, D_n$ without singular points with the following properties: 1) $z_0 \in D_0 \subset U$; 2) $z \in D_n$; 3) z_k, the center of D_k, belongs to D_{k-1} for all $k > 0$. Then by induction we prove that $f(D_k) = 0$. First step: if z_0 is a non-isolated zero of f, then the Taylor series of f vanishes at z_0 by the Uniqueness Theorem 3.6.9. But this series represents $f(z)$ on D_0 due to Theorem 3.6.10, since D_0 does not contain singular points. Hence, $f(D_0) = 0$. Suppose we have proved already that $f(D_k) = 0$. Then z_{k+1} is a non-isolated zero of f by the third property of the sequence $\{D_k\}_{k=0}^n$. Consequently, the same arguments as above for $k = 0$ prove that $f(D_{k+1}) = 0$. And finally we get $f(z) = 0$. □

Consider any formula which you know from school about trigonometric functions. For example, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. The above lemma implies that this formula remains true for complex x and y. Indeed, consider the function $T(x, y) = \tan(x + y) - \frac{\tan x + \tan y}{1 - \tan x \tan y}$. For a fixed x the function $T(x, y)$ is analytic and has finitely many singular points in any disk. This function has non-isolated zeroes in all real points, hence this function is zero in any disk intersecting the real line. This implies that $T(x, y)$ is zero for all y. The same arguments applied to $T(x, y)$ with fixed y and variable x prove that $T(x, y)$ is zero for all complex x, y.

The same arguments prove the following theorem.

THEOREM 3.6.14. *If some analytic relation between analytic functions holds on a curve Γ, it holds for any $z \in \mathbb{C}$, which can be connected with Γ by a paths avoiding singular points of the functions.*

LEMMA 3.6.15. $\sin t \geq \frac{2t}{\pi}$ *for* $t \in [0, \pi/2]$.

PROOF. Let $f(t) = \sin t - \frac{2t}{\pi}$. Then $f'(x) = \cos t - \frac{2}{\pi}$. Set $y = \arccos \frac{2}{\pi}$. Then $f'(x) \geq 0$ for $x \in [0, y]$. Therefore f is nondecreasing on $[0, y]$, and nonnegative, because $f(0) = 0$. On the interval $[y, \pi/2]$ the derivative of f is negative. Hence $f(x)$ is non-increasing and nonnegative, because its value on the end of the interval is 0. □

LEMMA 3.6.16 (Jordan). *Let $f(z)$ be an analytic function in the upper half-plane such that $\lim_{z\to\infty} f(z) = 0$. Denote by Γ_R the upper half of the circle $|z| = R$. Then for any natural m*

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) \exp(miz)\, dz = 0. \tag{3.6.6}$$

PROOF. Consider the parametrization $z(t) = R\cos t + Ri \sin t$, $t \in [0, \pi]$ of Γ_R. Then the integral (3.6.6) turns into

$$\begin{aligned} &\int_0^\pi f(z) \exp(iRm \cos t - Rm \sin t)\, d(R\cos t + Ri \sin t) \\ &\qquad = \int_0^\pi Rf(z) \exp(iRm \cos t) \exp(-Rm \sin t)(-\sin t + i \cos t)\, dt. \end{aligned} \tag{3.6.7}$$

If $|f(z)| \leq B$ on Γ_R, then $|f(z) \exp(iRm \cos t)(-\sin t + i \cos t)| \leq B$ on Γ_R. And the module of the integral (3.6.7) can be estimated from above by

$$BR \int_0^\pi \exp(-Rm \sin t)\, dt.$$

Since $\sin(\pi - t) = \sin t$, the latter integral is equal to $2BR \int_0^{\pi/2} \exp(-Rm \sin t)\, dt$. Since $\sin t \geq \frac{2t}{\pi}$, the latter integral does not exceed

$$2BR \int_0^{\pi/2} \exp(-2Rmt/\pi)\, dt = 2BR \frac{1 - \exp(-Rm)}{2Rm} \leq \frac{B}{m}.$$

Since B can be chosen arbitrarily small for sufficiently large R, this proves the lemma. □

Evaluation of $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \lim_{N\to\infty} \int_{-N}^{N} \frac{\sin x}{x}\, dx$. Since $\sin x = \operatorname{Im} e^{ix}$ our integral is equal to $\operatorname{Im} \int_{-\infty}^{+\infty} \frac{e^{iz}}{z}\, dz$. Set $\Gamma(r) = \{z \mid |z| = r, \operatorname{Im} z \geq 0\}$. This is a semicircle. Let us orient it counter-clockwise, so that its initial point is r.

Consider the domain $D(R)$ bounded by the semicircles $-\Gamma(r)$, $\Gamma(R)$ and the intervals $[-R, -r]$, $[r, R]$, where $r = \frac{1}{R}$ and $R > 1$. The function $\frac{e^{iz}}{z}$ has no singular points inside $D(R)$. Hence $\oint_{\partial D(R)} \frac{e^{iz}}{z} dz = 0$. Hence for any R

$$\int_{-r}^{-R} \frac{e^{iz}}{z}\, dz + \int_r^R \frac{e^{iz}}{z}\, dz = \int_{\Gamma(r)} \frac{e^{iz}}{z}\, dz - \int_{\Gamma(R)} \frac{e^{iz}}{z}\, dz. \tag{3.6.8}$$

The second integral on the right-hand side tends to 0 as R tends to infinity due to Jordan's Lemma 3.6.16. The function $\frac{e^{iz}}{z}$ has a simple pole at 0, hence the first

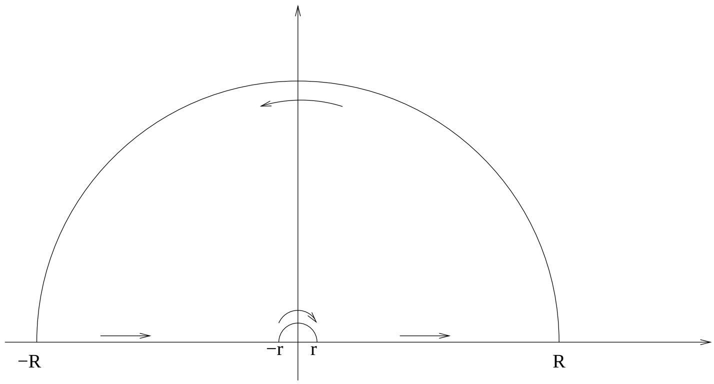

FIGURE 3.6.2. The domain $D(R)$

integral on the right-hand side of (3.6.8) tends to $\pi i \operatorname{res} \frac{e^{iz}}{z} = \pi i$ due to Remark 3.5.8. As a result, the right-hand side of (3.6.8) tends to πi as R tends to infinity. Consequently the left-hand side of (3.6.8) also tends to πi as $R \to \infty$. The imaginary part of left-hand side of (3.6.8) is equal to $\int_{-R}^{R} \frac{\sin x}{x}\, dx - \int_{-r}^{r} \frac{\sin x}{x}\, dx$. The last integral tends to 0 as $r \to 0$, because $|\frac{\sin x}{x}| \le 1$. Hence $\int_{-R}^{R} \frac{\sin x}{x}\, dx$ tends to π as $R \to \infty$. Finally $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \pi$.

Problems.

1. Prove that an even analytic function f, i.e., a function such that $f(z) = f(-z)$, has a Taylor series at 0 consisting only of even powers.

2. Prove that analytic function which has a Taylor series only with even powers is an even function.

3. Prove: If an analytic function $f(z)$ takes real values on $[0, 1]$, then $f(x)$ is real for any real x.

4. Evaluate $\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx$.

5. Evaluate $\int_{-\pi}^{+\pi} \frac{d\phi}{5+3\cos\phi}$.

6. Evaluate $\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2}\, dx$ $(a > 0)$.

7. Evaluate $\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+4x+20}\, dx$.

8. Evaluate $\int_{0}^{\infty} \frac{\cos ax}{x^2+b^2}\, dx$ $(a, b > 0)$.

CHAPTER 4

Differences

4.1. Newton Series

On the contents of the lecture. The formula with the binomial series was engraved on Newton's gravestone in 1727 at Westminster Abbey.

Interpolation problem. Suppose we know the values of a function f at some points called *interpolation nodes* and we would like to interpolate the value of f at some point, not contained in the data. This is the so-called *interpolation problem.* Interpolation was applied in the computation of logarithms, maritime navigation, astronomical observations and in a lot of other things.

A natural idea is to construct a polynomial which takes given values at the interpolation nodes and consider its value at the point of interest as the interpolation. Values at $n+1$ points define a unique polynomial of degree n, which takes just these values at these points. In 1676 Newton discovered a formula for this polynomial, which is now called *Newton's interpolation formula.*

Consider the case, when interpolation nodes are natural numbers. Recall that the difference of a function f is the function denoted δf and defined by $\delta f(x) = f(x+1) - f(x)$. Define iterated differences $\delta^k f$ by induction: $\delta^0 f = f$, $\delta^{k+1} f = \delta(\delta^k f)$. Recall that $x^{\underline{k}}$ denotes the k-th factorial power $x^{\underline{k}} = x(x-1)\dots(x-k+1)$.

LEMMA 4.1.1. *For any polynomial $P(x)$, its difference $\Delta P(x)$ is a polynomial of degree one less.*

PROOF. The proof is by induction on the degree of $P(x)$. The difference is constant for any polynomial of degree 1. Indeed, $\delta(ax+b) = a$. Suppose the lemma is proved for polynomials of degree $\leq n$ and let $P(x) = \sum_{k=0}^{n+1} a_k x^k$ be a polynomial of degree $n+1$. Then $P(x) - a_{n+1}x^{\underline{n+1}} = Q(x)$ is a polynomial of degree $\leq n$. $\Delta P(x) = \Delta a x^{\underline{n+1}} + \Delta Q(x)$. By the induction hypothesis, $\Delta Q(x)$ has degree $\leq n-1$ and, as we know, $\Delta x^{\underline{n+1}} = (n+1)x^{\underline{n}}$ has degree n. □

LEMMA 4.1.2. *If $\Delta P(x) = 0$, and $P(x)$ is a polynomial, then $P(x)$ is constant.*

PROOF. If $\Delta P(x) = 0$, then degree of $P(x)$ cannot be positive by Lemma 4.1.1, hence $P(x)$ is constant. □

LEMMA 4.1.3 (Newton Polynomial Interpolation Formula). *For any polynomial $P(x)$*

$$P(x) = \sum_{k=0}^{\infty} \frac{\Delta^k P(0)}{k!} x^{\underline{k}}. \tag{4.1.1}$$

PROOF. If $P(x) = ax+b$, then $\Delta^0 P(0) = b$, $\Delta^1 P(0) = a$ and $\delta^k P(x) = 0$ for $k > 1$. Hence the Newton series (4.1.1) turns into $b + ax$. This proves our assertion for polynomials of degree ≤ 1. Suppose it is proved for polynomials of degree n. Consider $P(x)$ of degree $n+1$. Then $\Delta P(x) = \sum_{k=1}^{\infty} \frac{\Delta^k P(0)}{k!} x^{\underline{k}}$ by the induction hypothesis. Denote by $Q(x)$ the Newton series $\sum_{k=0}^{\infty} \frac{\Delta^k P(0)}{k!} x^{\underline{k}}$ for $P(x)$.

Then

$$\begin{aligned}\Delta Q(x) &= \sum_{k=0}^{\infty}\frac{\Delta^k P(0)}{k!}(x+1)^{\underline{k}} - \sum_{k=0}^{\infty}\frac{\Delta^k P(0)}{k!}x^{\underline{k}} \\ &= \sum_{k=0}^{\infty}\frac{\Delta^k P(0)}{k!}\Delta x^{\underline{k}} \\ &= \sum_{k=0}^{\infty}\frac{\Delta^k P(0)}{k!}kx^{\underline{k-1}} \\ &= \sum_{k=0}^{\infty}\frac{\Delta^k P(0)}{(k-1)!}x^{\underline{k-1}} \\ &= \sum_{k=0}^{\infty}\frac{\delta^k(\delta P(0))}{k!}x^{\underline{k}} \\ &= \Delta P(x).\end{aligned}$$

Hence $\Delta(P(x) - Q(x)) = 0$ and $P(x) = Q(x) + c$. Since $P(0) = Q(0)$, one gets $c = 0$. This proves $P(x) = Q(x)$. □

LEMMA 4.1.4 (Lagrange Formula). *For any sequence* $\{y_k\}_{k=0}^n$, *the polynomial* $L_n(x) = \sum_{k=0}^n (-1)^{n-k}\frac{y_k}{k!(n-k)!}\frac{x^{\underline{n+1}}}{x-k}$ *has the property* $L_n(k) = y_k$ *for* $0 \le k \le n$.

PROOF. For $x = k$, all terms of the sum $\sum_{k=0}^n (-1)^{n-k}\frac{y_k}{k!(n-k)!}\frac{x^{\underline{n+1}}}{x-k}$ but the k-th vanish, and $\frac{x^{\underline{k}}}{x-k}$ is equal to $k!(n-k)!(-1)^{n-k}$. □

LEMMA 4.1.5. *For any function* f *and for any natural number* $m \le n$ *one has* $f(m) = \sum_{k=0}^n \frac{\delta^k f(0)}{k!}m^{\underline{k}}$.

PROOF. Consider the Lagrange polynomial L_n such that $L_n(k) = f(k)$ for $k \le n$. Then $\delta^k L_n(0) = \delta^k f(0)$ for all $k \le n$ and $\delta^k L_n(0) = 0$ for $k > n$, because the degree of L_n is n. Hence, $L_n(x) = \sum_{k=0}^{\infty}\frac{\delta^k f(0)}{k!}x^{\underline{k}} = \sum_{k=0}^{n}\frac{\delta^k f(0)}{k!}x^{\underline{k}}$ by Lemma 4.1.3. Putting $x = m$ in the latter equality, one gets $f(m) = L_n(m) = \sum_{k=0}^{n}\frac{\delta^k f(0)}{k!}m^{\underline{k}}$. □

We see that the Newton polynomial gives a solution for the interpolation problem and our next goal is to estimate the interpolation error.

Theorem on extremal values. The least upper bound of a set of numbers A is called the *supremum* of A and denoted by $\sup A$. In particular, the ultimate sum of a positive series is the supremum of its partial sums. And the variation of a function on an interval is the supremum of its partial variations.

Dually, the greatest lower bound of a set A is called the *infinum* and denoted by $\inf A$.

THEOREM 4.1.6 (Weierstrass). *If a function* f *is continuous on an interval* $[a, b]$, *then it takes maximal and minimal values on* $[a, b]$.

PROOF. The function f is bounded by Lemma 3.6.3. Denote by B the supremum of the set of values of f on $[a, b]$. If f does not take the maximum value, then $f(x) \ne B$ for all $x \in [a, b]$. In this case $\frac{1}{B-f(x)}$ is a continuous function on $[a, b]$.

Hence it is bounded by Lemma 3.6.3. But the difference $B - f(x)$ takes arbitrarily small values, because $B - \varepsilon$ does not bound $f(x)$. Therefore $\frac{1}{B-f(x)}$ is not bounded. This is in contradiction to Lemma 3.6.3, which states that a locally bounded function is bounded. The same arguments prove that $f(x)$ takes its minimal value on $[a, b]$. □

THEOREM 4.1.7 (Rolle). *If a function f is continuous on the interval $[a, b]$, differentiable in interval (a, b) and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.*

PROOF. If the function f is not constant on $[a, b]$ then either its maximal value or its minimal value differs from $f(a) = f(b)$. Hence at least one of its extremal values is taken in some point $c \in (a, b)$. Then $f'(c) = 0$ by Lemma 3.2.1. □

LEMMA 4.1.8. *If an n-times differentiable function $f(x)$ has $n + 1$ roots in the interval $[a, b]$, then $f^{(n)}(\xi) = 0$ for some $\xi \in (a, b)$.*

PROOF. The proof is by induction. For $n = 1$ this is Rolle's theorem. Let $\{x_k\}_{k=0}^n$ be a sequence of roots of f. By Rolle's theorem any interval (x_i, x_{i+1}) contains a root of f'. Hence f' has $n - 1$ roots, and its $(n - 1)$-th derivative has a root. But the $(n - 1)$-th derivative of f' is the n-th derivative of f. □

THEOREM 4.1.9 (Newton Interpolation Formula). *Let f be an $n + 1$ times differentiable function on $I \supset [0, n]$. Then for any $x \in I$ there is $\xi \in I$ such that*

$$f(x) = \sum_{k=0}^{n} \frac{\delta^k f(0)}{k!} x^{\underline{k}} + \frac{f^{(k+1)}(\xi)}{(k+1)!} x^{\underline{k+1}}.$$

PROOF. The formula holds for $x \in \{0, 1, \dots n\}$ and any ξ, due to Lemma 4.1.5, because $x^{\underline{n+1}} = 0$ for such x. For other x one has $x^{\underline{n+1}} \neq 0$, hence there is C such that $f(x) = \sum_{k=0}^{n} \frac{\delta^k f(0)}{k!} x^{\underline{k}} + C x^{\underline{k+1}}$. The function $F(y) = f(y) - \sum_{k=0}^{n} \frac{\delta^k f(0)}{k!} x^{\underline{k}} - C y^{\underline{k+1}}$ has roots $0, 1, \dots, n, x$. Hence its $(n + 1)$-th derivative has a root $\xi \in I$. Since $\sum_{k=0}^{n} \frac{\delta^k f(0)}{k!} x^{\underline{k}}$ is a polynomial of degree n its $(n + 1)$-th derivative is 0. And the $(n + 1)$-th derivatives of $C x^{\underline{n+1}}$ and $C x^{n+1}$ coincide, because their difference is a polynomial of degree n. Hence $0 = F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - C(n + 1)!$ and $C = \frac{f^{(n+1)}(\xi)}{(n+1)!}$. □

Binomial series. The series $\sum_{k=0}^{\infty} \frac{\delta^k f(0)}{k!} x^{\underline{k}}$ is called the *Newton series* of a function f. The Newton series coincides with the function at all natural points. And sometimes it converges to the function. The most important example of such convergence is given by the so-called *binomial series*.

Consider the function $(1 + x)^y$. This is a function of two variables. Fix x and evaluate its difference with respect to y. One has $\delta_y(1+x)^y = (1+x)^{y+1} - (1+x)^y = (1+x)^y(1+x-1) = x(1+x)^y$. This simple formula allows us immediately to evaluate $\delta_y^k(1 + x)^y = x^k(1 + x)^y$. Hence the Newton series for $(1 + x)^y$ as function of y is

$$(1 + x)^y = \sum_{k=0}^{\infty} \frac{x^k y^{\underline{k}}}{k!}. \tag{4.1.2}$$

For fixed y and variable x, the formula (4.1.2) represents the famous Newton binomial series. Our proof is not correct. We applied Newton's interpolation formula, proved only for polynomials, to an exponential function. But Newton's original proof was essentially of the same nature. Instead of interpolation of the whole function, he interpolated coefficients of its power series expansion. Newton considered the discovery of the binomial series as one of his greatest discoveries. And the role of the binomial series in further developments is very important.

For example, Newton expands into a power series $\arcsin x$ in the following way. One finds the derivative of $\arcsin x$ by differentiating identity $\sin\arcsin x = x$. This differentiation gives $\cos(\arcsin x)\arcsin' x = 1$. Hence $\arcsin' x = \frac{1}{\cos\arcsin x} = (1-x^2)^{-\frac{1}{2}}$. Since

$$(1-x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\frac{(-x^2)^k(-\frac{1}{2})^{\underline{k}}}{k!} = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{2k!!}x^{2k}, \tag{4.1.3}$$

one gets the series for arcsin by termwise integration of (4.1.3). The result is

$$\arcsin x = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{2k!!}\frac{x^{2k+1}}{2k+1}.$$

It was more than a hundred years after the discovery Newton's Binomial Theorem that it was first completely proved by Abel.

THEOREM 4.1.10. *For any complex z and ζ such that $|z| < 1$, the series $\sum_{k=0}^{\infty}\frac{z^k\zeta^{\underline{k}}}{k!}$ absolutely converges to $(1+z)^\zeta = \exp(\zeta\ln(1+z))$.*

PROOF. The analytic function $\exp\zeta\ln(1+z)$ of variable z has no singular points in the disk $|z| < 1$, hence its Taylor series converges to it. The derivative of $(1+z)^\zeta$ by z is $\zeta(1+z)^{\zeta-1}$. The k-th derivative is $\zeta^{\underline{k}}(1+z)^{\zeta-k}$. In particular, the value of k-th derivative for $z=0$ is equal to $\zeta^{\underline{k}}$. Hence the Taylor series of the function is $\sum_{k=0}^{\infty}\frac{\zeta^{\underline{k}}z^k}{k!}$. □

On the boundary of convergence. Since $(1+z)^\zeta$ has its only singular point on the circle $|z| = 1$, and this point is -1, the binomial series for all z on the circle has $(1+z)^\zeta$ as its Abel's sum. In particular, for $z = 1$ one gets

$$\sum_{k=0}^{\infty}\frac{x^{\underline{k}}}{k!} = 2^x.$$

The series on the left-hand side converges for $x > 0$. Indeed, the series becomes alternating starting with $k > x$. The ratio $\frac{k-x}{k+1}$ of modules of terms next to each other is less then one. Hence the moduli of the terms form a monotone decreasing sequence onward $k > x$. And to apply the Leibniz Theorem 2.4.3, one needs only to prove that $\lim_{n\to\infty}\frac{x^{\underline{n}}}{n!} = 0$. Transform this limit into $\lim_{n\to\infty}\frac{x}{n}\prod_{k=1}^{n-1}(\frac{x}{k}-1)$. The product $\prod_{k=1}^{n-1}(\frac{x}{k}-1)$ contains at most x terms which have moduli greater than 1, and all terms of the product do not exceed x. Hence the absolute value of this product does not exceed x^x. And our sequence $\{\frac{x^{\underline{n}}}{n!}\}$ is majorized by an infinitesimally small $\{\frac{x^{x+1}}{n}\}$. Hence it is infinitesimally small.

Plain binomial theorem. For a natural exponent the binomial series contains only finitely many nonzero terms. In this case it turns into $(1+x)^n = \sum_{k=0}^{n}\frac{n^{\underline{k}}x^k}{k!}$.

Because $(a+b)^n = a^n(1+\frac{b}{a})^n$, one gets the following famous formula

$$(a+b)^n = \sum_{k=0}^{n+1} \frac{n^{\underline{k}}}{k!} a^k b^{n-k}.$$

This is the formula that is usually called Newton's Binomial Theorem. But this simple formula was known before Newton. In Europe it was proved by Pascal in 1654. Newton's discovery concerns the case of non integer exponents.

Symbolic calculus. One defines the *shift operation* $\mathbf{S}^a$ for a function f by the formula $\mathbf{S}^a f(x) = f(x+a)$. Denote by $\mathbf{1}$ the identity operation and by $\mathbf{S} = \mathbf{S}^1$. Hence $\mathbf{S}^0 = \mathbf{1}$. The composition of two operations is written as a product. So, for any a and b one has the following sum formula $\mathbf{S}^a\mathbf{S}^b = \mathbf{S}^{a+b}$.

We will consider only so-called *linear* operations. An operation O is called linear if $O(f+g) = O(f)+O(g)$ for all f, g and $O(kf) = kO(f)$ for any constant k. Define the sum $A+B$ of operations A and B by the formula $(A+B)f = Af+Bf$. Further, define the product of an operation A by a number k as $(kA)f = k(Af)$. For linear operations O, U, V one has the distributivity law $O(U+V) = OU+OV$. If the operations under consideration commute $UV = VU$, (for example, all iterations of the same operation commute) then they obey all usual numeric laws, and all identities which hold for numbers extend to operations. For example, $U^2 - V^2 = (U-V)(U+V)$, or the plain binomial theorem.

Let us say that an operation O is *decreasing* if for any polynomial P the degree of $O(P)$ is less than the degree of P. For example, the operation of difference $\delta = \mathbf{S} - \mathbf{1}$ and the operation $\mathbf{D}$ of differentiation $\mathbf{D}f(x) = f'(x)$ are decreasing. For a decreasing operation O, any power series $\sum_{k=0}^{\infty} a_k O^k$ defines an operation at least on polynomials, because this series applied to a polynomial contains only finitely many terms. Thus we can apply analytic functions to operations.

For example, the binomial series $(1+\delta)^y = \sum_{k=0}^{\infty} \frac{\delta^k y^{\underline{k}}}{k!}$ represents $\mathbf{S}^y$. And the equality $\mathbf{S}^y = \sum_{k=0}^{\infty} \frac{\delta^k y^{\underline{k}}}{k!}$, which is in fact the Newton Polynomial Interpolation Formula, is a direct consequence of binomial theorem. Another example, consider $\delta_n = \mathbf{S}^{\frac{x}{n}} - \mathbf{1}$. Then $\mathbf{S}^{\frac{x}{n}} = \mathbf{1} + \delta_n$ and $\mathbf{S}^x = (\mathbf{1}+\delta_n)^n$. Further, $\mathbf{S}^x = \sum_{k=0}^{n} \frac{n^{\underline{k}} \delta_n^k}{k!} = \sum_{k=0}^{\infty} \frac{n^{\underline{k}}}{n^k} \frac{(n\delta_n)^k}{k!}$. Now we follow Euler's method to "substitute $n = \infty$". Then $n\delta_n$ converts into $x\mathbf{D}$, and $\frac{n^{\underline{k}}}{n^k}$ turns into 1. As result we get the Taylor formula $\mathbf{S}^x = \sum_{k=0}^{\infty} \frac{x^k \mathbf{D}^k}{k!}$. Our proof is copied from the Euler proof in his *Introductio* of $\lim_{n\to\infty}(1+\frac{x}{n})^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. This substitution of infinity means passing to the limit. This proof is sufficient for decreasing operations on polynomials because the series contains only finitely many nonzero terms. In the general case problems of convergence arise.

The binomial theorem was the main tool for the expansion of functions into power series in Euler's times. Euler also applied it to get power expansions for trigonometric functions.

The Taylor expansion for $x = 1$ gives a symbolic equality $\mathbf{S} = \exp \mathbf{D}$. Hence $\mathbf{D} = \ln \mathbf{S} = \ln(\mathbf{1}+\delta) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\delta^k}{k}$. We get a formula for numerical differentiation. Symbolic calculations produce formulas which hold at least for polynomials.

Problems.

1. Prove $(x+y)^{\underline{n}} = \sum_{k=0}^{n} \frac{n^{\underline{k}} x^{\underline{k}} y^{\underline{n-k}}}{k!}$.

2. Evaluate $\sum_{k=0}^{n} \frac{n^{\underline{k}}}{k!} 2^{n-k}$.

3. Prove: If p is prime, then $\frac{p^{\underline{k}}}{k!}$ is divisible by p.

4. Prove: $\frac{n^{\underline{k}}}{k!} = \frac{n^{\underline{n-k}}}{(n-k)!}$.

5. Deduce the plain binomial theorem from multiplication of series for exponenta.

6. One defines the Catalan number c_n as the number of correct placement of brackets in the sum $a_1 + a_2 + \cdots + a_n$. Prove that Catalan numbers satisfy the following recursion equation $c_n = \sum_{k=0}^{n-1} c_k c_{n-k}$ and deduce a formula for Catalan numbers.

7. Prove that $\Delta^k x^{\underline{n}} x^{\underline{m}} = 0$ for $x = 0$ and $k < n$.

8. Prove that $\sum_{k=0}^{n} (-1)^k \frac{n^{\underline{k}}}{k!} = 0$.

9. Get a differential equation for the binomial series and solve it.

10. Prove $(a+b)^{\underline{n}} = \sum_{k=0}^{n} \frac{n^{\underline{k}}}{k!} a^{\underline{k}} b^{\underline{n-k}}$.

11. Prove: A sequence $\{a_k\}$ such that $\Delta^2 a_k \geq 0$ satisfies the inequality $\max\{a_1, \ldots, a_n\} \geq a_k$ for any k between 1 and n.

12. Prove $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k!} = 2^{x/2} \cos \frac{x\pi}{4}$.

13. Prove $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = 2^{x/2} \sin \frac{x\pi}{4}$.

14. Prove $\Delta^n 0^p$ is divisible by $p!$.

*__15.__ Prove that $\Delta^n 0^p = \sum_{k=0}^{n-1} (-1)^{n-k} \frac{n^{\underline{k}}}{k!} k^p$.

16. Prove $\cos^2 x + \sin^2 x = 1$ via power series.

4.2. Bernoulli Numbers

On the contents of the lecture. In this lecture we give explicit formulas for telescoping powers. These formulas involve a remarkable sequence of numbers, which were discovered by Jacob Bernoulli. They will appear in formulas for sums of series of reciprocal powers. In particular, we will see that $\frac{\pi^2}{6}$, the sum of Euler series, contains the second Bernoulli number $\frac{1}{6}$.

Summation Polynomials. Jacob Bernoulli found a general formula for the sum $\sum_{k=1}^n k^q$. To be precise he discovered that there is a sequence of numbers $B_0, B_1, B_2, \dots, B_n, \dots$ such that

$$\sum_{k=1}^{n} k^q = \sum_{k=0}^{q+1} B_k \frac{q^{\underline{k-1}} n^{q+1-k}}{k!}. \tag{4.2.1}$$

The first 11 of the *Bernoulli numbers* are $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}$. The right-hand side of (4.2.1) is a polynomial of degree $q+1$ in n. Let us denote this polynomial by $\psi_{q+1}(n)$. It has the following remarkable property: $\delta\psi_{q+1}(x) = (1+x)^q$. Indeed the latter equality holds for any natural value n of the variable, hence it holds for all x, because two polynomials coinciding in infinitely many points coincide. Replacing in (4.2.1) $q+1$ by m, n by x and reversing the order of summation, one gets the following:

$$\begin{aligned}\psi_m(x) &= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{m-k-1}}}{(m-k)!} x^k \\ &= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)!}{k!(m-k)!} x^k \\ &= \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{k-1}}}{k!} x^k.\end{aligned}$$

Today's lecture is devoted to the proof of this Bernoulli theorem.

Telescoping powers. Newton's Formula represents x^m as a factorial polynomial $\sum_{k=0}^n \frac{\delta^k 0^m}{k!} x^{\underline{k}}$, where $\Delta^k 0^m$ denotes the value of $\delta^k x^m$ at $x = 0$. Since $\delta x^{\underline{k}} = k x^{\underline{k-1}}$, one immediately gets a formula for a polynomial $\phi_{m+1}(x)$ which telescopes x^m in the form

$$\phi_{m+1}(x) = \sum_{k=0}^{\infty} \frac{\Delta^k 0^m}{(k+1)!} x^{\underline{k+1}}$$

This polynomial has the property $\phi_{m+1}(n) = \sum_{k=0}^{n-1} k^m$ for all n.

The polynomials $\phi_m(x)$, as follows from Lemma 4.1.2, are characterized by two conditions:

$$\Delta\phi_m(x) = x^{m-1}, \qquad \phi_m(1) = 0.$$

LEMMA 4.2.1 (on differentiation). $\phi'_{m+1}(x) = \phi'_{m+1}(0) + m\phi_m(x)$.

PROOF. Differentiation of $\Delta\phi_{m+1}(x) = x^m$ gives $\Delta\phi'_{m+1}(x) = mx^{m-1}$. The polynomial $m\phi_m$ has the same differences, hence $\Delta(\phi'_{m+1}(x) - m\phi_m(x)) = 0$. By Lemma 4.1.2 this implies that $\phi'_{m+1}(x) - m\phi_m(x)$ is constant. Therefore, $\phi'_{m+1}(x) -$

$m\phi_m(x) = \phi'_{m+1}(0) - m\phi_m(0)$. But $\phi_m(1) = 0$ and $\phi_m(0) = \phi_m(1) - \delta\phi_m(0) = 0 - 0^{m-1} = 0$. □

Bernoulli polynomials. Let us introduce the m-th *Bernoulli number* B_m as $\phi'_{m+1}(0)$, and define the *Bernoulli polynomial* of degree $m > 0$ as $B_m(x) = m\phi_m(x) + B_m$. The Bernoulli polynomial $B_0(x)$ of degree 0 is defined as identically equal to 1. Consequently $B_m(0) = B_m$ and $B'_{m+1}(0) = (m+1)B_m$.

The Bernoulli polynomials satisfy the following condition:

$$\Delta B_m(x) = mx^{m-1} \quad (m > 0).$$

In particular, $\Delta B_m(0) = 0$ for $m > 1$, and therefore we get the following *boundary conditions* for Bernoulli polynomials:

$$\begin{aligned} B_m(0) = B_m(1) = B_m \quad &\text{for } m > 1, \text{ and} \\ B_1(0) = -B_1(1) = B_1. & \end{aligned}$$

The Bernoulli polynomials, in contrast to $\phi_m(x)$, are *normed*: their leading coefficient is equal to 1 and they have a simpler rule for differentiation:

$$B'_m(x) = mB_{m-1}(x)$$

Indeed, $B'_m(x) = m\phi'_m(x) = m((m-1)\phi_{m-1}(x) + \phi'_m(0)) = mB_{m-1}(x)$, by Lemma 4.2.1.

Differentiating $B_m(x)$ at 0, k times, we get $B_m^{(k)}(0) = m^{\underline{k-1}}B'_{m-k+1}(0) = m^{\underline{k-1}}(m-k+1)B_{m-k} = m^{\underline{k}}B_{m-k}$. Hence the Taylor formula gives the following representation of the Bernoulli polynomial:

$$B_m(x) = \sum_{k=0}^{m} \frac{m^{\underline{k}} B_{m-k}}{k!} x^k.$$

Characterization theorem. The following important property of Bernoulli polynomials will be called the *Balance property*:

$$\int_0^1 B_m(x)\,dx = 0 \quad (m > 0). \tag{4.2.2}$$

Indeed, $\int_0^1 B_m(x)\,dx = \int_0^1 (m+1)B'_{m+1}(x)\,dx = \Delta B_{m+1}(0) = 0$.

The Balance property and the Differentiation rule allow us to evaluate Bernoulli polynomials recursively. Thus, $B_1(x)$ has 1 as leading coefficient and zero integral on $[0,1]$; this allows us to identify $B_1(x)$ with $x - 1/2$. Integration of $B_1(x)$ gives $B_2(x) = x^2 - x + C$, where C is defined by (4.2.2) as $-\int_0^1 x^2\,dx = \frac{1}{6}$. Integrating $B_2(x)$ we get $B_3(x)$ modulo a constant which we find by (4.2.2) and so on. Thus we obtain the following theorem:

THEOREM 4.2.2 (characterization). *If a sequence of polynomials $\{P_n(x)\}$ satisfies the following conditions:*

- $P_0(x) = 1$,
- $\int_0^1 P_n(x)\,dx = 0$ *for* $n > 0$,
- $P'_n(x) = nP_{n-1}(x)$ *for* $n > 0$,

then $P_n(x) = B_n(x)$ for all n.

Analytic properties.

LEMMA 4.2.3 (on reflection). *$B_n(x) = (-1)^n B_n(1-x)$ for any n.*

PROOF. We prove that the sequence $T_n(x) = (-1)^n B_n(1-x)$ satisfies all the conditions of Theorem 4.2.2. Indeed, $T_0 = B_0 = 1$,

$$\int_0^1 T_n(x)\,dx = (-1)^n \int_1^0 B_n(x)\,dx = 0$$

and

$$\begin{aligned} T_n(x)' &= (-1)^n B_n'(1-x) \\ &= (-1)^n n B_{n-1}(1-x)(1-x)' \\ &= (-1)^{n+1} n B_{n-1}(x) \\ &= n T_{n-1}(x). \end{aligned}$$

□

LEMMA 4.2.4 (on roots). *For any odd $n > 1$ the polynomial $B_n(x)$ has on $[0,1]$ just three roots: $0, \frac{1}{2}, 1$.*

PROOF. For odd n, the reflection Lemma 4.2.3 implies that $B_n(\frac{1}{2}) = -B_n(\frac{1}{2})$, that is $B_n(\frac{1}{2}) = 0$. Furthermore, for $n > 1$ one has $B_n(1) - B_n(0) = n0^{n-1} = 0$. Hence $B_n(1) = B_n(0)$ for any Bernoulli polynomial of degree $n > 1$. By the reflection formula for an odd n one obtains $B_n(0) = -B_n(1)$. Thus any Bernoulli polynomial of odd degree greater than 1 has roots $0, \frac{1}{2}, 1$.

The proof that there are no more roots is by contradiction. In the opposite case consider $B_n(x)$, of the least odd degree > 1 which has a root α different from the above mentioned numbers. Say $\alpha < \frac{1}{2}$. By Rolle's Theorem 4.1.7 $B_n'(x)$ has at least three roots $\beta_1 < \beta_2 < \beta_3$ in $(0,1)$. To be precise, $\beta_1 \in (0,\alpha)$, $\beta_2 \in (\alpha, \frac{1}{2})$, $\beta_3 \in (\frac{1}{2}, 1)$. Then $B_{n-1}(x)$ has the same roots. By Rolle's Theorem $B_{n-1}'(x)$ has at least two roots in $(0,1)$. Then at least one of them differs from $\frac{1}{2}$ and is a root of $B_{n-2}(x)$. By the minimality of n one concludes $n-2 = 1$. However, $B_1(x)$ has the only root $\frac{1}{2}$. This is a contradiction. □

THEOREM 4.2.5. *$B_n = 0$ for any odd $n > 1$. For $n = 2k$, the sign of B_n is $(-1)^{k+1}$. For any even n one has either $B_n = \max_{x\in[0,1]} B_n(x)$ or $B_n = \min_{x\in[0,1]} B_n(x)$. The first occurs for positive B_n, the second for negative.*

PROOF. $B_{2k+1} = B_{2k+1}(0) = 0$ for $k > 0$ by Lemma 4.2.4. Above we have found that $B_2 = \frac{1}{6}$. Suppose we have established that $B_{2k} > 0$ and that this is the maximal value for $B_{2k}(x)$ on $[0,1]$. Let us prove that $B_{2k+2} < 0$ and it is the minimal value for $B_{2k+2}(x)$ on $[0,1]$. The derivative of B_{2k+1} in this case is positive at the ends of $[0,1]$, hence $B_{2k+1}(x)$ is positive for $0 < x < \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$, by Lemma 4.2.4 on roots and the Theorem on Intermediate Values. Hence, $B_{2k+2}'(x) > 0$ for $x < \frac{1}{2}$ and $B_{2k+2}'(x) < 0$ for $x > \frac{1}{2}$. Therefore, $B_{2k+2}(x)$ takes the maximal value in the middle of $[0,1]$ and takes the minimal values at the ends of $[0,1]$. Since the integral of the polynomial along $[0,1]$ is zero and the polynomial is not constant, its minimal value has to be negative. The same arguments prove that if B_{2k} is negative and minimal, then B_{2k+2} is positive and maximal. □

Lemma 4.2.6 (Lagrange Formula). *If f is a differentiable function on $[a, b]$, then there is a $\xi \in (a, b)$, such that*

$$f(b) = f(a) + f'(\xi)\frac{f(b) - f(a)}{b - a}. \tag{4.2.3}$$

Proof. The function $g(x) = f(x) - (x - a)\frac{f(b)-f(a)}{b-a}$ is differentiable on $[a, b]$ and $g(b) = g(a) = 0$. By Rolle's Theorem $g'(\xi) = 0$ for some $\xi \in [a, b]$. Hence $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. Substitution of this value of $f'(\xi)$ in (4.2.3) gives the equality. $\square$

Generating function. The following function of two variables is called the *generating function of Bernoulli polynomials.*

$$B(x, t) = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \tag{4.2.4}$$

Since $B_k \leq \frac{k!}{2^k}$, the series on the right-hand side converges for $t < 2$ for any x. Let us differentiate it termwise as a function of x, for a fixed t. We get $\sum_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x, t)$. Consequently $(\ln B(x, t))'_x = \frac{B'_x(x,t)}{B(x,t)} = t$ and $\ln B(x, t) = xt + c(t)$, where the constant $c(t)$ depends on t. It follows that $B(x, t) = \exp(xt)k(t)$, where $k(t) = \exp(c(t))$. For $x = 0$ we get $B(0, t) = k(t) = \sum_{k=0}^{\infty} B_k\frac{t^k}{k!}$. To find $k(t)$ consider the difference $B(x + 1, t) - B(x, t)$. It is equal to $\exp(xt + t)k(t) - \exp(xt)$. On the other hand the difference is $\sum_{k=0}^{\infty} \Delta B_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x, t)$. Comparing these expressions we get explicit formulas for the generating functions of Bernoulli numbers:

$$k(t) = \frac{t}{\exp t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}t^k,$$

and Bernoulli polynomials:

$$B(x, t) = \sum_{k=+}^{0-1} B_k(x)\frac{t^k}{k!} = \frac{t\exp(tx)}{\exp t - 1}.$$

From (4.2.4) one gets $t = (\exp t - 1)\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}$. Substituting $\exp t - 1 = \sum_{k=1}^{\infty} \frac{t^k}{k!}$ in this equality, by the Uniqueness Theorem 3.6.9, one gets the equalities for the coefficients of the power series

$$\sum_{k=1}^{n} \frac{B_{n-k}}{(n-k)!k!} = 0 \quad \text{for } n > 1.$$

Add $\frac{B_n}{n!}$ to both sides of this equality and multiply both sides by $n!$ to get

$$B_n = \sum_{k=0}^{n} \frac{B_k n^{\underline{k}}}{k!} \quad \text{for } n > 1. \tag{4.2.5}$$

The latter equality one memorizes via the formula $B^n = (B + 1)^n$, where after expansion of the right hand side, one should move down all the exponents at B turning the powers of B into Bernoulli numbers.

Now we are ready to prove that

$$\phi_m(1 + x) = \frac{B_m(x + 1)}{m} - \frac{B_m}{m} = \sum_{k=0}^{m} B_{m-k}\frac{(m-1)^{\underline{k-1}}}{k!}x^k = \psi_m(x). \tag{4.2.6}$$

Putting $x = 0$ in the right hand side one gets $\psi_m(0) = B_m(m-1)^{\underline{-1}} = \frac{B_m}{m}$. The left-hand side takes the same value at $x = 0$, because $B_m(1) = B_m(0) = B_m$. It remains to prove the equality of the coefficients in (4.2.6) for positive degrees.

$$\begin{aligned}\frac{B_m(x+1)}{m} &= \frac{1}{m}\sum_{k=0}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}(1+x)^k\\ &= \frac{1}{m}\sum_{k=0}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}\sum_{j=0}^{k}\frac{k^{\underline{j}}x^j}{j!}\end{aligned}$$

Now let us change the summation order and change the summation index of the interior sum by $i = m - k$.

$$\begin{aligned}&= \frac{1}{m}\sum_{j=0}^{m}\frac{x^j}{j!}\sum_{k=j}^{m}\frac{m^{\underline{k}}B_{m-k}}{k!}k^{\underline{j}}\\ &= \frac{1}{m}\sum_{j=0}^{m}\frac{x^j}{j!}\sum_{i=0}^{m-j}\frac{m^{\underline{m-i}}B_i}{(m-i)!}(m-i)^{\underline{j}}\end{aligned}$$

Now we change $\frac{m^{\underline{m-i}}(m-i)^{\underline{j}}}{(m-i)!}$ by $\frac{(m-j)^{\underline{i}}m^{\underline{j}}}{i!}$ and apply the identity (4.2.5).

$$\begin{aligned}&= \sum_{j=0}^{m}\frac{x^j m^{\underline{j}}}{mj!}\sum_{i=0}^{m-j}\frac{B_i(m-j)^{\underline{i}}}{i!}\\ &= \sum_{j=0}^{m}\frac{(m-1)^{\underline{j-1}}x^j}{j!}B_{m-j}.\end{aligned}$$

Problems.

1. Evaluate $\int_0^1 B_n(x)\sin 2\pi x\,dx$.
2. Expand $x^4 - 3x^2 + 2x - 1$ as a polynomial in $(x-2)$.
3. Calculate the first 20 Bernoulli numbers.
4. Prove the inequality $|B_n(x)| \le |B_n|$ for even n.
5. Prove the inequality $|B_n(x)| \le \frac{n}{4}|B_{n-1}|$ for odd n.
6. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \int_0^1 f'(x)B_1(x)\,dx$.
7. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \frac{\Delta f'(0)}{2} - \int_0^1 f''(x)B_2(x)\,dx$.
8. Deduce $\Delta B_n(x) = nx^{\underline{n-1}}$ from the balance property and the differentiation rule.
9. Prove that $B_n(x) = B_n(1-x)$, using the generating function.
10. Prove that $B_{2n+1} = 0$, using the generating function.
11. Prove that $B_m(nx) = n^{m-1}\sum_{k=0}^{n-1}B_m\left(x + \frac{k}{n}\right)$.
12. Evaluate $B_n(\frac{1}{2})$.
13. Prove that $B_{2k}(x) = P(B_2(x))$, where $P(x)$ is a polynomial with positive coefficient (Jacobi Theorem).
14. Prove that $B_n = \sum_{k=0}^{\infty}(-1)^k\frac{\Delta^k 0^n}{k+1}$.
***15.** Prove that $B_m + \sum\frac{1}{k+1}[k+1$ is prime and k is divisor of $m]$ is an integer (Staudt Theorem).

4.3. Euler-Maclaurin Formula

On the contents of the lecture. From this lecture we will learn how Euler managed to calculate eighteen digit places of the sum $\sum_{k=0}^{\infty} \frac{1}{k^2}$.

Symbolic derivation. Taylor expansion of a function f at point x gives

$$f(x+1) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}.$$

Hence

$$\delta f(x) = \sum_{k=1}^{\infty} \frac{\mathbf{D}^k f(x)}{k!},$$

where $\mathbf{D}$ is the operation of differentiation. One expresses this equality symbolically as

$$\delta = \exp \mathbf{D} - \mathbf{1}. \tag{4.3.1}$$

We are searching for F such that $F(n) = \sum_{k=1}^{n-1} f(k)$ for all n. Then $\delta F(x) = f(x)$, or symbolically $F = \delta^{-1} f$. So we have to invert the operation of the difference. From (4.3.1), the inversion is given formally by the formula $(\exp \mathbf{D} - \mathbf{1})^{-1}$. This function has a singularity at 0 and cannot be expanded into a power series in $\mathbf{D}$. However we know the expansion

$$\frac{t}{\exp t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$

This allows us to give a symbolic solution of our problem in the form

$$\delta^{-1} = \mathbf{D}^{-1} \frac{\mathbf{D}}{\exp \mathbf{D} - \mathbf{1}} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathbf{D}^{k-1} = \mathbf{D}^{-1} - \frac{1}{2}\mathbf{1} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} \mathbf{D}^{2k-1}.$$

Here we take into account that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for $k > 0$. Since $\sum_{k=1}^{n-1} f(k) = F(n) - F(1)$, the latter symbolic formula gives the following summation formula:

$$\sum_{k=1}^{n-1} f(k) = \int_1^n f(x)\, dx - \frac{f(n) - f(1)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(1)). \tag{4.3.2}$$

For $f(x) = x^m$ this formula gives the Bernoulli polynomial ϕ_{m+1}.

Euler's estimate. Euler applied this formula to $f(x) = \frac{1}{(x+9)^2}$ and estimated the sum $\sum_{k=10}^{\infty} \frac{1}{k^2}$. In this case the k-th derivative of $\frac{1}{(x+9)^2}$ at 1 has absolute value $\frac{(k+1)!}{10^{k+2}}$. Hence the module of the k-th term of the summation formula does not exceed $\frac{B_k}{k10^{k+2}}$. For an accuracy of eighteen digit places it is sufficient to sum up the first fourteen terms of the series, only eight of them do not vanish. Euler conjectured, and we will prove, that the value of error does not exceed of the value of the first rejected term, which is $\frac{B_{16}}{16 \cdot 10^{18}}$. Since $B_{16} = -\frac{3617}{510}$ this gives the promised accuracy.

B_1	B_2	B_4	B_6	B_8	B_{10}	B_{12}	B_{14}	B_{16}	B_{18}	B_{20}
$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

FIGURE 4.3.1. Bernoulli numbers

We see from the table (Figure 4.3.1) that increasing of the number of considered terms does not improve accuracy noticeably.

Summation formula with remainder. In this lecture we assume that all functions under consideration are differentiable as many times as needed.

LEMMA 4.3.1. *For any function* $f(x)$ *on* $[0,1]$ *one has*

$$\frac{1}{2}(f(1)+f(0)) = \int_0^1 f(x)\,dx - \int_0^1 f'(x)B_1(x)\,dx.$$

PROOF. Recall that $B_1(x) = x - \frac{1}{2}$, hence $\int_0^1 f'(x)B_1(x)\,dx = \int_0^1 (x-\frac{1}{2})\,df(x)$. Now, integration by parts gives

$$\int_0^1 (x-\frac{1}{2})\,df(x) = \frac{1}{2}(f(1)+f(0)) - \int_0^1 f(x)\,dx.$$

□

Consider the *periodic Bernoulli polynomials* $B_m\{x\} = B_m(x-[x])$. Then $B'_m\{x\} = mB_{m-1}\{x\}$ for non integer x.

Let us denote by $\sum_m^n a_k$ the sum $\frac{1}{2}a_m + \sum_{k=m+1}^{n-1} a_k + \frac{1}{2}a_n$.

LEMMA 4.3.2. *For any natural* p, q *and any function* $f(x)$ *one has*

$$\sum_p^q f(k) = \int_p^q f(x)\,dx - \int_p^q f'(x)B_1\{x\}\,dx.$$

PROOF. Applying Lemma 4.3.1 to $f(x+k)$ one gets

$$\begin{aligned}\frac{1}{2}(f(k+1)+f(k)) &= \int_0^1 f(x+k)\,dx + \int_0^1 f'(x+k)B_1(x)\,dx\\ &= \int_k^{k+1} f(x)\,dx + \int_k^{k+1} f'(x)B_1\{x\}\,dx.\end{aligned}$$

Summing up these equalities for k from p to q, one proves the lemma. □

LEMMA 4.3.3. *For* $m>0$ *and a function* f *one has*

$$\int_p^q f(x)B_m\{x\}\,dx = \frac{B_{m+1}}{m+1}(f(q)-f(p)) - \int_p^q f'(x)B_{m+1}\{x\}\,dx. \tag{4.3.3}$$

PROOF. Since $B_m\{x\}dx = d\frac{B_{m+1}\{x\}}{m+1}$ and $B_{m+1}\{k\} = B_{m+1}$ for any natural k, the formula (4.3.3) is obtained by a simple integration by parts. □

THEOREM 4.3.4. *For any function f and natural numbers n and m one has:*

$$\sum_1^n f(k) = \int_1^n f(x)\,dx + \sum_{k=1}^{m-1} \frac{B_{k+1}}{(k+1)!}\left(f^{(k)}(n) - f^{(k)}(1)\right) + \frac{(-1)^{m+1}}{m!}\int_1^n f^{(m)}(x)B_m\{x\}\,dx. \tag{4.3.4}$$

PROOF. The proof is by induction on m. For $m = 1$, formula (4.3.4) is just given by Lemma 4.3.2. Suppose (4.3.4) is proved for m. The *remainder*

$$\frac{(-1)^{m+1}}{m!}\int_1^n f^{(m)}(x)B_m\{x\}\,dx$$

can be transformed by virtue of Lemma 4.3.3 into

$$\frac{(-1)^{m+1}B_{m+1}}{(m+1)!}(f^{(m)}(n) - f^{(m)}(1)) + \frac{(-1)^{m+2}}{(m+1)!}\int_1^n B_{m+1}\{x\}f^{(m+1)}(x)\,dx.$$

Since odd Bernoulli numbers vanish, $(-1)^{m+1}B_{m+1} = B_{m+1}$ for $m > 0$. □

Estimation of the remainder. For $m = \infty$, (4.3.4) turns into (4.3.2). Denote

$$R_m = \frac{(-1)^{m+1}}{m!}\int_1^n f^{(m)}(x)B_m\{x\}\,dx.$$

This is the so-called *remainder* of Euler-Maclaurin formula.

LEMMA 4.3.5. *$R_{2m} = R_{2m+1}$ for any $m > 1$.*

PROOF. Because $B_{2m+1} = 0$, the only thing which changes in (4.3.4) when one passes from $2m$ to $2m+1$ is the remainder. Hence its value does not change either. □

LEMMA 4.3.6. *If $f(x)$ is monotone on $[0, 1]$ then*

$$\operatorname{sgn}\int_0^1 f(x)B_{2m+1}(x)\,dx = \operatorname{sgn}(f(1) - f(0))\operatorname{sgn} B_{2m}.$$

PROOF. Since $B_{2m+1}(x) = -B_{2m+1}(1-x)$, the change $x \to 1-x$ transforms the integral $\int_{0.5}^1 f(x)B_{2m+1}(x)\,dx$ to $-\int_0^{0.5} f(1-x)B_{2m+1}(x)\,dx$:

$$\begin{aligned}\int_0^1 f(x)B_{2m+1}(x)\,dx &= \int_0^{0.5} f(x)B_{2m+1}(x)\,dx + \int_{0.5}^1 f(x)B_{2m+1}(x)\,dx\\ &= \int_0^{0.5}(f(x) - f(1-x))B_{2m+1}(x)\,dx.\end{aligned}$$

$B_{2m+1}(x)$ is equal to 0 at the end-points of $[0, 0.5]$ and has constant sign on $(0, 0.5)$, hence its sign on the interval coincides with the sign of its derivative at 0, that is, it is equal to $\operatorname{sgn} B_{2m}$. The difference $f(x) - f(1-x)$ also has constant sign as $x < 1-x$ on $(0, 0.5)$ and its sign is $\operatorname{sgn}(f(1) - f(0))$. Hence the integrand has constant sign. Consequently the integral itself has the same sign as the integrand has. □

LEMMA 4.3.7. *If $f^{(2m+1)}(x)$ and $f^{(2m+3)}(x)$ are comonotone for $x \geq 1$ then*

$$R_{2m} = \theta_m \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(n) - f^{(2m+1)}(1)), \quad 0 \leq \theta_m \leq 1.$$

PROOF. The signs of R_{2m+1} and R_{2m+3} are opposite. Indeed, by Lemma 4.2.5 $\operatorname{sgn} B_{2m} = -\operatorname{sgn} B_{2m+2}$, and $\operatorname{sgn}(f^{(2m+1)}(n) - f^{(2m+1)}(1)) = \operatorname{sgn}(f^{(2m+3)}(n) - f^{(2m+3)}(1)$ due to the comonotonity condition. Hence $\operatorname{sgn} R_{2m+1} = -\operatorname{sgn} R_{2m+3}$ by Lemma 4.3.6.

Set

$$T_{2m+2} = \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(n) - f^{(2m+1)}(1)).$$

Then $T_{2m+2} = R_{2m+1} - R_{2m+2}$. By Lemma 4.3.5, $T_{2m+2} = R_{2m+1} - R_{2m+3}$. Since R_{2m+3} and R_{2m+1} have opposite signs, it follows that $\operatorname{sgn} T_{2m+2} = \operatorname{sgn} R_{2m+1}$ and $|T_{2m+2}| \geq |R_{2m+1}|$. Hence $\theta_m = \frac{R_{2m+1}}{T_{2m+2}} = \frac{R_{2m}}{T_{2m+2}}$ belongs to $[0, 1]$. □

THEOREM 4.3.8. *If $f^{(k)}$ and $f^{(k+2)}$ are comonotone for any $k > 1$, then*

$$\left|\sum_1^n f(k) - \int_1^n f(x)\,dx - \sum_{k=1}^m \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(1)\right)\right| \leq \left|\frac{B_{2m+2}}{(2m+2)!}\left(f^{(2m+1)}(n) - f^{(2m+1)}(1)\right)\right|.$$

Hence the value of the error which gives the summation formula (4.3.2) with m terms has the same sign as the first rejected term, and its absolute value does not exceed the absolute value of the term.

THEOREM 4.3.9. *Suppose that $\int_1^\infty |f^{(k)}(x)|\,dx < \infty$, $\lim_{x\to\infty} f^{(k)}(x) = 0$ and $f^{(k)}$ is comonotone with $f^{(k+2)}$ for all $k \geq K$ for some K. Then there is a constant C such that for any $m > K$ for some $\theta_m \in [0, 1]$*

$$\sum_{k=1}^n f(k) = C + \frac{f(n)}{2} + \int_1^n f(x)\,dx + \sum_{k=1}^m \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n) + \theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(n). \tag{4.3.5}$$

LEMMA 4.3.10. *Under the condition of the theorem, for any $m \geq K$,*

$$\frac{(-1)^m}{m!}\int_p^\infty f^{(m)}(x) B_m\{x\}\,dx = -\theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(p). \tag{4.3.6}$$

PROOF. By Lemma 4.3.7,

$$\frac{(-1)^{m+1}}{m!}\int_p^q f^{(m)}(x) B_m\{x\}\,dx = \theta_m \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(q) - f^{(2m+1)}(p)).$$

To get (4.3.6), pass to the limit as q tends to infinity. □

PROOF OF THEOREM 4.3.9. To get (4.3.5) we change the form of the remainder R_K for (4.3.4). Since

$$\int_1^n B_K\{x\} f^{(K)}\,dx = \int_1^\infty B_K\{x\} f^{(K)}(x)\,dx - \int_n^\infty B_K\{x\} f^{(K)}(x)\,dx,$$

applying the equality (4.3.3) to the interval $[n, \infty)$, one gets

$$-\frac{(-1)^{k+1}B_k}{k!}\int_n^\infty B_k\{x\}f^{(k)}(x)\,dx = \frac{(-1)^{k+1}B_{k+1}}{(k+1)!}f^{(k)}(n) - \frac{(-1)^{k+2}B_{k+1}}{(k+1)!}\int_n^\infty B_{k+1}\{x\}f^{(k+1)}(x)\,dx.$$

Iterating this formula one gets

$$R_K = \int_1^\infty B_K\{x\}f^{(K)}\,dx + \sum_{k=K}^{m}\frac{B_{k+1}}{(k+1)!}f^{(k)}(n) + \frac{(-1)^m}{m!}\int_n^\infty B_m\{x\}f^{(m)}(x)\,dx.$$

Here we take into account the equalities $(-1)^k B_k = B_k$ and $(-1)^{m+2} = (-1)^m$. Now we substitute this expression of R_K into (4.3.4) and set

$$C = (-1)^{K+1}\int_1^\infty B_K\{x\}f^{(K)}(x)\,dx - \frac{f(1)}{2} - \sum_{k=1}^{K-1}\frac{B_{k+1}}{(k+1)!}f^{(k)}(1). \tag{4.3.7}$$

□

Stirling formula. The logarithm satisfies all the conditions of Theorem 4.3.9 with $K = 2$. Its k-th derivative at n is equal to $\frac{(-1)^{k+1}(k-1)!}{n^k}$. Thus (4.3.5) for the logarithm turns into

$$\sum_{k=1}^{n}\ln k = n\ln n - n + \sigma + \frac{\ln n}{2} + \sum_{k=1}^{m}\frac{B_{2k}}{2k(2k-1)n^{2k-1}} + \frac{\theta_m B_{2m+2}}{(2m+2)(2m+1)n^{2k-1}}.$$

By (4.3.7), the constant is

$$\sigma = \int_1^\infty \frac{B_2\{x\}}{x^2}\,dx - \frac{B_2}{2}.$$

But we already know this constant as $\sigma = \frac{1}{2}\ln 2\pi$. For $m = 0$, the above formula gives the most common form of Stirling formula:

$$n! = \sqrt{2\pi n}\,n^n e^{-n+\frac{\Theta}{12n}}.$$

Problems.

1. Write the Euler-Maclaurin series telescoping $\frac{1}{x}$.
2. Prove the uniqueness of the constant in Euler-Maclaurin formula.
3. Calculate ten digit places of $\sum_{k=1}^{\infty}\frac{1}{n^3}$.
4. Calculate eight digit places of $\sum_{k=1}^{1000000}\frac{1}{k}$.
5. Evaluate $\ln 1000!$ with accuracy 10^{-4}.

4.4. Gamma Function

On the contents of the lecture. Euler's Gamma-function is the function responsible for infinite products. An infinite product whose terms are values of a rational function at integers is expressed in terms of the Gamma-function. In particular it will help us prove Euler's factorization of sin.

Telescoping problem. Given a function $f(x)$, find a function $F(x)$ such that $\delta F = f$. This is the *telescoping problem* for functions. In particular, for $f = 0$ any periodic function of period 1 is a solution. In the general case, to any solution of the problem we can add a 1-periodic function and get another solution. The general solution has the form $F(x) + k(t)$ where $F(x)$ is a particular solution and $k(t)$ is a 1-periodic function, called the periodic constant.

The Euler-Maclaurin formula gives a formal solution of the problem, but the Euler-Maclaurin series rarely converges. Another formal solution is

$$F(x) = -\sum_{k=0}^{\infty} f(x+k). \tag{4.4.1}$$

Trigamma. Now let us try to telescope the Euler series. The series (4.4.1) converges for $f(x) = \frac{1}{x^m}$ provided $m \geq 2$ and $x \neq -n$ for natural $n > 1$. In particular, the function

$$\mathit{\Gamma}(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} \tag{4.4.2}$$

is analytic; it is called the *trigamma* function and it telescopes $-\frac{1}{(1+x)^2}$. Its value $\mathit{\Gamma}(0)$ is just the sum of the Euler series.

This function is distinguished among others functions telescoping $-\frac{1}{(1+x)^2}$ by its finite variation.

THEOREM 4.4.1. *There is a unique function $\mathit{\Gamma}(x)$ such that $\delta\mathit{\Gamma}(x) = -\frac{1}{(1+x)^2}$, $\operatorname{var}_{\mathit{\Gamma}}[0,\infty] < \infty$ and $\mathit{\Gamma}(0) = \sum_{k=1}^{\infty} \frac{1}{k^2}$.*

PROOF. Since $\mathit{\Gamma}$ is monotone, one has $\operatorname{var}_{\mathit{\Gamma}}[0,\infty] = \sum_{k=0}^{\infty} |\delta\mathit{\Gamma}| = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Suppose $f(x)$ is another function of finite variation telescoping $\frac{1}{(1+x)^2}$. Then $f(x) - \mathit{\Gamma}(x)$ is a periodic function of finite variation. It is obvious that such a function is constant, and this constant is 0 if $f(1) = \mathit{\Gamma}(1)$. ☐

Digamma. The series $-\sum_{k=0}^{\infty} \frac{1}{x+k}$, which formally telescopes $\frac{1}{x}$, is divergent. However the series $-\sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k}[k \neq 0]\right)$ is convergent and it telescopes $\frac{1}{x}$, because adding a constant does not affect the differences. Indeed,

$$-\sum_{k=0}^{\infty} \left(\tfrac{1}{x+1+k} - \tfrac{1}{k}[k \neq 0]\right) + \sum_{k=0}^{\infty} \left(\tfrac{1}{x+k} - \tfrac{1}{k}[k \neq 0]\right) = -\sum_{k=0}^{\infty} \delta \tfrac{1}{x+k} = \tfrac{1}{x}.$$

The function

$$\digamma(x) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k}\right) \tag{4.4.3}$$

is called the*digamma* function. Here γ is the Euler constant. The digamma function is an analytic function, whose derivative is the trigamma function, and whose difference is $\frac{1}{1+x}$.

Monotonicity distinguishes $\digamma$ among others function telescoping $\frac{1}{1+x}$.

THEOREM 4.4.2. *There is a unique monotone function $\digamma(x)$ such that $\delta\digamma(x) = \frac{1}{1+x}$ and $\digamma(0) = -\gamma$.*

PROOF. Suppose $f(x)$ is a monotone function telescoping $\frac{1}{1+x}$. Denote by v the variation of $f-\digamma$ on $[0,1]$. Then the variation of $f-\digamma$ over $[1,n]$ is nv. On the other hand, $\operatorname{var}_f[1,n] = \sum_{k=1}^{n} \frac{1}{k} < \ln n + \gamma$. Hence the variation of $f(x) - \digamma(x)$ on $[1,n]$ is less than $2(\gamma + \ln n)$. Hence v for any n satisfies the inequality $nv \leq 2(\gamma + \ln n)$. Since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, we get $v = 0$. Hence $f - \digamma$ is constant, and it is zero if $f(1) = \digamma(1)$. □

LEMMA 4.4.3. $\digamma' = \Gamma$.

PROOF. To prove that $\digamma'(x) = \Gamma(x)$, consider $F(x) = \int_1^x \Gamma(t)\,dt$. This function is monotone, because $F'(x) = \Gamma(x) \geq 0$. Further $(\delta F)' = \delta F' = \delta\Gamma(x) = -\frac{1}{(1+x)^2}$. It follows that $\delta F = \frac{1}{1+x} + c$, where c is a constant. By Theorem 4.4.2 it follows that $F(x+1) - cx - \gamma = \digamma(x)$. Hence $\digamma(x)' = F'(x+1) + c = \Gamma(x)$. This proves that $\digamma'$ is differentiable and has finite variation. As $\delta\digamma(x) = \frac{1}{1+x}$ it follows that $\delta\digamma'(x) = -\frac{1}{(1+x)^2}$. We get that $\digamma'(x) = \Gamma(x)$ by Theorem 4.4.1. □

Telescoping the logarithm. To telescope the logarithm, we start with the formal solution $-\sum_{k=0}^{\infty} \ln(x+k)$. To decrease the divergence, add $\sum_{k=1}^{\infty} \ln k$ termwise. We get $-\ln x - \sum_{k=1}^{\infty}(\ln(x+k) - \ln k) = -\ln x - \sum_{k=1}^{\infty} \ln(1+\frac{x}{k})$. We know that $\ln(1+x)$ is close to x, but the series still diverges. Now convergence can be reached by the subtraction of $\frac{x}{k}$ from the k-th term of the series. This substraction changes the difference. Let us evaluate the difference of $F(x) = -\ln x - \sum_{k=1}^{\infty}(\ln(1+\frac{x}{k}) - \frac{x}{k})$. The difference of the n-th term of the series is

$$\begin{aligned}
&\left(\ln\left(1+\tfrac{x+1}{k}\right) - \tfrac{x+1}{k}\right) - \left(\ln\left(1+\tfrac{x}{k}\right) - \tfrac{x}{k}\right)\\
&\quad = \left(\ln(x+k+1) - \ln k - \tfrac{x+1}{k}\right) - \left(\ln(x+k) - \ln k - \tfrac{x}{k}\right)\\
&\quad = \delta\ln(x+k) - \tfrac{1}{k}.
\end{aligned}$$

Hence

$$\begin{aligned}
\delta F(x) &= -\delta\ln x - \textstyle\sum_{k=1}^{\infty}\left(\delta\ln(x+k) - \frac{1}{k}\right)\\
&= \lim_{n\to\infty}\left(-\delta\ln x - \textstyle\sum_{k=1}^{n-1}\left(\delta\ln(x+k) - \frac{1}{k}\right)\right)\\
&= \lim_{n\to\infty}\left(\ln x - \ln(n+x) + \textstyle\sum_{k=1}^{n-1}\frac{1}{k}\right)\\
&= \ln x + \lim_{n\to\infty}(\ln(n) - \ln(n+x)) + \lim_{n\to\infty}\left(\textstyle\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right)\\
&= \ln x + \gamma.
\end{aligned}$$

As a result, we get the following formula for a function, which telescopes the logarithm:

$$\Theta(x) = -\gamma x - \ln x - \sum_{k=1}^{\infty}\left(\ln\left(1+\frac{x}{k}\right) - \frac{x}{k}\right). \tag{4.4.4}$$

THEOREM 4.4.4. *The series* (4.4.4) *converges absolutely for all* x *except negative integers. It presents a function* $\Theta(x)$ *such that* $\Theta(1) = 0$ *and* $\delta\Theta(x) = \ln x$.

PROOF. The inequality $\frac{x}{1+x} \leq \ln(1+x) \leq x$ implies

$$|\ln(1+x) - x| \leq \left|\frac{x}{1+x} - x\right| = \left|\frac{x^2}{1+x}\right|. \tag{4.4.5}$$

Denote by ε the distance from x to the closest negative integer. Then due to (4.4.5), the series $\sum_{k=1}^{\infty} \ln\left(\left(1+\frac{y}{k}\right) - \frac{y}{k}\right)$ is termwise majorized by the convergent series $\sum_{k=1}^{\infty} \frac{x^2}{\varepsilon k^2}$. This proves the absolute convergence of (4.4.4).

Since $\lim_{n\to\infty} \sum_{k=1}^{n-1}(\ln(1+\frac{1}{k}) - \frac{1}{k}) = \lim_{n\to\infty}(\ln n - \sum_{k=1}^{n-1} \frac{1}{k}) = -\gamma$, one gets $\Theta(1) = 0$. □

Convexity. There are a lot of functions that telescope the logarithm. The property which distinguishes Θ among others is convexity.

Throughout the lecture θ and $\overline{\theta}$ are nonnegative and *complementary* to each other, that is $\theta + \overline{\theta} = 1$. The function f is called *convex* if, for any x, y, it satisfies the inequality:

$$f(\theta x + \overline{\theta} y) \leq \theta f(x) + \overline{\theta} f(y) \quad \forall\theta \in [0,1]. \tag{4.4.6}$$

Immediately from the definition it follows that

LEMMA 4.4.5. *Any linear function* $ax + b$ *is convex.*

LEMMA 4.4.6. *Any sum (even infinite) of convex functions is a convex function. The product of a convex function by a positive constant is a convex function.*

LEMMA 4.4.7. *If* $f(p) = f(q) = 0$ *and* f *is convex, then* $f(x) \geq 0$ *for all* $x \notin [p, q]$.

PROOF. If $x > q$ then $q = x\theta + p\overline{\theta}$ for $\theta = \frac{q-p}{x-p}$. Hence $f(q) \leq f(x)\theta + f(p)\overline{\theta} = f(x)$, and it follows that $f(x) \geq f(q) = 0$. For $x < p$ one has $p = x\theta + q\overline{\theta}$ for $\theta = \frac{q-p}{q-x}$. Hence $0 = f(p) \leq f(x)\theta + f(q)\overline{\theta} = f(x)$. □

LEMMA 4.4.8. *If* f'' *is nonnegative then* f *is convex.*

PROOF. Consider the function $F(t) = f(l(t))$, where $l(t) = x\overline{\theta} + y\theta$. Newton's formula for $F(t)$ with nodes 0, 1 gives $F(t) = F(0) + \delta F(0)t + \frac{1}{2}F''(\xi)t^{\underline{2}}$. Since $F''(\xi) = (y-x)^2 f''(\xi) > 0$, and $t^{\underline{2}} = t(t-1) < 0$ we get the inequality $F(t) \leq F(0)\overline{\theta} + tF(1)$. Since $F(\theta) = f(x\overline{\theta} + y\theta)$ this is just the inequality of convexity. □

LEMMA 4.4.9. *If* f *is convex, then* $0 \leq f(a) + \theta\delta f(a) - f(a+\theta) \leq \delta^2 f(a-1)$ *for any* a *and any* $\theta \in [0,1]$

PROOF. Since $a + \theta = \overline{\theta}a + \theta(a+1)$ we get $f(a+\theta) \leq f(a)\overline{\theta} + f(a+1)\theta = f(a) + \theta\delta f(a)$. On the other hand, the convex function $f(a+x) - f(a) - x\delta f(a-1)$ has roots -1 and 0. By Lemma 4.4.7 it is nonnegative for $x > 0$. Hence $f(a+\theta) \geq f(a) + \theta\delta f(a-1)$. It follows that $f(a) + \theta\delta f(a) - f(a+\theta) \geq f(a) + \theta\delta f(a) - f(a) - \theta\delta f(a-1) = \theta\delta^2 f(a-1)$. □

THEOREM 4.4.10. $\Theta(x)$ *is the unique convex function that telescopes* $\ln x$ *and satisfies* $\Theta(1) = 1$.

PROOF. Convexity of Θ follows from the convexity of the summands of its series. The summands are convex because their second derivatives are nonnegative.

Suppose there is another convex function $f(x)$ which telescopes the logarithm too. Then $\phi(x) = f(x) - \Theta(x)$ is a periodic function, $\delta\phi = 0$. Let us prove that $\phi(x)$ is convex. Consider a pair c, d, such that $|c-d| \leq 1$. Since $f(c\theta + d\overline{\theta}) - \theta f(c) - \overline{\theta} f(d) \leq 0$, as f is convex, one has

$$\begin{aligned}\phi(c\theta + d\overline{\theta}) - \theta\phi(c) - \overline{\theta}\phi(d) &= (f(c\theta + d\overline{\theta}) - \theta f(c) - \overline{\theta} f(d)) \\ &\quad - (\Theta(c\theta + d\overline{\theta}) - \theta\Theta(c) - \overline{\theta}\Theta(d)) \\ &\leq \theta\Theta(c) + \overline{\theta}\Theta(d) - \Theta(c\theta + d\overline{\theta}).\end{aligned}$$

First, prove that ϕ satisfies the following ε-relaxed inequality of convexity:

$$\phi(c\theta + d\overline{\theta}) \leq \theta\phi(c) + \overline{\theta}\phi(d) + \varepsilon. \tag{4.4.7}$$

Increasing c and d by 1, we do not change the inequality as $\delta\phi = 0$. Due to this fact, we can increase c and d to satisfy $\frac{1}{c-1} < \frac{\varepsilon}{3}$. Set $L(x) = \Theta(c) + (x - c)\ln c$. By Lemma 4.4.9 for $x \in [c, c+1]$ one has $|\Theta x - L(x)| \leq \delta^2\Theta(c-1) = \ln c - \ln(c-1) = \ln(1 + \frac{1}{c-1}) \leq \frac{1}{c-1} < \frac{\varepsilon}{3}$. Since $|\Theta(x) - L(x)| < \frac{\varepsilon}{3}$ for $x = c, d, \frac{c+d}{2}$, it follows that $\theta\Theta(c) + \overline{\theta}\Theta(d) - \Theta(c\theta + d\overline{\theta})$ differs from $\theta L(c) + \overline{\theta} L(d) - L(c\theta + d\overline{\theta}) = 0$ by less than by ε. The inequality (4.4.7) is proved. Passing to the limit as ε tends to 0, one eliminates ε.

Hence $\phi(x)$ is convex on any interval of length 1 and has period 1. Then $\phi(x)$ is constant. Indeed, consider a pair a, b with condition $b - 1 < a < b$. Then $a = (b-1)\theta + b\overline{\theta}$ for $\theta = b - a$. Hence $f(a) \leq f(b)\theta + f(b-1)\overline{\theta} = f(b)$. □

LEMMA 4.4.11. $\Theta''(1 + x) = \mathit{\Gamma}(x)$.

PROOF. The function $F(x) = \int_1^x \mathit{\Gamma}(t)\,dt$ is convex because its second derivative is $\mathit{\Gamma}$. The difference of $F' = \mathit{\Gamma}$ is $\frac{1}{1+x}$. Hence $\delta F(x) = \ln(x+1) + c$, where c is some constant. It follows that $F(x-1) - cx + c = \Theta(x)$. Hence Θ is twice differentiable and its second derivative is $\mathit{\Gamma}$. □

Gamma function. Now we define Euler's *gamma function* $\Gamma(x)$ as $\exp(\Theta(x))$, where $\Theta(x)$ is the function telescoping the logarithm. Exponentiating (4.4.4) gives a representation of the Gamma function in so-called *canonical Weierstrass form*:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)^{-1} e^{\frac{x}{k}}. \tag{4.4.8}$$

Since $\delta \ln\Gamma(x) = \ln x$, one gets the following *characteristic equation* of the Gamma function

$$\Gamma(x + 1) = x\Gamma(x). \tag{4.4.9}$$

Since $\Theta(1) = 0$, according to (4.4.4), one proves by induction that $\Gamma(n) = (n-1)!$ using (4.4.9).

A nonnegative function f is called *logarithmically convex* if $\ln f(x)$ is convex.

THEOREM 4.4.12 (characterization). $\Gamma(x)$ *is the unique logarithmically convex function defined for* $x > 0$, *which satisfies equation* (4.4.9) *for all* $x > 0$ *and takes the value* 1 *at* 1.

PROOF. Logarithmical convexity of $\Gamma(x)$ follows from the convexity of $\Theta(x)$. Further $\Gamma(1) = \exp\Theta(1) = 1$. If f is a logarithmically convex function satisfying the gamma-equation, then $\ln f$ satisfies all the conditions of Theorem 4.4.4. Hence, $\ln f(x) = \Theta(x)$ and $f(x) = \Gamma(x)$. □

THEOREM 4.4.13 (Euler). *For any $x \geq 0$ one has $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.*

Let us check that the integral satisfies all the conditions of Theorem 4.4.12. For $x = 1$ the integral gives $\int_0^\infty e^{-t}\,dt = -e^{-t}\,|_0^\infty = 1$. The integration by parts $\int_0^\infty t^x e^{-t}\,dt = -\int_0^\infty t^x\,de^{-t} = -t^x e^{-t}\,|_0^\infty + \int_0^\infty e^{-t}xt^{x-1}\,dx$ proves that it satisfies the gamma-equation (4.4.9). It remains to prove logarithmic convexity of the integral.

LEMMA 4.4.14 (mean criterium). *If f is a monotone function which satisfies the following* mean inequality $2f(\frac{x+y}{2}) \leq f(x) + f(y)$ *for all x, y then f is convex.*

PROOF. We have to prove the inequality $f(x\theta + y\overline{\theta}) \leq \theta f(x) + \overline{\theta} f(y) = L(\overline{\theta})$ for all θ, x and y. Set $F(t) = f(x + (y-x)t)$; than F also satisfies the mean inequality. And to prove our lemma it is sufficient to prove that $F(t) \leq L(t)$ for all $t \in [0,1]$.

First we prove this inequality only for all *binary rational* numbers t, that is for numbers of the type $\frac{m}{2^n}$, $m \leq 2^n$. The proof is by induction on n, the degree of the denominator. If $n = 0$, the statement is true. Suppose the inequality $F(t) \leq L(t)$ is already proved for fractions with denominators of degree $\leq n$. Consider $r = \frac{m}{2^{n+1}}$, with odd $m = 2k+1$. Set $r^- = \frac{k}{2^n}$, $r^+ = \frac{k+1}{2^n}$. By the induction hypothesis $F(r^\pm) \leq L(r^\pm)$. Since $r = \frac{r^+ + r^-}{2}$, by the mean inequality one has $F(r) \leq \frac{f(r^+)+f(r^-)}{2} \leq \frac{L(r^+)+L(r^-)}{2} = L(\frac{r^+ + r^-}{2}) = L(r)$.

Thus our inequality is proved for all binary rational t. Suppose $F(t) > L(t)$ for some t. Consider two binary rational numbers p, q such that $t \in [p,q]$ and $|q-p| < \frac{F(t)-L(t)}{|f(y)-f(x)|}$. In this case $|L(p) - L(t)| \leq |p-t||f(y) - f(x)| < |F(t) - L(t)|$. Therefore $F(p) \leq L(p) < F(t)$. The same arguments give $F(q) < F(t)$. This is a contradiction, because t is between p and q and its image under a monotone mapping has to be between images of p and q. □

LEMMA 4.4.15 (Cauchy-Bunyakovski-Schwarz).

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx. \tag{4.4.10}$$

PROOF. Since $\int_a^b (f(x) + tg(x))^2\,dx \geq 0$ for all t, the discriminant of the following quadratic equation is non-negative:

$$t^2 \int_a^b g^2(x)\,dx + 2t \int_a^b f(x)g(x)\,dx + \int_a^b f^2(x)\,dx = 0. \tag{4.4.11}$$

This discriminant is $4\left(\int_a^b f(x)g(x)\,dx\right)^2 - 4\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx$. □

Now we are ready to prove the logarithmic convexity of the Euler integral. The integral is obviously an increasing function, hence by the mean criterion it is sufficient to prove the following inequality:

$$\left(\int_0^\infty t^{\frac{x+y}{2}-1}e^{-t}\,dt\right)^2 \leq \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty t^{y-1}e^{-t}\,dt. \tag{4.4.12}$$

This inequality turns into the Cauchy-Bunyakovski-Schwarz inequality (4.4.10) for $f(x) = t^{\frac{x-1}{2}} e^{-t/2}$ and $g(t) = t^{\frac{y-1}{2}} e^{-t/2}$.

Evaluation of products. From the canonical Weierstrass form it follows that

$$\prod_{n=1}^{\infty}\{(1-x/n)\exp(x/n)\} = \frac{-e^{\gamma x}}{x\Gamma(-x)}, \tag{4.4.13}$$

$$\prod_{n=1}^{\infty}\{(1+x/n)\exp(-x/n)\} = \frac{e^{-\gamma x}}{x\Gamma(x)}.$$

One can evaluate a lot of products by splitting them into parts which have this canonical form (4.4.13). For example, consider the product $\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$. Division by n^2 transforms it into $\prod_{k=1}^{\infty}(1-\frac{1}{2n})^{-1}(1+\frac{1}{2n})^{-1}$. Introducing multipliers $e^{\frac{1}{2n}}$ and $e^{-\frac{1}{2n}}$, one gets a canonical form

$$\prod_{n=1}^{\infty}\left\{\left(1-\frac{1}{2n}\right)e^{\frac{1}{2n}}\right\}^{-1}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{2n}\right)e^{-\frac{1}{2n}}\right\}^{-1}. \tag{4.4.14}$$

Now we can apply (4.4.13) for $x = \frac{1}{2}$. The first product of (4.4.14) is equal to $-\frac{1}{2}\Gamma(-1/2)e^{-\gamma/2}$, and the second one is $\frac{1}{2}\Gamma(1/2)e^{\gamma/2}$. Since according to the characteristic equation for Γ-function, $\Gamma(1/2) = -\frac{1}{2}\Gamma(1/2)$, one gets $\Gamma(1/2)^2/2$ as the value of Wallis product. Since the Wallis product is $\frac{\pi}{2}$, we get $\Gamma(1/2) = \sqrt{\pi}$.

Problems.

1. Evaluate the product $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)\left(1+\frac{2x}{n}\right)\left(1-\frac{3x}{n}\right)$.
2. Evaluate the product $\prod_{k=1}^{\infty}\frac{k(5+k)}{(3+k)(2+k)}$.
3. Prove: The sum of logarithmically convex functions is logarithmically convex.
4. Prove $\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x x^{-n}}{x}$.
5. Prove $\prod_{k=1}^{\infty}\frac{k}{x+k}\left(\frac{k+1}{k}\right)^x = \Gamma(x+1)$.
6. Prove Legendre's doubling formula $\Gamma(2x)\Gamma(0.5) = 2^{2x-1}\Gamma(x+0.5)\Gamma(x)$.

4.5. The Cotangent

On the contents of the lecture. In this lecture we perform what was promised at the beginning: we sum up the Euler series and expand $\sin x$ into the product. We will see that sums of series of reciprocal powers are expressed via Bernoulli numbers. And we will see that the function responsible for the summation of the series is the cotangent.

. An ingenious idea, which led Euler to finding the sum $\sum_{k=1}^{\infty}\frac{1}{k^2}$, is the following. One can consider $\sin x$ as a polynomial of infinite degree. This polynomial has as roots all points of the type $k\pi$. Any ordinary polynomial can be expanded into a product $\prod(x-x_k)$ where x_k are its roots. By analogy, Euler conjectured that $\sin x$ can be expanded into the product

$$\sin x = \prod_{k=-\infty}^{\infty}(x-k\pi).$$

This product diverges, but can be modified to a convergent one by division of the n-th term by $-n\pi$. The division does not change the roots. The modified product is

$$\prod_{k=-\infty}^{\infty}\left(1-\frac{x}{k\pi}\right) = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right). \tag{4.5.1}$$

Two polynomials with the same roots can differ by a multiplicative constant. To find the constant, consider $x=\frac{\pi}{2}$. In this case we get the inverse to the Wallis product in (4.5.1) multiplied by $x=\frac{\pi}{2}$. Hence the value of (4.5.1) is 1, which coincides with $\sin\frac{\pi}{2}$. Thus it is natural to expect that $\sin x$ coincides with the product (4.5.1).

There is another way to tame $\prod_{k=-\infty}^{\infty}(x-k\pi)$. Taking the logarithm, we get a divergent series $\sum_{k=-\infty}^{\infty}\ln(x-k\pi)$, but achieve convergence by termwise differentiation. Since the derivative of $\ln\sin x$ is $\cot x$, it is natural to expect that $\cot x$ coincides with the following function

$$\operatorname{ctg}(x) = \sum_{k=-\infty}^{\infty}\frac{1}{x-k\pi} = \frac{1}{x}+\sum_{k=1}^{\infty}\frac{2x}{x^2-k^2\pi^2}. \tag{4.5.2}$$

Cotangent expansion. The expansion $\frac{z}{e^z-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}z^k$ allows us to get a power expansion for $\cot z$. Indeed, representing $\cot z$ by Euler's formula one gets

$$i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{e^{2iz}+1}{e^{2iz}-1} = i+\frac{2i}{e^{2iz}-1} = i+\frac{1}{z}\frac{2iz}{e^{2iz}-1} = i+\frac{1}{z}\sum_{k=0}^{\infty}\frac{B_k}{k!}(2iz)^k.$$

The term of the last series corresponding to $k=1$ is $2izB_1=-iz$. Multiplied by $\frac{1}{z}$, it turns into $-i$, which eliminates the first i. The summand corresponding to $k=0$ is 1. Taking into account that $B_{2k+1}=0$ for $k>0$, we get

$$\cot z = \frac{1}{z}+\sum_{k=1}^{\infty}(-1)^k\frac{4^kB_{2k}}{(2k)!}z^{2k-1}.$$

Power expansion of $\operatorname{ctg}(z)$**.** Substituting

$$\frac{1}{z^2 - n^2\pi^2} = -\frac{1}{n^2\pi^2}\frac{1}{1-\frac{z^2}{n^2\pi^2}} = -\sum_{k=0}^{\infty}\frac{z^{2k}}{(n\pi)^{2k+2}}$$

into (4.5.2) and changing the order of summation, one gets:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{z^{2k}}{(n\pi)^{2k+2}} = \sum_{k=0}^{\infty}\frac{z^{2k}}{\pi^{2k+2}}\sum_{n=1}^{\infty}\frac{1}{n^{2k+2}}.$$

The change of summation order is legitimate in the disk $|z| < 1$, because the series absolutely converges there. This proves the following:

LEMMA 4.5.1. $\operatorname{ctg}(z) - \frac{1}{z}$ *is an analytic function in the disk* $|z| < 1$. *The* n*-th coefficient of the Taylor series of* $\operatorname{ctg}(z) - \frac{1}{z}$ *at* 0 *is equal to* 0 *for even* n *and is equal to* $\frac{1}{\pi^{n+1}}\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}$ *for any odd* n.

Thus the equality $\cot z = \operatorname{ctg}(z)$ would imply the following remarkable equality:

$$\boxed{(-1)^n\frac{4^n B_{2n}}{2n!} = -\frac{1}{\pi^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}}$$

In particular, for $n = 1$ it gives the sum of Euler series as $\frac{\pi^2}{6}$.

Exploring the cotangent.

LEMMA 4.5.2. $|\cot z| \leq 2$ *provided* $|\operatorname{Im} z| \geq 1$.

PROOF. Set $z = x + iy$. Then $|e^{iz}| = |e^{ix-y}| = e^{-y}$. Therefore if $y \geq 1$, then $|e^{2iz}| = e^{-2y} \leq \frac{1}{e^2} < \frac{1}{3}$. Hence $|e^{2iz} + 1| \leq \frac{1}{e^2} + 1 < \frac{4}{3}$ and $|e^{2iz} - 1| \geq 1 - \frac{1}{e^2} > \frac{2}{3}$. Thus the absolute value of

$$\cot z = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i\frac{e^{2iz} + 1}{e^{2iz} - 1}$$

is less than 2. For $y \geq 1$ the same arguments work for the representation of $\cot z$ as $i\frac{1+e^{-2iz}}{1-e^{-2iz}}$. □

LEMMA 4.5.3. $|\cot(\pi/2 + iy)| \leq 4$ *for all* y.

PROOF. $\cot(\pi/2 + iy) = \frac{\cos(\pi/2+iy)}{\sin(\pi/2+iy)} = \frac{-\sin iy}{\cos iy} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$. The module of the numerator of this fraction does not exceed $e - e^{-1}$ for $t \in [-1, 1]$ and the denominator is greater than 1. This proves the inequality for $y \in [-1, 1]$. For other y this is the previous lemma. □

Let us denote by $\pi\mathbb{Z}$ the set $\{k\pi \mid k \in \mathbb{Z}\}$ of π-*integers*.

LEMMA 4.5.4. *The set of singular points of* $\cot z$ *is* $\pi\mathbb{Z}$. *All these points are simple poles with residue* 1.

PROOF. The singular points of $\cot z$ coincide with the roots of $\sin z$. The roots of $\sin z$ are roots of the equation $e^{iz} = e^{-iz}$ which is equivalent to $e^{2iz} = 1$. Since $|e^{2iz}| = |e^{-2\operatorname{Im} z}|$ one gets $\operatorname{Im} z = 0$. Hence $\sin z$ has no roots beyond the real line. And all its real roots as we know have the form $\{k\pi\}$. Since $\lim_{z\to 0} z\cot z = \lim_{z\to 0}\frac{z\cos z}{\sin z} = \lim_{z\to 0}\frac{z}{\sin z} = \frac{1}{\sin' 0} = 1$, we get that 0 is a simple pole of $\cot z$

with residue 1 and the other poles have the same residue because of periodicity of $\cot z$. □

LEMMA 4.5.5. *Let $f(z)$ be an analytic function on a domain D. Suppose that f has in D finitely many singular points, they are not π-integers and D has no π-integer point on its boundary. Then*

$$\oint_{\partial D} f(\zeta)\cot\zeta d\zeta = 2\pi i \sum_{k=-\infty}^{\infty} f(k\pi)[k\pi\in D] + 2\pi i \sum_{z\in D} \operatorname{res}_z(f(z)\cot z)[z\notin \pi\mathbb{Z}].$$

PROOF. In our situation every singular point of $f(z)\cot z$ in D is either a π-integer or a singular point of $f(z)$. Since $\operatorname{res}_{z=k\pi}\cot z = 1$, it follows that $\operatorname{res}_{z=k\pi} f(z)\cot z = f(k\pi)$. Hence the conclusion of the lemma is a direct consequence of Residue Theory. □

Exploring $\operatorname{ctg}(z)$**.**

LEMMA 4.5.6. $\operatorname{ctg}(z+\pi) = \operatorname{ctg}(z)$ *for any z.*

PROOF.

$$\begin{aligned}
\operatorname{ctg}(z+\pi) &= \lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{z+\pi-k\pi}\\
&= \lim_{n\to\infty}\sum_{k=-n-1}^{n-1}\frac{1}{z+k\pi}\\
&= \lim_{n\to\infty}\frac{1}{z-(n+1)\pi} + \lim_{n\to\infty}\frac{1}{z-n\pi} + \lim_{n\to\infty}\sum_{k=-(n-1)}^{(n-1)}\frac{1}{z+\pi-k\pi}\\
&= 0+0+\operatorname{ctg}(z).
\end{aligned}$$

□

LEMMA 4.5.7. *The series representing* $\operatorname{ctg}(z)$ *converges for any z which is not a π-integer.* $|\operatorname{ctg}(z)|\le 2$ *for all z such that* $|\operatorname{Im} z|>\pi$.

PROOF. For any z one has $|z^2-k^2\pi^2|\ge k^2$ for $k>|z|$. This provides the convergence of the series. Since $\operatorname{ctg}(z)$ has period π, it is sufficient to prove the inequality of the lemma in the case $x\in[0,\pi]$, where $z=x+iy$. In this case $|y|\ge|x|$ and $\operatorname{Re} z^2 = x^2-y^2\le 0$. Then $\operatorname{Re}(z^2-k^2\pi^2)\le -k^2\pi^2$. It follows that $|z^2-k^2\pi^2|\ge k^2\pi^2$. Hence $|\operatorname{ctg}(z)|$ is termwise majorized by $\frac{1}{\pi}+\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}<2$. □

LEMMA 4.5.8. $|\operatorname{ctg}(z)|\le 3$ *for any z with* $\operatorname{Re} z = \frac{\pi}{2}$.

PROOF. In this case $\operatorname{Re}(z^2-k^2\pi^2) = \frac{\pi^2}{4}-y^2-k^2\pi^2\le -k^2$ for all $k\ge 1$. Hence $|C(z)|\le\frac{2}{\pi}+\sum_{k=1}^{\infty}\frac{1}{k^2}\le 1+2=3$. □

LEMMA 4.5.9. *For any $z\ne k\pi$ and domain D which contains z and whose boundary does not contain π-integers, one has*

$$\oint_{\partial D}\frac{\operatorname{ctg}(\zeta)}{\zeta-z}d\zeta = 2\pi i\operatorname{ctg}(z) + 2\pi i\sum_{k=-\infty}^{\infty}\frac{1}{k\pi-z}[k\pi\in D]. \tag{4.5.3}$$

PROOF. As was proved in Lecture 3.6, the series $\sum_{k=-\infty}^{\infty} \frac{1}{(\zeta-z)(\zeta-k\pi)}$ admits termwise integration. The residues of $\frac{1}{(\zeta-z)(\zeta-k\pi)}$ are $\frac{1}{k\pi-z}$ at $k\pi$ and $\frac{1}{z-k\pi}$ at z. Hence

$$\oint_{\partial D} \frac{1}{(\zeta-z)(\zeta-k\pi)} d\zeta = \begin{cases} 2\pi i \frac{1}{z-k\pi} & \text{for } k\pi \notin D, \\ 0 & \text{if } k\pi \in D. \end{cases}$$

It follows that

$$\oint_{\partial D} \frac{\operatorname{ctg}(\zeta)}{\zeta - z} d\zeta = 2\pi i \sum_{k=-\infty}^{\infty} \frac{1}{z-k\pi}[k\pi \notin D]$$
$$= 2\pi i \operatorname{ctg}(z) - \sum_{k=-\infty}^{\infty} \frac{1}{z-k\pi}[k\pi \in D].$$

□

LEMMA 4.5.10. $\operatorname{ctg}(z)$ *is an analytic function defined on the whole plane, having all* π*-integers as its singular points, where it has residues* 1.

PROOF. Consider a point $z \notin \pi\mathbb{Z}$. Consider a disk D, not containing π-integers with center at z. Then formula (4.5.3) transforms to the Cauchy Integral Formula. And our assertion is proved by termwise integration of the power expansion of $\frac{1}{\zeta-z}$ just with the same arguments as was applied there. The same formula (4.5.3) allows us to evaluate the residues. □

THEOREM 4.5.11. $\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2\pi^2}$.

PROOF. Consider the difference $R(z) = \cot z - \operatorname{ctg}(z)$. This is an analytic function which has π-integers as singular points and has residues 0 in all of these. Hence $R(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{R(\zeta)}{\zeta - z} d\zeta$ for any $z \notin \pi\mathbb{Z}$. We will prove that $R(z)$ is constant. Let z_0 and ζ be a pair of different points not belonging to $\pi\mathbb{Z}$. Then for any D such that $\partial D \cap \pi\mathbb{Z} = \varnothing$ one has

$$R(z) - R(z_0) = \frac{1}{2\pi i} \oint_{\partial D} R(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) d\zeta$$
$$= \frac{1}{2\pi i} \oint_{\partial D} \frac{R(z)(z-z_0)}{(\zeta-z)(\zeta-z_0)}. \tag{4.5.4}$$

Let us define D_n for a natural $n > 3$ as the rectangle bounded by the lines $\operatorname{Re} z = \pm(\pi/2 - n\pi)$, $\operatorname{Im} z = \pm n\pi$. Since $|R(z)| \leq 7$ by Lemmas 4.5.2, 4.5.3, 4.5.7, and 4.5.8 the integrand of (4.5.4) eventually is bounded by $\frac{7|z-z_0|}{n^2}$. The contour of integration consists of four monotone curves of diameter $< 2n\pi$. By the Estimation Lemma 3.5.4, the integral can be estimated from above by $\frac{32\pi n 7|z-z_0|}{n^2}$. Hence the limit of our integral as n tends to infinity is 0. This implies $R(z) = R(z_0)$. Hence $R(z)$ is constant and the value of the constant we find by putting $z = \pi/2$. As $\cot \pi/2 = 0$, the value of the constant is

$$\operatorname{ctg}(\pi/2) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{\pi/2 - k\pi} = \frac{2}{\pi} \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{1-2k}.$$

This limit is zero because

$$\sum_{k=-n}^{n} \frac{1}{1-2k} = \sum_{k=-n}^{0} \frac{1}{1-2k} + \sum_{k=1}^{n} \frac{1}{1-2k} = \sum_{k=0}^{n} \frac{1}{2k+1} + \sum_{k=1}^{n} -\frac{1}{2k-1} = \frac{1}{2n+1}.$$

□

Summation of series by $\cot z$**.**

THEOREM 4.5.12. *For any rational function* $R(z)$*, which is not singular in integers and has degree* ≤ -2*, one has* $\sum_{k=-\infty}^{\infty} R(n) = -\sum_z \operatorname{res} \pi \cot(\pi z) R(z)$*.*

PROOF. In this case the integral $\lim_{n\to\infty} \oint_{\partial D_n/pi} R(z)\pi \cot \pi z = 0$. Hence the sum of all residues of $R(z)\pi\cot\pi z$ is zero. The residues at π-integers gives $\sum_{k=-\infty}^{\infty} R(k)$. The rest gives $-\sum_z \operatorname{res} \pi \cot(\pi z) R(z)$. □

Factorization of $\sin x$**.** Theorem 4.5.11 with πz substituted for z gives the series $\pi \cot \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{z-k}$. The half of the series on the right-hand side consisting of terms with nonnegative indices represents a function, which formally telescopes $-\frac{1}{z}$. The negative half telescopes $\frac{1}{z}$. Let us bisect the series into nonnegative and negative halves and add $\sum_{k=-\infty}^{\infty} \frac{1}{k}[k \neq 0]$ to provide convergence:

$$\sum_{k=-\infty}^{-1} \left(\frac{1}{z-k} + \frac{1}{k}\right) + \sum_{k=0}^{\infty} \left(\frac{1}{z-k} + \frac{1}{k+1}\right)$$
$$= \sum_{k=1}^{\infty} \left(-\frac{1}{k} + \frac{1}{z+k}\right) + \sum_{k=1}^{\infty} \left(\frac{1}{z+1-k} + \frac{1}{k}\right).$$

The first of the series on the right-hand side represents $-\digamma(z) - \gamma$, the second is $\digamma(-z+1)+\gamma$. We get the following *complement formula* for the digamma function:

$$-\digamma(z) + \digamma(1-z) = \pi \cot \pi z.$$

Since $\Theta''(z+1) = \digamma'(z) = \mathit{\Gamma}(z)$ (Lemma 4.4.11) it follows that $\Theta'(1+z) = \digamma(z)+c$ and $\Theta'(-z) = -(\digamma(1-z)+c)$. Therefore $\Theta'(1+z)+\Theta'(-z) = \pi\cot\pi z$. Integration of the latter equality gives $-\Theta(1+z) - \Theta(-z) = \ln \sin \pi z + c$. Changing z by $-z$ we get $\Theta(1-z) + \Theta(z) = -\ln\sin\pi z + c$. Exponentiating gives $\Gamma(1-z)\Gamma(-z) = \frac{1}{\sin \pi z} c$. One defines the constant by putting $z = \frac{1}{2}$. On the left-hand side one gets $\Gamma(\frac{1}{2})^2 = \pi$, on the right-hand side, c. Finally we get the *complement formula for the Gamma-function*:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}. \tag{4.5.5}$$

Now consider the product $\prod_{k=1}^{\infty}(1 - \frac{x^2}{k^2})$. Its canonical form is

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \right\}^{-1} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \right\}^{-1}. \tag{4.5.6}$$

The first product of (4.5.6) is equal to $-\frac{e^{\gamma x}}{x\Gamma(-x)}$, and the second one is $\frac{e^{-\gamma x}}{x\Gamma(-x)}$. Therefore the whole product is $-\frac{1}{x^2\Gamma(x)\Gamma(-x)}$. Since $\Gamma(1-x) = -x\Gamma(-x)$ we get the following result

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right).$$

Comparing this to (4.5.5) and substituting πx for x we get the Euler formula:

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right).$$

Problems.

1. Expand $\tan z$ into a power series.

2. Evaluate $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

3. Evaluate $\sum_{k=1}^{\infty} \frac{1}{1+k^4}$.

4.6. Divergent Series

On the contents of the lecture. "Divergent series is a pure handiwork of Diable. It is a full nonsense to say that $1^{2n} - 2^{2n} + 3^{2n} - \cdots = 0$. Do you keep to die laughing about this?" (N.H. Abel letter to ...). The twist of fate: now one says that that the above mentioned equality holds in *Abel's sense*.

The earliest analysts thought that any series, convergent or divergent, has a sum given by God and the only problem is to find it correctly. Sometimes they disagreed what is the correct answer. In the nineteenth century divergent series were expelled from mathematics as a "handiwork of Diable" (N.H. Abel). Later they were rehabilitated (see G.H. Hardy's book *Divergent Series*[1]). Euler remains the unsurpassed master of divergent series. For example, with the help of divergent series he discovered Riemann's functional equation of the ζ-function a hundred years before Riemann.

Evaluations with divergent series. Euler wrote: "My pen is clever than myself." Before we develop a theory let us simply follow to Euler's pen. The fundamental equality is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \tag{4.6.1}$$

Now we, following Euler, suppose that this equality holds for all $x \neq 1$. In the second lecture we were confused by some unexpected properties of divergent series. But now in contrast with the second lecture we do not hurry up to land. Let us look around.

Substituting $x = -e^y$ in (4.6.1) one gets

$$1 - e^y + e^{2y} - e^{3y} + \cdots = \frac{1}{1+e^y}.$$

On the other hand

$$\frac{1}{1+e^y} = \frac{1}{e^y - 1} - \frac{2}{e^{2y} - 1}. \tag{4.6.2}$$

Since

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k. \tag{4.6.3}$$

One derives from (4.6.2) via (4.6.3)

$$\frac{1}{e^y + 1} = \sum_{k=1}^{\infty} \frac{B_k(1-2^k)}{k!} y^{k-1}. \tag{4.6.4}$$

Let us differentiate repeatedly n-times the equality (6) by y. The left-hand side gives $\sum_{k=0}^{\infty}(-1)^k k^n e^{ky}$. In particular for $y = 0$ we get $\sum_{k=0}^{\infty}(-1)^k k^n$. We get on the right-hand side by virtue of (4.6.4) the following

$$\left(\frac{d}{dy}\right)^n \frac{1}{1+e^y} = \frac{B_{n+1}(1-2^{n+1})}{n+1}.$$

Combining these results we get the following equality

$$1^n - 2^n + 3^n - 4^n + \cdots = \frac{B_{n+1}(2^{n+1}-1)}{n+1}. \tag{4.6.5}$$

[1] G.H. Hardy, *Divergent Series*, Oxford University Press, 1949.

Since odd Bernoulli numbers vanish, we get

$$1^{2n} - 2^{2n} + 3^{2n} - 4^{2n} + \dots = 0.$$

Consider an even analytic function $f(x)$, such that $f(0) = 0$. In this case $f(x)$ is presented by a power series $a_1x^2 + a_2x^4 + a_3x^6 + \dots$, then

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{f(kx)}{k^2} &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\sum_{n=1}^{\infty}a_nx^{2n}k^{2n}\\ &= \sum_{n=1}^{\infty}a_nx^{2n}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-2}\\ &= a_1x^2(1-1+1-1+\dots)\\ &= \frac{a_1x^2}{2}.\end{aligned}$$

In particular, for $f(x) = 1 - \cos x$ this equality turns into

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1-\cos kx}{k^2} = \frac{x^2}{4}. \tag{4.6.6}$$

For $x = \pi$ the equality (4.6.6) gives

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

Since

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \left(1-\frac{1}{4}\right)\sum_{k=1}^{\infty}\frac{1}{k^2}$$

one derives the sum of the Euler series:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

We see that calculations with divergent series sometimes give brilliant results. But sometimes they give the wrong result. Indeed the equality (4.6.6) generally is untrue, because on the left-hand side we have a periodic function and on the right-hand side a non-periodic one. But it is true for $x \in [-\pi, \pi]$. Termwise differentiation of (4.6.6) gives the true equality (3.4.2), which we know from Lecture 3.4.

Euler's sum of a divergent series. Now we develop a theory justifying the above evaluations. Euler writes that the value of an infinite expression (in particular the sum of a divergent series) is equal to the value of a finite expression whose expansion gives this infinite expression. Hence, numerical equalities arise by substituting a numerical value for a variable in a generating functional identity. To evaluate the sum of a series $\sum_{k=0}^{\infty} a_k$ Euler usually considers its *power generating function* $g(z)$ represented by the power series $\sum_{k=0}^{\infty} a_k z^k$, and supposes that the sum of the series is equal to $g(1)$.

To be precise suppose that the power series $\sum_{k=0}^{\infty} a_k z^k$ converges in a neighborhood of 0 and there is an analytic function $g(z)$ defined in a domain U containing a path p from 0 to 1 and such that $g(z) = \sum_{k=0}^{\infty} a_k z^k$ for z sufficiently close to 0 and 1 is a regular point of g. Then the series $\sum_{k=0}^{\infty} a_k$ is called *Euler summable* and the value $g(1)$ is called its *analytic Euler sum* with respect to p. And we will use a special sign $\simeq$ to denote the analytical sum.

By the Uniqueness Theorem 3.6.9 the value of analytic sum of a series is uniquely defined for a fixed p. But this value generally speaking depends on the path. For example, let us consider the function $\sqrt{1+x}$. Its binomial series for $x=-2$ turns into

$$-1+1-\frac{1}{2!}-\frac{1\cdot 3}{3!}-\frac{1\cdot 3\cdot 5}{4!}-\dots-\frac{(2k-1)!!}{(k+1)!}-\dots.$$

For $p(t)=e^{i\pi t}$ one sums up this series to i, because it is generated by the function $\exp\frac{\ln(1+z)}{2}$ defined in the upper half-plane. And along $p(t)=e^{-i\pi t}$ this series is summable to $-i$ by $\exp\frac{-\ln(1+z)}{2}$ defined in the lower half-plane.

For a fixed path the analytic Euler sum evidently satisfies the Shift, Multiplication and Addition Formulas of the first lecture. But we see that the analytic sum of a real series may be purely imaginary. Hence the rule $\operatorname{Im}\sum_{k=0}^{\infty}a_k\simeq\sum_{k=0}^{\infty}\operatorname{Im}a_k$ fails for the analytic sum. The Euler sum along $[0,1]$ coincides with the Abel sum of the series in the case when both of them exist.

In above evaluations we apply termwise differentiation to functional series. If the Euler sum $\sum_{k=1}^{\infty}f_k(z)$ is equal to $F(z)$ for all z in a domain this does not guarantee the possibility of termwise differentiation. To guarantee it we suppose that the function generating the equality $\sum_{k=1}^{\infty}f_k(z)\simeq F(z)$ analytically depends on z. To formalize the last condition we have to introduce analytic functions of two variables.

Power series of two variables. A power series of two variables z,w is defined as a formal unordered sum $\sum_{k,m}a_{km}z^kw^m$, over $\mathbb{N}\times\mathbb{N}$ — the set of all pairs of nonnegative integers.

For a function of two variables $f(z,w)$ one defines its *partial derivative* $\frac{\partial f(z_0,w_0)}{\partial z}$ with respect to z at the point (z_0,w_0) as the limit of $\frac{f(z_0+\Delta z,w_0)-f(z_0,w_0)}{\Delta z}$ as Δz tends to 0.

LEMMA 4.6.1. *If $\sum a_{km}z_1^kw_1^m$ absolutely converges, then both $\sum a_{km}z^kw^m$ and $\sum ma_{km}z^kw^{m-1}$ absolutely converge provided $|z|<|z_1|$, $|w|<|w_1|$. And for any fixed z, such that $|z|<|z_1|$ the function $\sum ma_kz^kw^{m-1}$ is the partial derivative of $\sum a_{km}z^kw^m$ with respect to w.*

PROOF. Since $\sum|a_{km}||z_1|^k|w_1|^m<\infty$ the same is true for $\sum|a_{km}||z|^k|w|^m$ for $|z|<|z_1|$, $|w|<|w_1|$. By the Sum Partition Theorem we get the equality

$$\sum a_{km}z^kw^m=\sum_{m=0}^{\infty}w^m\sum_{k=0}^{\infty}a_{km}z^k.$$

For any fixed z the right-hand side of this equality is a power series with respect to w as the variable. By Theorem 3.3.9 its derivative by w, which coincides with the partial derivative of the left-hand side, is equal to

$$\sum_{m=0}^{\infty}mw^{m-1}\sum_{k=0}^{\infty}a_{km}z^k=\sum ma_{km}w^{m-1}z^k.$$

□

Analytic functions of two variables. A function of two variables $F(z,w)$ is called analytic at the point (z_0,w_0) if for (z,w) sufficiently close to (z_0,w_0) it can be presented as a sum of a power series of two variables.

THEOREM 4.6.2.

(1) *If $f(z,w)$ and $g(z,w)$ are analytic functions, then $f+g$ and fg are analytic functions.*
(2) *If $f_1(z), f_2(z)$ and $g(z,w)$ are analytic functions, then $g(f_1(z), f_2(w))$ and $f_1(g(z,w))$ are analytic functions.*
(3) *The partial derivative of any analytic function is an analytic function.*

PROOF. The third statement follows from Lemma 4.6.1. The proofs of the first and the second statements are straightforward and we leave them to the reader. □

Functional analytical sum. Let us say that a series $\sum_{k=1}^{\infty} f_k(z)$ of analytic functions is *analytically summable* to a function $F(z)$ in a domain $U \subset \mathbb{C}$ along a path p in $\mathbb{C} \times \mathbb{C}$, such that $p(0) \in U \times 0$ and $p(1) \in U \times 1$, if there exists an analytic function of two variables $F(z,w)$, defined on a domain W containing p, $U \times 0$, $U \times 1$, such that for any $z_0 \in U$ the following two conditions are satisfied:

(1) $F(z_0, 1) = F(z_0)$.
(2) $F(z,w) = \sum \frac{f_m^{(k)}(z_0)}{k!}(z-z_0)^k w^m$ for sufficiently small $|w|$ and $|z-z_0|$.

Let us remark that the analytic sum does not change if we change p keeping it inside W. That is why one says that the sum is evaluated along the domain W.

To denote the functional analytical sum we use the sign $\cong$. And we will write also $\cong_W$ and $\cong_p$ to specify the domain or the path of summation.

The function $F(z,w)$ will be called the *generating function* for the analytical equality $\sum_{k=1}^{\infty} f_k(z) \cong F(z)$.

LEMMA 4.6.3. *If $f(z)$ is an analytic function in a domain U containing 0, such that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for sufficiently small $|z|$, then $f(z) \cong_W \sum_{k=0}^{\infty} a_k z^k$ in U for $W = \{(z,w) \mid wz \in U\}$.*

PROOF. The generating function of this analytical equality is $f((z-z_0)w)$. □

LEMMA 4.6.4 (on substitution). *If $F(z) \cong_p \sum_{k=0}^{\infty} f_k(z)$ in U and $g(z)$ is an analytic function, then $F(g(z)) \cong_{g(p)} \sum_{k=0}^{\infty} f_k(g(z))$ in $g^{-1}(U)$.*

PROOF. Indeed, if $F(z,w)$ generates $F(z) \cong_p \sum_{k=0}^{\infty} f_k(z)$, then $F(g(z),w))$ generates $F(g(z)) \cong_{g(p)} \sum_{k=0}^{\infty} f_k(g(z))$. □

N. H. Abel was the first to have some doubts about the legality of termwise differentiation of functional series. The following theorem justifies this operation for analytic functions.

THEOREM 4.6.5. *If $\sum_{k=1}^{\infty} f_k(z) \cong_p F(z)$ in U then $\sum_{k=1}^{\infty} f_k'(z) \cong_p F'(z)$ in U.*

PROOF. Let $F(z,w)$ be a generating function for $\sum_{k=1}^{\infty} f_k(z) \cong_p F(z)$. We demonstrate that its partial derivative by z (denoted $F'(z,w)$) is the generating function for $\sum_{k=1}^{\infty} f_k'(z) \cong_p F'(z)$. Indeed, locally in a neighborhood of $(z_0, 0)$ one has $F(z,w) = \sum \frac{f_m^{(k)}(z_0)}{k!} w^m (z-z_0)^k$. By virtue of Lemma 4.6.1 its derivative by z is $F'(z,w) = \sum \frac{f_m^{(k)}(z_0)}{(k-1)!} w^m (z-z_0)^{k-1} = \sum \frac{f_m'^{(k)}(z_0)}{k!} w^m (z-z_0)^k$. □

The dual theorem on termwise integration is the following one.

THEOREM 4.6.6. *Let $\sum_{k=1}^{\infty} f_k \cong F$ be generated by $F(z,w)$ defined on $W = U \times V$. Then for any path q in U one has $\int_q F(z)\,dz \simeq \sum_{k=1}^{\infty} \int_q f_k(z)\,dz$.*

PROOF. The generating function for integrals is defined as $\int_q F(z,w)\,dz$. □

The proof of the following theorem is left to the reader.

THEOREM 4.6.7. *If* $\sum_{k=0}^{\infty} f_k \cong_p F$ *and* $\sum_{k=0}^{\infty} g_k \cong_p G$ *then* $\sum_{k=0}^{\infty}(f_k+g_k) \cong_p F+G$, $\sum_{k=1}^{\infty} f_k \cong_p F - f_0$, $\sum_{k=0}^{\infty} cf_k \cong_p cF$

Revision of evaluations. Now we are ready to revise the above evaluation equipped with the theory of analytic sums. Since all considered generating functions in this paragraph are single valued, the results do not depend on the choice of the path of summation. That is why we drop the indications of path below.

The equality (4.6.1) is the analytical equivalence generated by $\frac{1}{1-tx}$. The next equality (4.6.7) is the analytical equivalence by Lemma 4.6.4. The equality (4.6.3) is analytical equivalence due to Lemma 4.6.3. Termwise differentiation of (4.6.7) is correct by virtue of Theorem 4.6.5. Therefore the equality (4.6.5) is obtained by the restriction of an analytical equivalence. Hence the Euler sum of $\sum_{k=1}^{\infty}(-1)^k k^{2n}$ is equal to 0. Since the series $\sum_{k=1}^{\infty}(-1)^k k^{2n} z^k$ converges for $|z|<1$ its value coincides with the value of the generating function. And the limit $\lim\limits_{z\to 1-0}\sum_{k=1}^{\infty}(-1)^k k^{2n} z^k$ gives the Euler sum, which is zero. Hence as a result of our calculations we have found *Abel's sum* $\sum_{k=1}^{\infty}(-1)^k k^{2n} = 0$.

Now we choose another way to evaluate the Euler series. Substituting $x = e^{\pm i\theta}$ in (4.6.1) for $0<\theta<2\pi$ one gets

$$\begin{aligned} 1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+\ldots &\cong \frac{1}{1-e^{i\theta}}, \\ 1+e^{-i\theta}+e^{-2i\theta}+e^{-3i\theta}+\ldots &\cong \frac{1}{1-e^{-i\theta}}. \end{aligned} \tag{4.6.7}$$

Termwise addition of the above lines gives for $\theta\in(0,2\pi)$ the following equality

$$\cos\theta+\cos 2\theta+\cos 3\theta+\cdots \cong -\frac{1}{2}. \tag{4.6.8}$$

Integration of (4.6.8) from π to x with subsequent replacement of x by θ gives by Theorem 4.6.6:

$$\sum_{k=1}^{\infty}\frac{\sin k\theta}{k} \cong \frac{\pi-\theta}{2} \quad (0<\theta<2\pi).$$

A second integration of the same type gives

$$\sum_{k=1}^{\infty}\frac{\cos k\theta-(-1)^k}{k^2} \cong \frac{(\pi-\theta)^2}{4}.$$

Putting $\theta=\frac{\pi}{2}$ we get

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} \simeq \frac{\pi^2}{16}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} + 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$

one gets

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6}.$$

Problems.

1. Prove that the analytic sum of convolution of two series is equal to the product of analytic sums of the series.

2. Suppose that for all $n \in \mathbb{N}$ one has $A_n \simeq \sum_{k=0}^{\infty} a_{n,k}$ and $B_n \simeq \sum_{k=0}^{\infty} a_{k,n}$. Prove that the equality $\sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} B_k$ holds provided there is an analytic function $F(z,w)$ coinciding with $\sum a_{k,n} z^k w^n$ for sufficiently small $|w|$,$|z|$ which is defined on a domain containing a path joining $(0,0)$ with $(1,1)$ analytically extended to $(1,1)$ (i.e., $(1,1)$ is a regular point of $F(z,w)$).