CHAPTER 2

Integrals

2.1. Natural Logarithm

On the contents of the lecture.

> In the beginning of Calculus was the Word, and the Word was with Arithmetic, and the Word was *Logarithm*[1]

Logarithmic tables. Multiplication is much more difficult than addition. A logarithm reduces multiplication to addition. The invention of logarithms was one of the great achievements of our civilization.

In early times, when logarithms were unknown instead of them one used trigonometric functions. The following identity

$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$

can be applied to calculate products via tables of cosines. To multiply numbers x and y, one represents them as cosines $x = \cos a$, $y = \cos b$ using the cosine table. Then evaluate $(a+b)$ and $(a-b)$ and find their cosines in the table. Finally, the results are summed and divided by 2. That is all. A single multiplication requires four searches in the table of cosines, two additions, one subtraction and one division by 2.

A logarithmic function $l(x)$ is a function such that $l(xy) = l(x) + l(y)$ for any x and y. If one has a logarithmic table, to evaluate the product xy one has to find in the logarithmic table $l(x)$ and $l(y)$ then sum them and find the antilogarithm of the sum. This is much easier.

The idea of logarithms arose in 1544, when M. Stiefel compared geometric and arithmetic progressions. The addition of exponents corresponds to the multiplication of powers. Hence consider a number close to 1, say, 1.000001. Calculate the sequence of its powers and place them in the left column. Place in the right column the corresponding values of exponents, which are just the line numbers. The logarithmic table is ready.

Now to multiply two numbers x and y, find them (or their approximations) in the left column of the logarithmic table, and read their logarithms from the right column. Sum the logarithms and find the value of the sum in the right column. Next to this sum in the left column the product xy stands. The first tables of such logarithms were composed by John Napier in 1614.

Area of a curvilinear trapezium. Recall that a sequence is said to be monotone, if it is either increasing or decreasing. The minimal interval which contains all elements of a given sequence of points will be called *supporting interval* of the sequence. And a sequence is called *exhausting* for an interval I if I is the supporting interval of the sequence.

Let f be a non-negative function defined on $[a,b]$. The set $\{(x,y) \mid x \in [a,b] \text{ and } 0 \le y \le f(x)\}$ is called a *curvilinear trapezium* under the graph of f over the interval $[a,b]$.

To estimate the area of a curvilinear trapezium under the graph of f over $[a,b]$, choose an exhausting sequence $\{x_i\}_{i=0}^{n}$ for $[a,b]$ and consider the following sums:

$$\sum_{k=0}^{n-1} f(x_k)|\delta x_k|, \qquad \sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k| \quad (\text{where } \delta x_k = x_{k+1} - x_k). \tag{2.1.1}$$

[1] $\lambda o\gamma o\varsigma$ is Greek for "word", $\alpha\varrho\iota\theta\mu o\varsigma$ means "number".

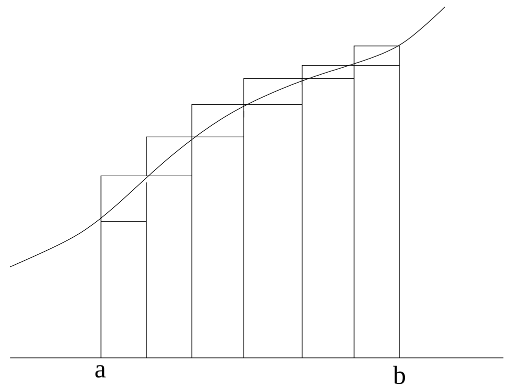

FIGURE 2.1.1. A curvilinear trapezium

We will call the first of them the *receding sum*, and the second the *advancing sum*, of the sequence $\{x_k\}$ for the function f. If the function f is monotone the area of the curvilinear trapezium is contained between these two sums. To see this, consider the following step-figures: $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_k)]$ and $\bigcup_{k=0}^{n-1}[x_k, x_{k+1}] \times [0, f(x_{k+1})]$. If f and $\{x_k\}$ both increase or both decrease the first step-figure is contained in the curvilinear trapezium and the second step-figure contains the trapezium with possible exception of a vertical segment $[a \times [0, f(a)]$ or $[b \times [0, f(b)]$. If one of f and $\{x_k\}$ increases and the other decreases, then the step-figures switch the roles. The rededing sum equals the area of the first step-figure, and the advancing sum equals the area of the second one. Thus we have proved the following lemma.

LEMMA 2.1.1. *Let f be a monotone function and let S be the area of the curvilinear trapezium under the graph of f over $[a, b]$. Then for any sequence $\{x_k\}_{k=0}^{n}$ exhausting $[a, b]$ the area S is contained between $\sum_{k=0}^{n-1} f(x_k)|\delta x_k|$ and $\sum_{k=0}^{n-1} f(x_{k+1})|\delta x_k|$.*

Fermat's quadratures of parabolas. In 1636 Pierre Fermat proposed an ingenious trick to determine the area below the curve $y = x^a$.

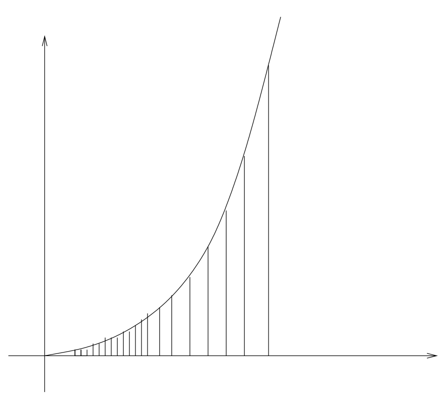

FIGURE 2.1.2. Fermat's quadratures of parabolas

If $a > -1$ then consider any interval of the form $[0, B]$. Choose a positive $q < 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[0, B]$ and the values of the function for this sequence also form a geometric progression $B^a, q^aB^a, q^{2a}B^a, q^{3a}B^a, \dots$. Then both the receding and advancing sums turn into geometric progressions:

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^kB - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(1-q)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^kB - q^{k+1}B) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(1-q)q^a}{1-q^{a+1}}.\end{aligned}$$

For a natural a, one has $\frac{1-q}{1-q^{a+1}} = \frac{1}{1+q+q^2+\dots+q^a}$. As q tends to 1 both sums converge to $\frac{B^{a+1}}{a+1}$. This is the area of the curvilinear trapezium. Let us remark that for $a < 0$ this trapezium is unbounded, nevertheless it has finite area if $a > -1$.

If $a < -1$, then consider an interval in the form $[B, \infty]$. Choose a positive $q > 1$. Then the infinite geometric progression $B, Bq, Bq^2, Bq^3, \dots$ exhausts $[B, \infty]$ and the values of the function for this sequence also form a geometric progression $B^a, q^aB^a, q^{2a}B^a, q^{3a}B^a, \dots$. The receding and advancing sums are

$$\begin{aligned}\sum_{k=0}^{\infty} B^a q^{ka}(q^{k+1}B - q^kB) &= B^{a+1}(q-1)\sum_{k=0}^{\infty} q^{k(a+1)}\\ &= \frac{B^{a+1}(q-1)}{1-q^{a+1}},\\ \sum_{k=0}^{\infty} B^a q^{(k+1)a}(q^{k+1}B - q^kB) &= B^{a+1}(1-q)\sum_{k=0}^{\infty} q^{(k+1)(a+1)}\\ &= \frac{B^{a+1}(q-1)q^a}{1-q^{a+1}}.\end{aligned}$$

If a is an integer set $p = q^{-1}$. Then $\frac{q-1}{1-q^{a+1}} = q\frac{1-p}{1-p^{|a|-1}} = q\frac{1}{1+p+p^2+\dots+p^{n-2}}$. As q tends to 1 both sums converge to $\frac{B^{a+1}}{|a|-1}$. This is the area of the curvilinear trapezium.

For $a > -1$ the area of the curvilinear trapezium under the graph of x^a over $[A, B]$ is equal to the difference between the areas of trapezia over $[0, B]$ and $[0, A]$. Hence this area is $\frac{B^{a+1}-A^{a+1}}{a+1}$.

For $a < -1$ one can evaluate the area of the curvilinear trapezium under the graph of x^a over $[A, B]$ as the difference between the areas of trapezia over $[A, \infty]$ and $[B, \infty]$. The result is expressed by the same formula $\frac{B^{a+1}-A^{a+1}}{a+1}$.

THEOREM 2.1.2 (Fermat). *The area below the curve $y = x^a$ over the interval $[A, B]$ is equal to $\frac{B^{a+1}-A^{a+1}}{a+1}$ for $a \neq 1$.*

We have proved this theorem for integer a, but Fermat proved it for all real $a \neq -1$.

The Natural Logarithm. In the case $a = -1$ the geometric progression for areas of step-figures turns into an arithmetic progression. This means that the area below a hyperbola is a logarithm! This discovery was made by Gregory in 1647.

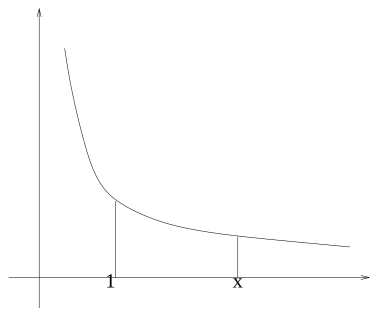

FIGURE 2.1.3. The hyperbolic trapezium over $[1, x]$

The figure bounded from above by the graph of hyperbola $y = 1/x$, from below by segment $[a, b]$ of the axis of abscissas, and on each side by vertical lines passing through the end points of the interval, is called a *hyperbolic trapezium over* $[a, b]$.

The area of hyperbolic trapezium over $[1, x]$ with $x > 1$ is called the *natural logarithm* of x, and it is denoted by $\ln x$. For a positive number $x < 1$ its logarithm is defined as the negative number whose absolute value coincides with the area of hyperbolic trapezium over $[x, 1]$. At last, $\ln 1$ is defined as 0.

THEOREM 2.1.3 (on logarithm). *The natural logarithm is an increasing function defined for all positive numbers. For each pair of positive numbers x, y*

$$\ln xy = \ln x + \ln y.$$

PROOF. Consider the case $x, y > 1$. The difference $\ln xy - \ln y$ is the area of the hyperbolic trapezium over $[y, xy]$. And we have to prove that it is equal to $\ln x$, the area of trapezium over $[1, x]$. Choose a large number n. Let $q = x^{1/n}$. Then $q^n = x$. The finite geometric progression $\{q^k\}_{k=0}^n$ exhausts $[1, x]$. Then the receding and advancing sums are

$$(2.1.2) \qquad \sum_{k=0}^{n-1} q^{-k}(q^{k+1} - q^k) = n(q-1) \qquad \sum_{k=0}^{n-1} q^{-k-1}(q^{k+1} - q^k) = \frac{n(q-1)}{q}.$$

Now consider the sequence $\{xq^k\}_{k=0}^n$ exhausting $[x, xy]$. Its receding sum

$$\sum_{k=0}^{n-1} x^{-1}q^{-k}(xq^{k+1} - xq^k) = n(q-1)$$

just coincides with the receding sum (2.1.2) for $\ln x$. The same is true for the advancing sum. As a result we obtain for any natural n the following inequalities:

$$n(q-1) \geq \ln x \geq \frac{n(q-1)}{q} \qquad n(q-1) \geq \ln xy - \ln y \geq \frac{n(q-1)}{q}$$

This implies that $|\ln xy - \ln x - \ln y|$ does not exceed the difference between the the receding and advancing sums. The statement of Theorem 2.1.3 in the case $x, y > 1$ will be proved when we will prove that this difference can be made arbitrarily small by a choice of n. This will be deduced from the following general lemma.

LEMMA 2.1.4. *Let f be a monotone function over the interval $[a,b]$ and let $\{x_k\}_{k=0}^n$ be a sequence that exhausts $[a,b]$. Then*

$$\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| \leq |f(b)-f(a)| \max_{k<n} |\delta x_k|$$

PROOF OF LEMMA. The proof of the lemma is a straightforward calculation. To shorten the notation, set $\delta f(x_k) = f(x_{k+1}) - f(x_k)$.

$$\begin{aligned}\left|\sum_{k=0}^{n-1} f(x_k)\delta x_k - \sum_{k=0}^{n-1} f(x_{k+1})\delta x_k\right| &= \left|\sum_{k=0}^{n-1} \delta f(x_k)\delta x_k\right| \\ &\leq \sum_{k=0}^{n-1} |\delta f(x_k)| \max|\delta x_k| \\ &= \max|\delta x_k| \sum_{k=0}^{n-1} |\delta f(x_k)| \\ &= \max|\delta x_k| \left|\sum_{k=0}^{n-1} \delta f(x_k)\right| \\ &= \max|\delta x_k||f(b)-f(a)|.\end{aligned}$$

The equality $\left|\sum_{k=0}^{n-1} \delta f(x_k)\right| = \sum_{k=0}^{n-1} |\delta f(x_k)|$ holds, as $\delta f(x_k)$ have the same signs due to the monotonicity of f. □

The value $\max|\delta x_k|$ is called *maximal step* of the sequence $\{x_k\}$. For the sequence $\{q^k\}$ of $[1,x]$ its maximal step is equal to $q^n - q^{n-1} = q^n(1-q^{-1}) = x(1-q)/q$. It tends to 0 as q tends to 1. In our case $|f(b)-f(a)| = 1 - \frac{1}{x} < 1$. By Lemma 2.1.4 the difference between the receding and advancing sums could be made arbitrarily small. This completes the proof in the case $x, y > 1$.

Consider the case $xy = 1$, $x > 1$. We need to prove the following

(inversion rule) $$\ln 1/x = -\ln x.$$

As above, put $q^n = x > 1$. The sequence $\{q^{-k}\}_{k=0}^n$ exhausts $[1/x, 1]$. The corresponding receding sum $\sum_{k=0}^{n-1} q^{k+1}(q^{-k} - q^{-k-1}) = \sum_{k=0}^{n-1}(q-1) = n(q-1)$ coincides with its counterpart for $\ln x$. The same is true for the advancing one. The same arguments as above prove $|\ln 1/x| = \ln x$. The sign of $\ln 1/x$ is defined as minus because $1/x < 1$. This proves the inversion rule.

Now consider the case $x < 1$, $y < 1$. Then $1/x > 1$ and $1/y > 1$ and by the first case $\ln 1/xy = (\ln 1/x + \ln 1/y)$. Replacing all terms of this equation according to the inversion rule, one gets $-\ln xy = -\ln x - \ln y$ and finally $\ln xy = \ln x + \ln y$.

The next case is $x > 1$, $y < 1$, $xy < 1$. Since both $1/x$ and xy are less then 1, then by the previous case $\ln xy + \ln 1/x = \ln \frac{xy}{x} = \ln y$. Replacing $\ln 1/x$ by $-\ln x$ one gets $\ln xy - \ln x = \ln y$ and finally $\ln xy = \ln x + \ln y$.

The last case, $x > 1$, $y < 1$, $xy > 1$ is proved by $\ln xy + \ln 1/y = \ln x$ and replacing $\ln 1/y$ by $-\ln y$. □

Base of a logarithm. Natural or hyperbolic logarithms are not the only logarithmic functions. Other popular logarithms are decimal ones. In computer science one prefers binary logarithms. Different logarithmic functions are distinguished by their *bases*. The base of a logarithmic function $l(x)$ is defined as the number b for which $l(b) = 1$. Logarithms with the base b are denoted by $\log_b x$. What is the base of the natural logarithm? This is the second most important constant in mathematics (after π). It is an irrational number denoted by e which is equal to $2.71828182845905\ldots$. It was Euler who introduced this number and this notation.

Well, e is the number such that the area of hyperbolic trapezium over $[1, e]$ is 1. Consider the geometric progression q^n for $q = 1 + \frac{1}{n}$. All summands in the corresponding hyperbolic receding sum for this progression are equal to $\frac{q^{k+1}-q^k}{q^k} = q - 1 = \frac{1}{n}$. Hence the receding sum for the interval $[1, q^n]$ is equal to 1 and it is greater than $\ln q^n$. Consequently $e > q^n$. The summands of the advancing sum in this case are equal to $\frac{q^{k+1}-q^k}{q^{k+1}} = 1 - \frac{1}{q} = \frac{1}{n+1}$. Hence the advancing sum for the interval $[1, q^{n+1}]$ is equal to 1. It is less than the corresponding logarithm. Consequently, $e < q^{n+1}$. Thus we have proved the following estimates for e:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

We see that $\left(1 + \frac{1}{n}\right)^n$ rapidly tends to e as n tends to infinity.

Problems.

1. Prove that $\ln x/y = \ln x - \ln y$.
2. Prove that $\ln 2 < 1$.
3. Prove that $\ln 3 > 1$.
4. Prove that $x > y$ implies $\ln x > \ln y$.
5. Is $\ln x$ bounded?
6. Prove that $\frac{1}{n+1} < \ln(1 + 1/n) < \frac{1}{n}$.
7. Prove that $\frac{x}{1+x} < \ln(1 + x) < x$.
8. Prove the Theorem 2.1.2 (Fermat) for $a = 1/2, 1/3, 2/3$.
9. Prove the unboundedness of $\frac{n}{\ln n}$.
10. Compare $\left(1 + \frac{1}{n}\right)^n$ and $(1 + \frac{1}{n+1})^{n+1}$.
11. Prove the monotonicity of $\frac{n}{\ln n}$.
12. Prove that $\sum_{k=2}^{n-1} \frac{1}{k} < \ln n < \sum_{k=1}^{n-1} \frac{1}{k}$.
13. Prove that $\ln(1 + x) > x - \frac{x^2}{2}$.
14. Estimate integral part of $\ln 1000000$.
15. Prove that $\ln \frac{x+y}{2} \geq \frac{\ln x + \ln y}{2}$.
16. Prove the convergence of $\sum_{k=1}^{\infty} (\frac{1}{k} - \ln(1 + \frac{1}{k}))$.
17. Prove that $(n + \frac{1}{2})^{-1} \leq \ln(1 + \frac{1}{n}) < \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})$.
***18.** Prove that $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \ln 2$.

2.2. Definite Integral

On the contents of the lecture. Areas of curvilinear trapezia play an extraordinary important role in mathematics. They generate a key concept of Calculus — the concept of the *integral*.

Three basic rules. For a nonnegative function f its integral $\int_a^b f(x)\,dx$ along the interval $[a,b]$ is defined just as the area of the curvilinear trapezium below the graph of f over $[a,b]$. We allow a function to take infinite values. Let us remark that changing of the value of function in one point does not affect the integral, because the area of the line is zero. That is why we allow the functions under consideration to be undefined in a finite number of points of the interval.

Immediately from the definition one gets the following three *basic rules of integration*:

Rule of constant	$\int_a^b f(x)\,dx = c(b-a)$, if $f(x) = c$	for $x \in (a,b)$,
Rule of inequality	$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$, if $f(x) \le g(x)$	for $x \in (a,b)$,
Rule of partition	$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	for $b \in (a,c)$.

Partition. Let $|J|$ denote the length of an interval J. Let us say that a sequence $\{J_k\}_{k=1}^n$ of disjoint open subintervals of an interval I is a *partition* of I, if $\sum_{k=1}^n |I_k| = |I|$. The boundary of a partition $P = \{J_k\}_{k=1}^n$ is defined as the difference $I \setminus \bigcup_{k=1}^n J_k$ and is denoted ∂P.

For any finite subset S of an interval I, which contains the ends of I, there is a unique partition of I which has this set as the boundary. Such a partition is called *generated* by S. For a monotone sequence $\{x_k\}_{k=0}^n$ the generated partition is $\{(x_{k-1}, x_k)\}_{k=1}^n$.

Piecewise constant functions. A function $f(x)$ is called *partially constant* on a partition $\{J_k\}_{k=1}^n$ of $[a,b]$ if it is constant on each J_k. The Rules of Constant and Partition immediately imply:

$$\int_a^b f(x)\,dx = \sum_{k=1}^n f(J_k)|J_k|. \tag{2.2.1}$$

Proof. Indeed, the integral splits into a sum of integrals over $J_k = [x_{k-1}, x_k]$, and the function takes the value $f(J_k)$ in (x_{k-1}, x_k). □

A function is called *piecewise constant* over an interval if it is partially constant with respect to some finite partition of the interval.

Lemma 2.2.1. *Let f and g be piecewise constant functions over $[a,b]$. Then $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.*

Proof. First, suppose $f(x) = c$ is constant on the interval (a,b). Let g take the value g_k over the interval (x_k, x_{k+1}) for an exhausting $\{x_k\}_{k=0}^n$. Then $f(x)+g(x)$ takes values $(c+g_k)$ over (x_k, x_{k+1}). Hence $\int_a^b (f(x)+g(x))\,dx = \sum_{k=0}^{n-1}(c+g_k)|\delta x_k|$ due to (2.2.1). Splitting this sum and applying (2.2.1) to both summands, one gets $\sum_{k=0}^{n-1} c|\delta x_k| + \sum_{k=0}^{n-1} g_k|\delta x_k| = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. This proves the case of a constant f.

Now let f be partially constant on the partition generated by $\{x_k\}_{k=0}^n$. Then, by the partition rule, $\int_a^b (f(x)+g(x))\,dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx$. As f is constant on any (x_{k-1}, x_k), for any k one gets $\int_{x_{k-1}}^{x_k} (f(x)+g(x))\,dx = \int_{x_{k-1}}^{x_k} f(x)\,dx + \int_{x_{k-1}}^{x_k} g(x)\,dx$. Summing up these equalities one completes the proof of Lemma 2.2.1 for the sum.

The statement about differences follows from the addition formula applied to $g(x)$ and $f(x) - g(x)$. □

LEMMA 2.2.2. *For any monotone nonnegative function f on the interval $[a,b]$ and for any $\varepsilon > 0$ there is such piecewise constant function f_ε such that $f_\varepsilon \leq f(x) \leq f_\varepsilon(x) + \varepsilon$.*

PROOF. $f_\varepsilon(x) = \sum_{k=0}^\infty k\varepsilon[k\varepsilon \leq f(x) < (k+1)\varepsilon]$. □

THEOREM 2.2.3 (Addition Theorem). *Let f and g be nonnegative monotone functions defined on $[a,b]$. Then*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

PROOF. Let f_ε and g_ε be ε-approximations of f and g respectively provided by Lemma 2.2.2. Set $f^\varepsilon(x) = f_\varepsilon(x) + \varepsilon$ and $g^\varepsilon(x) = g_\varepsilon(x) + \varepsilon$. Then $f_\varepsilon(x) \leq f(x) \leq f^\varepsilon(x)$ and $g_\varepsilon(x) \leq g(x) \leq g^\varepsilon(x)$ for $x \in (a,b)$. Summing and integrating these inequalities in different order gives

$$\int_a^b (f_\varepsilon(x)+g_\varepsilon(x))\,dx \leq \int_a^b (f(x)+g(x))\,dx \leq \int_a^b (f^\varepsilon(x)+g^\varepsilon(x))\,dx$$

$$\int_a^b f_\varepsilon(x)\,dx + \int_a^b g_\varepsilon(x)\,dx \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \leq \int_a^b f^\varepsilon(x)\,dx + \int_a^b g^\varepsilon(x)\,dx.$$

Due to Lemma 2.2.1, the left-hand sides of these inequalities coincide, as well as the right-hand sides. Hence the difference between the central parts does not exceed

$$\int_a^b (f^\varepsilon(x) - f_\varepsilon(x))\,dx + \int_a^b (g^\varepsilon(x) - g_\varepsilon(x))\,dx \leq 2\varepsilon(b-a).$$

Hence, for any positive ε

$$\left|\int_a^b (f(x)+g(x))\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| < 2\varepsilon(b-a).$$

This implies that the left-hand side vanishes. □

Term by term integration of a functional series.

LEMMA 2.2.4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of nonnegative nondecreasing functions and let p be a piecewise constant function. If $\sum_{k=1}^\infty f_k(x) \geq p(x)$ for all $x \in [a,b]$ then $\sum_{k=1}^\infty \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx$.*

PROOF. Let p be a piecewise constant function with respect to $\{x_i\}_{i=0}^n$. Choose any positive ε. Since $\sum_{k=1}^\infty f_k(x_i) \geq p(c)$, eventually one has $\sum_{k=1}^m f_k(x_i) > p(x_i) - \varepsilon$. Fix m such that this inequality holds simultaneously for all $\{x_i\}_{i=0}^n$. Let $[x_i, x_{i+1}]$ be an interval where $p(x)$ is constant. Then for any $x \in [x_i, x_{i+1}]$ one has these inequalities: $\sum_{k=1}^m f_k(x) \geq \sum_{k=1}^m f_k(x_k) > p(x_k) - \varepsilon = p(x) - \varepsilon$. Consequently for all $x \in [a,b]$ one has the inequality $\sum_{k=1}^{m} f_k(x) > p(x) - \varepsilon$. Taking integrals gives $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx \geq \int_a^b (p(x)-\varepsilon)\,dx = \int_a^b p(x)\,dx - \varepsilon(b-a)$. By the Addition Theorem $\int_a^b \sum_{k=1}^{m} f_k(x)\,dx = \sum_{k=1}^{m} \int_a^b f_k(x)\,dx \leq \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Therefore $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx - \varepsilon(b-a)$ for any positive ε. This implies the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b p(x)\,dx$. □

THEOREM 2.2.5. *For any sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative nondecreasing functions on an interval $[a,b]$*

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

PROOF. Since $\sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{\infty} f_k(x)$ for all x, by integrating one gets

$$\int_a^b \sum_{k=1}^{n} f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx.$$

By the the Addition Theorem the left-hand side is equal to $\sum_{k=1}^{n} \int_a^b f_k(x)\,dx$, which is a partial sum of $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$. Then by All-for-One one gets the inequality $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \leq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$.

To prove the opposite inequality for any positive ε, we apply Lemma 2.2.2 to find a piecewise constant function F_ε, such that $F_\varepsilon(x) \leq \sum_{k=1}^{\infty} f_k(x)dx$ and $\int_a^b \sum_{k=1}^{\infty} (f_k(x) - F_\varepsilon(x))\,dx < \varepsilon$. On the other hand, by Lemma 2.2.4 one gets

$$\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \geq \int_a^b F_\varepsilon(x)\,dx.$$

Together these inequalities imply $\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx + \varepsilon \geq \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx$. As the last inequality holds for all $\varepsilon > 0$, it holds also for $\varepsilon = 0$ □

THEOREM 2.2.6 (Mercator,1668). *For any $x \in (-1,1]$ one has*

$$\boxed{\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}} \tag{2.2.2}$$

PROOF. Consider $x \in [0,1)$. Since $\int_0^x t^k\,dt = \frac{t^{k+1}}{k+1}$ due to the Fermat Theorem 2.1.2, termwise integration of the geometric series $\sum_{k=0}^{\infty} t^k$ over the interval $[0,x]$ for $x < 1$ gives $\int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$.

LEMMA 2.2.7. $\int_0^x \frac{1}{1-t}\,dt = \ln(1-x)$.

PROOF OF LEMMA. Construct a translation of the plane which transforms the curvilinear trapezium below $\frac{1}{1-t}$ over $[0,x]$ into the trapezium for $\ln(1-x)$. Indeed, the reflection of the plane $((x,y) \to (2-x,y))$ along the line $x=1$ transforms this trapezium to the curvilinear trapezium under $\frac{1}{x-1}$ over $[2-x,2]$. The parallel translation by 1 to the left of the latter trapezium $(x,y) \to (x-1,y)$ transforms it just in to the ogarithmic trapezium for $\ln(1-x)$. □

The Lemma proves the Mercator Theorem for negative x. To prove it for positive x, set $f_k(x) = x^{2k-1} - x^{2k}$. All functions f_k are nonnegative on $[0,1]$ and

$\sum_{k=1}^{\infty} f_k(x) = \frac{1}{1+x}$. Termwise integration of this equality over $[0,x]$ gives (2.2.2), modulo the equality $\int_0^x \frac{1}{1+t}\,dt = \int_1^x \frac{1}{t}\,dt$. The latter is proved by parallel translation of the plane. Let us remark, that in the case $x=1$ the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$ is not absolutely convergent, and under its sum we mean $\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = \sum_{k=1}^{\infty} (\frac{1}{2k-1} - \frac{1}{2k})$. And the above proof proves just this fact. □

The arithmetic mean of Mercator's series evaluated at x and $-x$ gives *Gregory's Series*

$$\boxed{\frac{1}{2}\ln\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots.} \tag{2.2.3}$$

Gregory's series converges much faster than Mercator's one. For example, putting $x = \frac{1}{3}$ in (2.2.3) one gets

$$\ln 2 = \frac{2}{3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \frac{2}{7\cdot 3^7} + \dots.$$

Problems.

1. Prove that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

2. Prove the following formulas via piecewise constant approximations:

(multiplication formula) $$\int_a^b \lambda f(x)\,dx = \lambda\int_a^b f(x)\,dx$$

(shift formula) $$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

(reflection formula) $$\int_0^a f(x)\,dx = \int_{-a}^0 f(-x)\,dx$$

(compression formula) $$\int_0^a f(x)\,dx = \frac{1}{k}\int_0^{ka} f\left(\frac{x}{k}\right)\,dx$$

3. Evaluate $\int_0^{2\pi} (\sin x + 1)\,dx$.

4. Prove the inequality $\int_{-2}^{2} (2 + x^3 2^x)\,dx > 8$.

5. Prove $\int_0^{2\pi} x(\sin x + 1)\,dx < 2\pi$.

6. Prove $\int_{100\pi}^{200\pi} \frac{x+\sin(x)}{x}\,dx \le 100\pi + \frac{1}{50}\pi$.

7. Denote by s_n the area of $\{(x,y) \mid 0 \le x \le 1, (1-x)\ln n + x\ln(n+1) \le y \le \ln(1+x)\}$. Prove that $\sum_{k=1}^{\infty} s_k < \infty$.

8. Prove that $\sum_{k=1}^{2n} (-1)^{k+1}\frac{x^k}{k} < \ln(1+x) < \sum_{k=1}^{2n+1} (-1)^{k+1}\frac{x^k}{k}$ for $x > 0$.

9. Compute the logarithms of the primes $2, 3, 5, 7$ with accuracy 0.01.

10. Evaluate $\int_0^1 \sqrt{x}\,dx$.

***11.** Evaluate $\int_0^{\pi} \sin x\,dx$.

2.3. Stieltjes Integral

On the contents of the lecture. The Stieltjes relativization of the integral makes the integral flexible. We learn the main transformations of integrals. They allow us to evaluate a lot of integrals.

Basic rules. A *parametric curve* is a mapping of an interval into the plane. In cartesian coordinates a parametric curve can be presented as a pair of functions $x(t), y(t)$. The first function $x(t)$ represents the value of abscises at the moment t, and the second $y(t)$ is the ordinate at the same moment. We define the integral $\int_a^b f(t)\,dg(t)$ for a nonnegative function f, called the *integrand*, and with respect to a nondecreasing *continuous* function g, called the *differand*, as the area below the curve $f(t), g(t) \mid t \in [a,b]$.

A monotone function f is called *continuous* over the interval $[a,b]$ if it takes all intermediate values, that is, the image $f[a,b]$ of $[a,b]$ coincides with $[f(a), f(b)]$. If it is not continuous for some $y \in [f(a), f(b)] \setminus f[a,b]$, there is a point $x(y) \in [a,b]$ with the following property: $f(x) < y$ if $x < x(y)$ and $f(x) > y$ if $x > x(y)$. Let us define a *generalized preimage* $f^{[-1]}(y)$ of a point $y \in [f(a), f(b)]$ either as its usual preimage $f^{-1}(y)$ if it is not empty, or as $x(y)$ in the opposite case.

Now the curvilinear trapezium below the curve $f(t), g(t)$ over $[a,b]$ is defined as $\{(x,y) \mid 0 \le y \le g(f^{[-1]}(x))\}$.

The basic rules for relative integrals transform into:

Rule of constant	$\int_a^b f(t)\,dg(t) = c(g(b) - g(a))$, if $f(t) = c$	for $t \in (a,b)$,
Rule of inequality	$\int_a^b f_1(t)\,dg(t) \le \int_a^b f_2(t)\,dg(t)$, if $f_1(t) \le f_2(t)$	for $t \in (a,b)$,
Rule of partition	$\int_a^c f(t)\,dg(t) = \int_a^b f(t)\,dg(t) + \int_b^c f(t)\,dg(t)$	for $b \in (a,c)$.

Addition theorem. The proofs of other properties of the integral are based on piecewise constant functions. For any number x, let us define its ε-integral part as $\varepsilon[x/\varepsilon]$. Immediately from the definition one gets:

LEMMA 2.3.1. *For any monotone nonnegative function f on the interval $[a,b]$ and for any $\varepsilon > 0$, the function $[f]_\varepsilon$ is piecewise constant such that $[f(x)]_\varepsilon \le f(x) \le [f(x)]_\varepsilon + \varepsilon$ for all x.*

THEOREM 2.3.2 (on multiplication). *For any nonnegative monotone f, and any continuous nondecreasing g and any positive constant c one has*

$$\int_a^b cf(x)\,dg(x) = c\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dcg(x). \tag{2.3.1}$$

PROOF. For the piecewise constant $f_\varepsilon = [f]_\varepsilon$, the proof is by a direct calculation. Hence

$$\int_a^b cf_\varepsilon(x)\,dg(x) = c\int_a^b f_\varepsilon(x)\,dg(x) = \int_a^b f_\varepsilon(x)\,dcg(x) = I_\varepsilon. \tag{2.3.2}$$

Now let us estimate the differences between integrals from (2.3.1) and their approximations from (2.3.2). For example, for the right-hand side integrals one has:

$$\int_a^b f\,dcg - \int_a^b f_\varepsilon\,dcg = \int_a^b (f - f_\varepsilon)\,dcg \le \int_a^b \varepsilon\,dcg = \varepsilon(cg(b) - cg(a)). \tag{2.3.3}$$

Hence $\int_a^b f\,dcg = I_\varepsilon + \varepsilon_1$, where $\varepsilon_1 \le c\varepsilon(g(b) - g(a))$. The same argument proves $c\int_a^b f\,dg = I_\varepsilon + \varepsilon_2$ and $\int_a^b cf\,dg = I_\varepsilon + \varepsilon_3$, where $\varepsilon_2, \varepsilon_3 \le c\varepsilon(g(b) - g(a))$. Then the pairwise differences between the integrals of (2.3.1) do not exceed $2c\varepsilon(g(b) - g(a))$. Consequently they are less than any positive number, that is, they are zero. □

THEOREM 2.3.3 (Addition Theorem). *Let f_1, f_2 be nonnegative monotone functions and g_1, g_2 be nondecreasing continuous functions over $[a, b]$, then*

$$\int_a^b (f_1(t) + f_2(t))\,dg_1(t) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_2(t)\,dg_1(t), \tag{2.3.4}$$

$$\int_a^b f_1(t)\,d(g_1(t) + g_2(t)) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_1(t)\,dg_2(t). \tag{2.3.5}$$

PROOF. For piecewise constant integrands both the equalities follow from the Rule of Constant and the Rule of Partition. To prove (2.3.4) replace f_1 and f_2 in both parts by $[f_1]_\varepsilon$ and $[f_2]_\varepsilon$. We get equality and denote by I_ε the common value of both sides of this equality. Then by (2.3.3) both integrals on the right-hand side differ from they approximation at most by $\varepsilon(g_1(b) - g_1(a))$, therefore the right-hand side of (2.3.4) differs from I_ε at most by $2\varepsilon(g_1(b) - g_1(a))$. The same is true for the left-hand side of (2.3.4). This follows immediately from (2.3.3) in case $f = f_1 + f_2$, $f_\varepsilon = [f_1]_\varepsilon + [f_2]_\varepsilon$ and $g = g_1$. Consequently, the difference between left-hand and right-hand sides of (2.3.4) does not exceed $4\varepsilon(g_1(b) - g_1(a))$. As ε can be chosen arbitrarily small this difference has to be zero.

The proof of (2.3.5) is even simpler. Denote by I_ε the common value of both parts of (2.3.5) where f_1 is changed by $[f_1]_\varepsilon$. By (2.3.3) one can estimate the differences between the integrals of (2.3.5) and their approximations as being $\le \varepsilon(g_1(b) + g_2(b) - g_1(a) - g_2(a))$ for the left-hand side, and as $\le \varepsilon(g_1(b) - g_1(a))$ and $\le \varepsilon(g_2(b) - g_2(a))$ for the corresponding integrals of the right-hand side of (2.3.5). So both sides differ from I_ε by at most $\le \varepsilon(g_1(b) - g_1(a) + g_2(b) - g_2(a))$. Hence the difference vanishes. □

Differential forms. An expression of the type $f_1 dg_1 + f_2 dg_2 + \cdots + f_n dg_n$ is called a *differential form*. One can add differential forms and multiply them by functions. The integral of a differential form $\int_a^b (f_1\,dg_1 + f_2 dg_2 + \cdots + f_n dg_n)$ is defined as the sum of the integrals $\sum_{k=1}^n \int_a^b f_k\,dg_k$. Two differential forms are called equivalent on the interval $[a, b]$ if their integrals are equal for all subintervals of $[a, b]$. For the sake of brevity we denote the differential form $f_1 dg_1 + f_2 dg_2 + \cdots + f_n dg_n$ by FdG, where $F = \{f_1, \ldots, f_n\}$ is a collection of integrands and $G = \{g_1, \ldots, g_n\}$ is a collection of differands.

THEOREM 2.3.4 (on multiplication). *Let FdG and $F'dG'$ be two differential forms, with positive increasing integrands and continuous increasing differands, which are equivalent on $[a, b]$. Then their products by any increasing function f on $[a, b]$ are equivalent on $[a, b]$ too.*

PROOF. If f is constant then the statement follows from the multiplication formula. If f is piecewise constant, then divide $[a, b]$ into intervals where it is constant and prove the equality for parts and after collect the results by the Partition Rule. In the general case, $0 \le \int_a^b fF\,dG - \int_a^b [f]_\varepsilon F\,dG \le \int_a^b \varepsilon F\,dG = \varepsilon \int_a^b F\,dG$. Since $\int_a^b [f]_\varepsilon F'\,dG' = \int_a^b [f]_\varepsilon F\,dG$, one concludes that $\left|\int_a^b fF'\,dG' - \int_a^b fF\,dG\right| \le$

$\varepsilon \int_a^b F\,dG + \varepsilon \int_a^b F'\,dG'$. The right-hand side of this inequality can be made arbitrarily small. Hence the left-hand side is 0. □

Integration by parts.

THEOREM 2.3.5. *If f and g are continuous nondecreasing nonnegative functions on $[a,b]$ then $d(fg)$ is equivalent to $fdg + gdf$.*

PROOF. Consider $[c,d] \subset [a,b]$. The integral $\int_c^d f\,dg$ represents the area below the curve $(f(t), g(t))_{t\in[c,d]}$. And the integral $\int_c^d g\,df$ represents the area on the left of the same curve. Its union is equal to $[0, f(d)] \times [0, g(d)] \setminus [0, f(c)] \times [0, g(c)]$. The area of this union is equal to $(f(d)g(d) - f(c)g(c) = \int_c^d dfg$. On the other hand the area of this union is the sum of the areas of curvilinear trapezia representing the integrals $\int_c^d f\,dg$ and $\int_c^d g\,df$. □

Change of variable. Consider a Stieltjes integral $\int_a^b f(\tau)\,dg(\tau)$ and suppose there is a continuous nondecreasing mapping $\tau\colon [t_0, t_1] \to [a,b]$, such that $\tau(t_0) = a$ and $\tau(t_1) = b$. The composition $g(\tau(t))$ is a continuous nondecreasing function and the curve $\{(f(\tau(t), g(\tau(t))) \mid t \in [t_0, t_1]\}$ just coincides with the curve $\{(f(\tau), g(\tau)) \mid \tau \in [a,b]$. Hence, the following equality holds; it is known as the *Change of Variable* formula:

$$\int_{t_0}^{t_1} f(\tau(t))\,dg(\tau(t)) = \int_{\tau(t_0)}^{\tau(t_1)} f(\tau)\,dg(\tau).$$

For differentials this means that the equality $F(x)dG(x) = F'(x)dG'(x)$ conserves if one substitutes instead of an independent variable x a function.

Differential Transformations.

Case dx^n. Integration by parts for $f(t) = g(t) = t$ gives $dt^2 = tdt + tdt$. Hence $tdt = d\frac{t^2}{2}$. If we already know that $dx^n = ndx^{n-1}$, then $dx^{n+1} = d(xx^n) = xdx^n + x^n dx = nxx^{n-1}dx + x^n dx = (n+1)x^n dx$. This proves the Fermat Theorem for natural n.

Case $d\sqrt[n]{x}$. To evaluate $d\sqrt[n]{x}$ substitute $x = y^n$ into the equality $dy^n = ny^{n-1}dy$. One gets $dx = \frac{nx}{\sqrt[n]{x}}d\sqrt[n]{x}$, hence $d\sqrt[n]{x} = \frac{\sqrt[n]{x}}{nx}dx$.

Case $\ln x dx$. We know $d\ln x = \frac{1}{x}dx$. Integration by parts gives $\ln x dx = d(x\ln x) - xd\ln x = d(x\ln x) - dx = d(x\ln x - x)$.

Problems.

1. Evaluate $dx^{2/3}$.
2. Evaluate dx^{-1}.
3. Evaluate $x\ln x\,dx$.
4. Evaluate $d\ln^2 x$.
5. Evaluate $\ln^2 x\,dx$.
6. Evaluate de^x.
7. Investigate the convergence of $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$.

2.4. Asymptotics of Sums

On the contents of the lecture. We become at last acquainted with the fundamental concept of a *limit*. We extend the notion of the sum of a series and discover that a change of order of summands can affect the ultimate sum. Finally we derive the famous Stirling formula for $n!$.

Asymptotic formulas. The Mercator series shows how useful series can be for evaluating integrals. In this lecture we will use integrals to evaluate both partial and ultimate sums of series. Rarely one has an explicit formula for partial sums of a series. There are lots of important cases where such a formula does not exist. For example, it is known that partial sums of the Euler series cannot be expressed as a finite combination of elementary functions. When an explicit formula is not available, one tries to find a so-called *asymptotic formula*. An asymptotic formula for a partial sum S_n of a series is a formula of the type $S_n = f(n) + R(n)$ where f is a known function called the *principal part* and $R(n)$ is a *remainder*, which is small, in some sense, with respect to the principal part. Today we will get an asymptotic formula for partial sums of the harmonic series.

Infinitesimally small sequences. The simplest asymptotic formula has a constant as its principal part and an infinitesimally small remainder. One says that a sequence $\{z_k\}$ is *infinitesimally small* and writes $\lim z_k = 0$, if z_k tends to 0 as n tends to infinity. That is for any positive ε eventually (i.e., beginning with some n) $|z_k| < \varepsilon$. With Iverson notation, this definition can be expressed in the following clear form:

$$[\{z_k\}_{k=1}^{\infty} \text{ is infinitesimally small}] = \prod_{m=1}^{\infty} 2\left|\sum_{n=1}^{\infty}(-1)^n \prod_{k=1}^{\infty}[m[k>n]|z_k|<1]\right|.$$

Three basic properties of infinitesimally small sequences immediately follow from the definition:

- if $\lim a_k = \lim b_k = 0$ then $\lim(a_k + b_k) = 0$;
- if $\lim a_k = 0$ then $\lim a_k b_k = 0$ for any bounded sequence $\{b_k\}$;
- if $a_k \leq b_k \leq c_k$ for all k and $\lim a_k = \lim c_k = 0$, then $\lim b_k = 0$.

The third property is called the *squeeze rule*.

Today we need just one property of infinitesimally small sequences:

THEOREM 2.4.1 (Addition theorem). *If the sequences $\{a_k\}$ and $\{b_k\}$ are infinitesimally small, than their sum and their difference are infinitesimally small too.*

PROOF. Let ε be a positive number. Then $\varepsilon/2$ also is positive number. And by definition of infinitesimally small, the inequalities $|a_k| < \varepsilon/2$ and $|b_k| < \varepsilon/2$ hold eventually beginning with some n. Then for $k > n$ one has $|a_k \pm b_k| \leq |a_k| + |b_k| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Limit of sequence.

DEFINITION. *A sequence $\{z_k\}$ of (complex) numbers* converges *to a number z if* $\lim z - z_k = 0$. *The number z is called the* limit *of the sequence $\{z_k\}$ and denoted by* $\lim z_k$.

An infinite sum represents a particular case of a limit as demonstrated by the following.

THEOREM 2.4.2. *The partial sums of an absolutely convergent series* $\sum_{k=1}^{\infty} z_k$ *converge to its sum.*

PROOF. $|\sum_{k=1}^{n-1} z_k - \sum_{k=1}^{\infty} z_k| = |\sum_{k=n}^{\infty} z_k| \leq \sum_{k=n}^{\infty} |z_k|$. Since $\sum_{k=1}^{\infty} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$, there is a partial sum such that $\sum_{k=1}^{n-1} |z_k| > \sum_{k=1}^{\infty} |z_k| - \varepsilon$. Then for all $m \geq n$ one has $\sum_{k=m}^{\infty} |z_k| \leq \sum_{k=n}^{\infty} |z_k| < \varepsilon$. □

Conditional convergence. The concept of the limit of sequence leads to a notion of convergence generalizing absolute convergence.

A series $\sum_{k=1}^{\infty} a_k$ is called (conditionally) *convergent* if $\lim \sum_{k=1}^{n} a_k = A + \alpha_n$, where $\lim \alpha_n = 0$. The number A is called its ultimate sum.

The following theorem gives a lot of examples of conditionally convergent series which are not absolutely convergent. By $[[n]]$ we denote the even part of the number n, i.e., $[[n]] = 2[n/2]$.

THEOREM 2.4.3 (Leibniz). *For any of positive decreasing infinitesimally small sequence* $\{a_n\}$, *the series* $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ *converges.*

PROOF. Denote the difference $a_k - a_{k+1}$ by δa_k. The series $\sum_{k=1}^{\infty} \delta a_{2k-1}$ and $\sum_{k=1}^{\infty} \delta a_{2k}$ are positive and convergent, because their termwise sum is $\sum_{k=1}^{\infty} \delta a_k = a_1$. Hence $S = \sum_{k=1}^{\infty} \delta a_{2k-1} \leq a_1$. Denote by S_n the partial sum $\sum_{k=1}^{n-1}(-1)^{k+1} a_k$. Then $S_{2n} = \sum_{k=1}^{n-1} \delta a_{2n-1} = S + \alpha_n$, where $\lim \alpha_n = 0$. Then $S_n = S_{[[n]]} + a_n[n \text{ is odd}] + \alpha_{[[n]]}$. As $a_n[n \text{ is odd}] + \alpha_{[[n]]}$ is infinitesimally small, this implies the theorem. □

LEMMA 2.4.4. *Let* f *be a non-increasing nonnegative function. Then the series* $\sum_{k=1}^{\infty} (f(k) - \int_k^{k+1} f(x)\,dx)$ *is positive and convergent and has sum* $c_f \leq f(1)$.

PROOF. Integration of the inequalities $f(k) \geq f(x) \geq f(k+1)$ over $[k, k+1]$ gives $f(k) \geq \int_k^{k+1} f(x)\,dx \geq f(n+1)$. This proves the positivity of the series and allows us to majorize it by the telescopic series $\sum_{k=1}^{\infty}(f(k) - f(k+1)) = f(1)$. □

THEOREM 2.4.5 (Integral Test on Convergence). *If a nonnegative function* $f(x)$ *decreases monotonically on* $[1, +\infty)$, *then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if* $\int_1^{\infty} f(x)\,dx < \infty$.

PROOF. Since $\int_1^{\infty} f(x)\,dx = \sum_{k=1}^{\infty} \int_k^{k+1} f(x)\,dx$, one has $\sum_{k=1}^{\infty} f(k) = c_f + \int_1^{\infty} f(x)\,dx$. □

Euler constant. The sum $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right)$, which is c_f for $f(x) = \frac{1}{x}$, is called *Euler's constant* and denoted by γ. Its first ten digits are $0.5772156649\ldots$.

Harmonic numbers. The sum $\sum_{k=1}^{n} \frac{1}{k}$ is denoted H_n and is called the n-th *harmonic number.*

THEOREM 2.4.6. $H_n = \ln n + \gamma + o_n$ *where* $\lim o_n = 0$.

PROOF. Since $\ln n = \sum_{k=1}^{n-1}(\ln(k+1) - \ln k) = \sum_{k=1}^{n-1} \ln(1 + \frac{1}{k})$, one has $\ln n + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = H_{n-1}$. But $\sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1 + \frac{1}{k})\right) = \gamma + \alpha_n$, where $\lim \alpha_n = 0$. Therefore $H_n = \ln n + \gamma + (\frac{1}{n} + \alpha_n)$. □

Alternating harmonic series. The *alternating harmonic series* $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is a conditionally convergent series due to the Leibniz Theorem 2.4.3, and it is not absolutely convergent. To find its sum we apply our Theorem 2.4.6 on asymptotics of harmonic numbers.

Denote by $S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$ the partial sum. Then $S_n = H'_n - H''_n$, where $H'_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is odd}]$ and $H''_n = \sum_{k=1}^{n}\frac{1}{k}[k \text{ is even}]$. Since $H''_{2n} = \frac{1}{2}H_n$ and $H'_{2n} = H_{2n} - H''_{2n} = H_{2n} - \frac{1}{2}H_n$ one gets

$$\begin{aligned} S_{2n} &= H_{2n} - \tfrac{1}{2}H_n - \tfrac{1}{2}H_n \\ &= H_{2n} - H_n \\ &= \ln 2n + \gamma + o_{2n} - \ln n - \gamma - o_n \\ &= \ln 2 + (o_{2n} - o_n). \end{aligned}$$

Consequently $S_n = \ln 2 + (o_{[[n]]} - o_{[n/2]} + \frac{(-1)^{n+1}}{n}[n \text{ is odd}])$. As the sum in brackets is infinitesimally small, one gets

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \ln 2.$$

The same arguments for a permutated alternating harmonic series give

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots = \tfrac{3}{2}\ln 2. \tag{2.4.1}$$

Indeed, in this case its $3n$-th partial sum is

$$\begin{aligned} S_{3n} &= H'_{4n} - H''_{2n} \\ &= H_{4n} - \tfrac{1}{2}H_{2n} - \tfrac{1}{2}H_n \\ &= \ln 4n + \gamma + o_{4n} - \tfrac{1}{2}(\ln 2n + \gamma + o_{2n} + \ln n + \gamma + o_n) \\ &= \ln 4 - \tfrac{1}{2}\ln 2 + o'_n \\ &= \tfrac{3}{2}\ln 2 + o'_n, \end{aligned}$$

where $\lim o'_n = 0$. Since the difference between S_n and S_{3m} where $m = [n/3]$ is infinitesimally small, this proves (2.4.1).

Stirling's Formula. We will try to estimate $\ln n!$. Integration of the inequalities $\ln[x] \le \ln x \le \ln[x+1]$ over $[1,n]$ gives $\ln(n-1)! \le \int_1^n \ln x\,dx \le \ln n!$. Let us estimate the difference D between $\int_1^n \ln x\,dx$ and $\frac{1}{2}(\ln n! + \ln(n-1)!)$.

$$\begin{aligned} D &= \int_1^n (\ln x - \tfrac{1}{2}(\ln[x] + \ln[x+1]))\,dx \\ &= \sum_{k=1}^{n-1}\int_0^1 \left(\ln(k+x) - \ln\sqrt{k(k+1)}\right)\,dx. \end{aligned} \tag{2.4.2}$$

To prove that all summands on the left-hand side are nonnegative, we apply the following general lemma.

Lemma 2.4.7. *$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ for any function.*

Proof. The reflection of the plane across the line $y = \frac{1}{2}$ transforms the curvilinear trapezium of $f(x)$ over $[0,1]$ into curvilinear trapezium of $f(1-x)$ over $[0,1]$. □

LEMMA 2.4.8. $\int_0^1 \ln(k+x)\,dx \geq \ln\sqrt{k(k+1)}$.

PROOF. Due to Lemma 2.4.7 one has

$$\begin{aligned}\int_0^1 \ln(k+x)\,dx &= \int_0^1 \ln(k+1-x)\,dx \\ &= \int_0^1 \tfrac{1}{2}(\ln(k+x)+\ln(k+1-x))\,dx \\ &= \int_0^1 \ln\sqrt{(k+x)(k+1-x)}\,dx \\ &= \int_0^1 \ln\sqrt{k(k+1)+x-x^2}\,dx \\ &\geq \int_0^1 \ln\sqrt{k(k+1)}\,dx \\ &= \ln\sqrt{k(k+1)}.\end{aligned}$$

□

Integration of the inequality $\ln(1+x/k) \leq x/k$ over $[0,1]$ gives

$$\int_0^1 \ln(1+x/k)\,dx \leq \int_0^1 \frac{x}{k}\,dx = \frac{1}{2k}.$$

This estimate together with the inequality $\ln(1+1/k) \geq 1/(k+1)$ allows us to estimate the summands from the right-hand side of (2.4.2) in the following way:

$$\begin{aligned}\int_0^1 \ln(k+x) - \ln\sqrt{k(k+1)}\,dx &= \int_0^1 \ln(k+x) - \ln k - \tfrac{1}{2}(\ln(k+1)-\ln k)\,dx \\ &= \int_0^1 \ln\left(1+\tfrac{x}{k}\right) - \tfrac{1}{2}\ln\left(1+\tfrac{1}{k}\right)\,dx \\ &\leq \tfrac{1}{2k} - \tfrac{1}{2(k+1)}.\end{aligned}$$

We see that $D_n \leq \sum_{k=1}^{\infty} \frac{1}{2k} - \frac{1}{2(k+1)} = \frac{1}{2}$ for all n. Denote by D_∞ the sum (2.4.2) for infinite n. Then $R_n = D_\infty - D_n = \frac{\theta}{2n}$ for some nonnegative $\theta < 1$, and we get

$$\begin{aligned}D_\infty - \tfrac{\theta}{2n} &= \int_1^n \ln x\,dx - \tfrac{1}{2}\left(\ln n! + \ln(n-1)!\right) \\ &= \int_1^n \ln x\,dx - \ln n! + \tfrac{1}{2}\ln n.\end{aligned} \tag{2.4.3}$$

Substituting in (2.4.3) the value of the integral $\int_1^n \ln x\,dx = \int_1^n d(x\ln x - x) = (n\ln n - n) - (1\ln 1 - 1) = n\ln n - n + 1$, one gets

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + (1 - D_\infty) + \tfrac{\theta}{2n}.$$

Now we know that $1 \geq (1 - D_\infty) \geq \frac{1}{2}$, but it is possible to evaluate the value of D_∞ with more accuracy. Later we will prove that $1 - D_\infty = \sqrt{2\pi}$.

Problems.

1. Does $\sum_{k=1}^{\infty} \sin k$ converge?

2. Does $\sum_{k=1}^{\infty} \sin k^2$ converge?

3. Evaluate $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots - \frac{2}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} - \ldots$.

4. Prove: If $\lim \frac{a_{n+1}}{a_n} < 1$, then $\sum_{k=1}^{\infty} a_k$ converge.

5. Prove: If $\sum_{k=1}^{\infty} |a_k - a_{k-1}| < \infty$, then $\{a_k\}$ converges.

6. Prove the convergence of $\sum_{k=1}^{\infty} \dfrac{(-1)^{[\sqrt{k}]}}{k}$.

7. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{\ln^3 k}$.

8. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k \sqrt{\ln \ln k}}$.

9. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k (\ln \ln k)^2}$.

10. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ and find its asymptotic formula.

11. Prove the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$.

12. Which partial sum of the above series is 0.01 close to its ultimate sum?

13. Evaluate $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$ with precision 0.01.

14. Evaluate $\int_1^3 \ln x \, d[x]$.

15. Express the Stirling constant via the Wallis product $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1}$.

2.5. Quadrature of Circle

On the contents of the lecture. We extend the concept of the integral to complex functions. We evaluate a very important integral $\oint \frac{1}{z}dz$ by applying Archimedes' theorem on the area of circular sector. As a consequence, we evaluate the Wallis product and the Stirling constant.

Definition of a complex integral. To specify an integral of a complex function one has to indicate not only its limits, but also the *path of integration*. A path of integration is a mapping $p\colon [a,b] \to \mathbb{C}$, of an interval $[a,b]$ of the real line into complex plane. The integral of a complex differential form fdg (here f and g are complex functions of complex variable) along the path p is defined via separate integration of different combinations of real and imaginary parts in the following way:

$$\int_a^b \operatorname{Re} f(p(t))\, d\operatorname{Re} g(p(t)) - \int_a^b \operatorname{Im} f(p(t))\, d\operatorname{Im} g(p(t)) \\ + i\int_a^b \operatorname{Re} f(p(t))\, d\operatorname{Im} g(p(t)) + i\int_a^b \operatorname{Im} f(p(t))\, d\operatorname{Re} g(p(t))$$

Two complex differential forms are called equal if their integrals coincide for all paths. So, the definition above can be written shortly as $fdg = \operatorname{Re} f d\operatorname{Re} g - \operatorname{Im} f d\operatorname{Im} g + i\operatorname{Re} f d\operatorname{Im} g + i\operatorname{Im} f d\operatorname{Re} g$.

The integral $\int \frac{1}{z}dz$. The Integral is the principal concept of Calculus and $\int \frac{1}{z}dz$ is the principal integral. Let us evaluate it along the path $p(t) = \cos t + i\sin t$, $t \in [0,\phi]$, which goes along the arc of the circle of the length $\phi \leq \pi/2$. Since $\frac{1}{\cos t + i\sin t} = \cos t - i\sin t$, one has

$$\int_p \frac{1}{z}\,dz = \int_0^\phi \cos t\, d\cos t + \int_0^\phi \sin t\, d\sin t \\ - i\int_0^\phi \sin t\, d\cos t + i\int_0^\phi \cos t\, d\sin t. \tag{2.5.1}$$

Its real part transforms into $\int_0^\phi \frac{1}{2}\, d\cos^2 t + \int_0^\phi \frac{1}{2}\, d\sin^2 t = \int_0^\phi \frac{1}{2}\, d(\cos^2 t + \sin^2 t) = \int_0^\phi \frac{1}{2}\, d1 = 0$. An attentive reader has to object: integrals were defined only for differential forms with non-decreasing differands, while $\cos t$ decreases.

Sign rule. Let us define the integral for any differential form fdg with any continuous monotone differand g and any integrand f of a constant sign (i.e, non-positive or non-negative). The definition relies on the following *Sign Rule.*

$$\int_a^b -f\,dg = -\int_a^b f\,dg = \int_a^b f\,d(-g) \tag{2.5.2}$$

If f is of constant sign, and g is monotone, then among the forms fdg, $-fdg$, $fd(-g)$ and $-fd(-g)$ there is just one with non-negative integrand and non-decreasing differand. For this form, the integral was defined earlier, for the other cases it is defined by the Sign Rule.

Thus the integral of a negative function against an increasing differand and the integral of a positive function against a decreasing differand are negative. And the integral of a negative function against a decreasing differand is positive.

The Sign Rule agrees with the Constant Rule: the formula $\int_a^b c\,dg = c(g(b) - g(a))$ remains true either for negative c or decreasing g.

The Partition Rule also is not affected by this extension of the integral.

The Inequality Rule takes the following form: if $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$ then $\int_a^b f_1(x)\,dg(x) \leq \int_a^b f_2(x)\,dg(x)$ for non-decreasing g and $\int_a^b f_1(x)\,dg(x) \geq \int_a^b f_2(x)\,dg(x)$ for non-increasing g.

Change of variable. Now all integrals in (2.5.1) are defined. The next objection concerns transformation $\cos t d\cos t = \frac{1}{2} d\cos^2 t$. This transformation is based on a decreasing change of variable $x = \cos t$ in $dx^2/2 = xdx$. But what happens with an integral when one applies a decreasing change of variable? The curvilinear trapezium, which represents the integral, does not change at all under any change of variable, even for a non-monotone one. Hence the only thing that may happen is a change of sign. And the sign changes by the Sign Rule, simultaneously on both sides of equality $dx^2/2 = xdx$. If the integrals of xdx and dx^2 are positive, both integrals of $\cos t d\cos t$ and $\cos^2 t$ are negative and have the same absolute value. These arguments work in the general case:

A decreasing change of variable reverses the sign of the integral.

Addition Formula. The next question concerns the legitimacy of addition of differentials, which appeared in the calculation $d\cos^2 t + d\sin^2 t = d(\cos^2 t + \sin^2 t) = 0$, where differands are not *comonotone*: $\cos t$ decreases, while $\sin t$ increases. The addition formula in its full generality will be proved in the next lecture, but this special case is not difficult to prove. Our equality is equivalent to $d\sin^2 t = -d\cos^2 t$. By the Sign Rule $-d\cos^2 t = d(-\cos^2 t)$, but $-\cos^2 t$ is increasing. And by the Addition Theorem $d(-\cos^2 t + 1) = d(-\cos^2 t) + d1 = d(-\cos^2 t)$. But $-\cos^2 t + 1 = \sin^2 t$. Hence our evaluation of the real part of (2.5.1) is justified.

Trigonometric integrals. We proceed to the evaluation of the imaginary part of (2.5.1), which is $\cos t\,d\sin t - \sin t\,d\cos t$. This is a simple geometric problem.

The integral of $\sin t\,d\cos t$ is negative as $\cos t$ is decreasing on $[0, \frac{\pi}{2}]$, and its absolute value is equal to the area of the curvilinear triangle $A'BA$, which is obtained from the circular sector OBA with area $\phi/2$ by deletion of the triangle $OA'B$, which has area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \sin t\,d\cos t$ is $\phi/2 - \frac{1}{2}\cos\phi\sin\phi$.

The integral of $\cos t\,d\sin t$ is equal to the area of curvilinear trapezium $OB'BA$. The latter consists of a circular sector OBA with area $\phi/2$ and a triangle $OB'B$ with area $\frac{1}{2}\cos\phi\sin\phi$. Thus $\int_0^\phi \cos t\,d\sin t = \phi/2 + \frac{1}{2}\cos\phi\sin\phi$.

As a result we get $\int_p \frac{1}{z}\,dz = i\phi$. This result has a lot of consequences. But today we restrict our attention to the integrals of $\sin t$ and $\cos t$.

Multiplication of differentials. We have proved

$$\cos t\,d\sin t - \sin t\,d\cos t = dt. \tag{2.5.3}$$

Multiplying this equality by $\cos t$, one gets

$$\cos^2 t\,d\sin t - \sin t\cos t\,d\cos t = \cos t\,dt.$$

Replacing $\cos^2 t$ by $(1 - \sin^2 t)$ and moving $\cos t$ into the differential, one transforms the left-hand side as

$$d\sin t - \sin^2 t\,d\sin t - \tfrac{1}{2}\sin t\,d\cos^2 t = d\sin t - \tfrac{1}{2}\sin t\,d\sin^2 t - \tfrac{1}{2}\sin t\,d\cos^2 t.$$

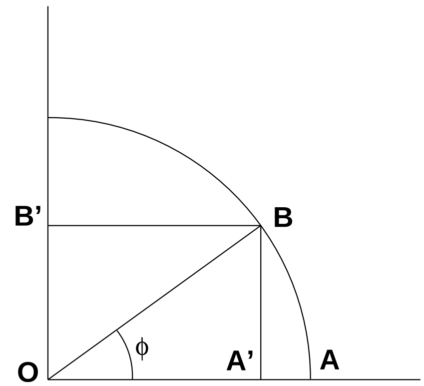

FIGURE 2.5.1. Trigonometric integrals

We already know that $d\sin^2 t + d\cos^2 t$ is zero. Now we have to prove the same for the product of this form by $\frac{1}{2}\sin t$. The arguments are the same: we multiply by $\frac{1}{2}\sin t$ the equivalent equality $d\sin^2 t = d(-\cos^2 t)$ whose differands are increasing. This is a general way to extend the theorem on multiplication of differentials to the case of any monotone functions. We will do it later. Now we get just $d\sin t = \cos t\,dt$.

Further, multiplication of the left-hand side of (2.5.3) by $\sin t$ gives

$$\sin t\cos t\,d\sin t - \sin^2 t\,d\cos t = \tfrac{1}{2}\cos t\,d\sin^2 t - d\cos t + \tfrac{1}{2}\cos t\,d\cos^2 t = -d\cos t.$$

So we get $d\cos t = -\sin t dt$.

THEOREM 2.5.1. *$d\sin t = \cos t\,dt$ and $d\cos t = -\sin t\,dt$.*

We have proved this equality only for $[0,\pi/2]$. But due to well-known symmetries this suffices.

Application of trigonometric integrals.

LEMMA 2.5.2. *For any convergent infinite product of factors ≥ 1 one has*

$$\lim \prod_{k=1}^{n} p_k = \prod_{k=1}^{\infty} p_k. \tag{2.5.4}$$

PROOF. Let ε be a positive number. Then $\prod_{k=1}^{\infty} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$, and by All-for-One there is n such that $\prod_{k=1}^{n} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Then for any $m > n$ one has the inequalities $\prod_{k=1}^{\infty} p_k \geq \prod_{k=1}^{m} p_k > \prod_{k=1}^{\infty} p_k - \varepsilon$. Therefore $|\prod_{k=1}^{m} p_k - \prod_{k=1}^{\infty} p_k| < \varepsilon$. □

Wallis product. Set $I_n = \int_0^\pi \sin^n x\,dx$. Then $I_0 = \int_0^\pi 1\,dx = \pi$ and $I_1 = \int_0^\pi \sin x\,dx = -\cos\pi + \cos 0 = 2$. For $n \geq 2$, let us replace the integrand $\sin^n x$ by

$\sin^{n-2} x(1-\cos^2 x)$ and obtain

$$\begin{aligned} I_n &= \int_0^\pi \sin^{n-2} x(1-\cos^2 x)\, dx \\ &= \int_0^\pi \sin^{n-2} x\, dx - \int_0^\pi \sin^{n-2} x \cos x\, d\sin x \\ &= I_{n-2} - \tfrac{1}{n-1}\int_0^\pi \cos x\, d\sin^{n-1}(x) \\ &= I_{n-2} - \int_0^\pi d(\cos x \sin^{n-1} x) + \int_0^\pi \sin^{n-1} x\, d\cos x \\ &= I_{n-2} - \tfrac{1}{n-1} I_n. \end{aligned}$$

We get the recurrence relation $I_n = \frac{n-1}{n} I_{n-2}$, which gives the formula

$$I_{2n} = \pi\frac{(2n-1)!!}{2n!!}, \quad I_{2n-1} = 2\frac{(2n-2)!!}{(2n-1)!!} \tag{2.5.5}$$

where $n!!$ denotes the product $n(n-2)(n-4)\cdots(n \mod 2+1)$. Since $\sin^n x \leq \sin^{n-1} x$ for all $x \in [0,\pi]$, the sequence $\{I_n\}$ decreases. Since $I_n \leq I_{n-1} \leq I_{n-2}$, one gets $\frac{n-1}{n} = \frac{I_n}{I_{n-2}} \leq \frac{I_{n-1}}{I_{n-2}} \leq 1$. Hence $\frac{I_{n-1}}{I_{n-2}}$ differs from 1 less than $\frac{1}{n}$. Consequently, $\lim \frac{I_{n-1}}{I_{n-2}} = 1$. In particular, $\lim \frac{I_{2n+1}}{I_{2n}} = 1$. Substituting in this last formula the expressions of I_n from (2.5.5) one gets

$$\lim \frac{\pi}{2}\frac{(2n+1)!!(2n-1)!!}{2n!!2n!!} = 1.$$

Therefore this is the famous Wallis Product

$$\frac{\pi}{2} = \lim \frac{2n!!2n!!}{(2n-1)!!(2n+1)!!} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}.$$

Stirling constant. In Lecture 2.4 we have proved that

$$\ln n! = n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n, \tag{2.5.6}$$

where o_n is infinitesimally small and σ is a constant. Now we are ready to determine this constant. Consider the difference $\ln 2n! - 2\ln n!$. By (2.5.6) it expands into

$$\begin{aligned} &(2n\ln 2n - 2n + \tfrac{1}{2}\ln 2n + \sigma + o_{2n}) - 2(n\ln n - n + \tfrac{1}{2}\ln n + \sigma + o_n) \\ &\qquad = 2n\ln 2 + \tfrac{1}{2}\ln 2n - \ln n - \sigma + o'_n, \end{aligned}$$

where $o'_n = o_{2n} - 2o_n$ is infinitesimally small. Then σ can be presented as

$$\sigma = 2\ln n! - \ln 2n! + 2n\ln 2 + \tfrac{1}{2}\ln n + \tfrac{1}{2}\ln 2 - \ln n + o'_n.$$

Multiplying by 2 one gets

$$2\sigma = 4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2 + 2o'_n.$$

Hence $2\sigma = \lim(4\ln n! - 2\ln 2n! + 2\ln 2^{2n} - \ln n + \ln 2)$. Switching to product and keeping in mind the identities $n! = n!!(n-1)!!$ and $n!2^n = 2n!!$ one gets

$$\sigma^2 = \lim \frac{n!^4 2^{4n+1}}{(2n!)^2 n} = \lim \frac{2\cdot(2n!!)^4}{(2n!!)^2(2n-1)!!^2 n} \lim \frac{2\cdot(2n!!)^2(2n+1)}{(2n-1)!!(2n+1)!!n} = 2\pi.$$

Problems.

1. Evaluate $\int \sqrt{1-x^2}\,dx$.
2. Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx$.
3. Evaluate $\int \sqrt{5-x^2}\,dx$.
4. Evaluate $\int \cos^2 x\,dx$.
5. Evaluate $\int \tan x\,dx$.
6. Evaluate $\int \sin^4 x\,dx$.
7. Evaluate $\int \sin x^2\,dx$.
8. Evaluate $\int \tan x\,dx$.
9. Evaluate $\int x^2 \sin x\,dx$.
10. Evaluate $d \arcsin x$.
11. Evaluate $\int \arcsin x\,dx$.
12. Evaluate $\int e^x \cos x\,dx$.

2.6. Virtually monotone functions

Monotonization of the integrand. Let us say that a pair of functions f_1, f_2 *monotonize* a function f, if f_1 is non-negative and non-decreasing, f_2 is non-positive and non-increasing and $f = f_1 + f_2$.

LEMMA 2.6.1. *Let $f = f_1 + f_2$ and $f = f_1' + f_2'$ be two monotonizations of f. Then for any monotone h one has $f_1 dh + f_2 dh = f_1 dh + f_2' dh$.*

PROOF. Our equality is equivalent to $f_1 dh - f_2' dh = f_1' dh - f_2 dh$. By the sign rule this turns into $f_1 dh + (-f_2')dh = f_1' dh + (-f_2)dh$. Now all integrands are nonnegative and for non-decreasing h we can apply the Addition Theorem and transform the inequality into $(f_1 - f_2')dh = (f_1' - f_2)dh$. This is true because $(f_1 - f_2') = (f_1' - f_2)$.

The case of a non-increasing differand is reduced to the case of a non-decreasing one by the transformation $f_1 d(-h) + f_2 d(-h) = f_1' d(-h) + f_2' d(-h)$, which is based on the Sign Rule. □

A function which has a monotonization is called *virtually monotone.*

We define the integral $\int_a^b f\,dg$ for any virtually monotone integrand f and any continuous monotone differand g via a monotonization $f = f_1 + f_2$ by

$$\int_a^b f\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$$

Lemma 2.6.1 demonstrates that this definition does not depend on the choice of a monotonization.

LEMMA 2.6.2. *Let f and g be virtually monotone functions; then $f + g$ is virtually monotone and $f dh + g dh = (f + g)dh$ for any continuous monotone h.*

PROOF. Let h be nondecreasing. Consider monotonizations $f = f_1 + f_2$ and $g = g_1 + g_2$. Then $f dh + g dh = f_1 dh + f_2 dh + g_1 dh + g_2 dh$ by definition via monotonization of the integrand. By virtue of the Addition Theorem 2.3.3 this turns into $(f_1 + g_1)dh + (f_2 + g_2)dh$. But the pair of brackets monotonize $f + g$. Hence $f + g$ is proved to be virtually monotone and the latter expression is $(f + g)dh$ by definition, via monotonization of the integrand. The case of non-increasing h is reduced to the previous case via $-fd(-h) - gd(-h) = -(f + g)d(-h)$. □

Lemma on locally constant functions. Let us say that a function $f(x)$ is *locally constant* at a point x if $f(y) = f(x)$ for all y sufficiently close to x, i.e., for all y from an interval $(x - \varepsilon, x + \varepsilon)$.

LEMMA 2.6.3. *A function f which is locally constant at each point of an interval is constant.*

PROOF. Suppose $f(x)$ is not constant on $[a, b]$. We will construct by induction a sequence of intervals $I_k = [a_k, b_k]$, such that $I_0 = [a, b]$, $I_{k+1} \subset I_k$, $|b_k - a_k| \geq 2|b_{k+1} - a_{k+1}|$ and the function f is not constant on each I_k. First step: Let $c = (a + b)/2$, as f is not constant $f(x) \neq f(c)$ for some x. Then choose $[x, c]$ or $[c, x]$ as for $[a_1, b_1]$. On this interval f is not constant. The same are all further steps. The intersection of the sequence is a point such that any of its neighborhoods contains some interval of the sequence. Hence f is not locally constant at this point. □

LEMMA 2.6.4. *If $f(x)$ is a continuous monotone function and $a < f(x) < b$ then $a < f(y) < b$ for all y sufficiently close to x.*

PROOF. If f takes values greater than b, then it takes value b and if $f(x)$ takes values less than a then it takes value a due to continuity. Then $[f^{-1}(a), f^{-1}(b)]$ is the interval where inequalities hold. □

LEMMA 2.6.5. *Let g_1, g_2 be continuous* comonotone *functions. Then $g_1 + g_2$ is continuous and monotone, and for any virtually monotone f one has*

$$fdg_1 + fdg_2 = fd(g_1 + g_2). \tag{2.6.1}$$

PROOF. Suppose $g_1(x) + g_2(x) < p$, let $\varepsilon = p - g_1(x) - g_2(x)$. Then $g_1(y) < g_1(y) + \varepsilon/2$ and $g_2(y) < g_2(y) + \varepsilon/2$ for all y sufficiently close to x. Hence $g_(y) + g_2(y) < p$ for all y sufficiently close to x. The same is true for the opposite inequality. Hence $\operatorname{sgn}(g_1(x)+g_2(x)-p)$ is locally constant at all points where it is not 0. But it is not constant if p is an intermediate value, hence it is not locally constant, hence it takes value 0. At this point $g_1(x) + g_2(x) = p$ and the continuity of $g_1 + g_2$ is proved.

Consider a monotonization $f = f_1 + f_2$. Let g_i be nondecreasing. By definition via monotonization of the integrand, the left-hand side of (2.6.1) turns into $(f_1dg_1 + f_2dg_1) + (f_1dg_2 + f_2dg_2) = (f_1dg_1 + f_1dg_2) + (f_2dg_1 + f_2dg_2)$. By the Addition Theorem 2.3.3 $f_1dg_1 + f_1dg_2 = f_1d(g_1 + g_2)$. And the equality $f_2dg_1 + f_2dg_2 = f_2d(g_1 + g_2)$ follows from $(-f_2)dg_1 + (-f_2)dg_2 = (-f_2)d(g_1 + g_2)$ by the Sign Rule. Hence the left-hand side is equal to $f_1d(g_1 + g_2) + f_2d(g_1 + g_2)$, which coincides with the right-hand side of (2.6.1) by definition via monotonization of integrand. The case of non-increasing differands is taken care of via transformation of (2.6.1) by the Sign Rule into $fd(-g_1) + fd(-g_2) = fd(-g_1 - g_2)$. □

LEMMA 2.6.6. *Let $g_1 + g_2 = g_3 + g_4$ where all $(-1)^k g_k$ are non-increasing continuous functions. Then $fdg_1 + fdg_2 = fdg_3 + fdg_4$ for any virtually monotone f.*

PROOF. Our equality is equivalent to $fdg_1 - fdg_4 = fdg_3 - fdg_2$. By the Sign Rule it turns into $fdg_1 + fd(-g_4) = fdg_3 + fd(-g_2)$. Now all differands are nondecreasing and by Lemma 2.6.5 it transforms into $fd(g_1 - g_4) = fd(g_3 - g_2)$. This is true because $g_1 - g_4 = g_3 - g_2$. □

Monotonization of the differand. A monotonization by continuous functions is called continuous. A virtually monotone function which has a continuous monotonization is called *continuous*. The integral for any virtually monotone integrand f against a virtually monotone continuous differand g is defined via a continuous virtualization $g = g_1 + g_2$ of the differand

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

The integral is well-defined because of Lemma 2.6.6.

THEOREM 2.6.7 (Addition Theorem). *For any virtually monotone functions f, f' and any virtually monotone continuous g, g', $fdg + f'dg = (f + f')dg$ and $fdg + fdg' = fd(g + g')$*

PROOF. To prove $fdg + f'dg = (f + f')dg$, consider a continuous monotonization $g = g_1 + g_2$. Then by definition of the integral for virtually monotone differands this equality turns into $(fdg_1 + fdg_2) + (f'dg_1 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. After rearranging it turns into $(fdg_1 + f'dg_1) + (fdg_2 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. But this is true due to Lemma 2.6.2.

To prove $fdg + fdg' = fd(g + g')$, consider monotonizations $g = g_1 + g_2$, $g' = g'_1 + g'_2$. Then $(g_1 + g'_1) + (g_2 + g'_2)$ is a monotonization for $g + g'$. And by the definition of the integral for virtually monotone differands our equality turns into $fdg_1 + fdg_2 + fdg'_1 + fdg'_2$ □

Change of variable.

LEMMA 2.6.8. *If f is virtually monotone and g is monotone, then $f(g(x))$ is virtually monotone.*

PROOF. Let $f_1 + f_2$ be a monotonization of f. If h is non-decreasing then $f_1(h(x)) + f_2(h(x))$ gives a monotonization of $f(g(x))$. If h is decreasing then the monotonization is given by $(f_2(h(x)) + c) + (f_1(h(x)) - c)$ where c is a sufficiently large constant to provide positivity of the first brackets and negativity of the second one. □

The following natural convention is applied to define an integral with reversed limits: $\int_a^b f(x)\,dg(x) = -\int_b^a f(x)\,dg(x)$.

THEOREM 2.6.9 (on change of variable). *If $h\colon [a, b] \to [h(a), h(b)]$ is monotone, $f(x)$ is virtually monotone and $g(x)$ is virtually monotone continuous then*

$$\int_a^b f(h(t))\,dg(h(t)) = \int_{h(a)}^{h(b)} f(x)\,dg(x).$$

PROOF. Let $f = f_1 + f_2$ and $g = g_1 + g_2$ be a monotonization and a continuous monotonization of f and g respectively. The $\int_a^b f(h(t))\,dg(h(t))$ splits into sum of four integrals: $\int_a^b f_i(h(t))\,dg_j(h(t))$ where f_i are of constant sign and g_j are monotone continuous. These integrals coincide with the corresponding integrals $\int_{h(a)}^{h(b)} f_i(x)\,dg_i(x)$. Indeed their absolute values are the areas of the same curvilinear trapezia. And their signs determined by the Sign Rule are the same. □

Integration by parts. We have established the Integration by Parts formula for non-negative and non-decreasing differential forms. Now we extend it to the case of continuous monotone forms. In the first case f and g are non-decreasing. In this case choose a positive constant c sufficiently large to provide positivity of $f + c$ and $g + c$ on the interval of integration. Then $d(f + c)(g + c) = (f + c)d(g + c) + (g + c)d(f + c)$. On the other hand $d(f + c)(g + c) = dfg + cdf + cdg$ and $(f + c)d(g + c) + (g + c)d(f + c) = fdg + cdg + cdf$. Compare these results to get $dfg = fdg + gdf$. Now if f is increasing and g is decreasing then $-g$ is increasing and we get $-dfg = df(-g) = fd(-g) + (-g)df = -fdg - gdf$, which leads to $dfg = fdg + gdf$. The other cases: f decreasing, g increasing and both decreasing are proved by the same arguments. The extension of the Integration by Parts formula to piecewise monotone forms immediately follows by the Partition Rule.

Variation. Define the *variation of a sequence* of numbers $\{x_k\}_{k=1}^n$ as the sum $\sum_{k=1}^{\infty} |x_{k+1} - x_k|$. Define the variation of a function f along a sequence $\{x_k\}_{k=0}^n$

as the variation of sequence $\{f(x_k)\}_{k=0}^n$. Define a *chain* on an interval $[a,b]$ as a nondecreasing sequence $\{x_k\}_{k=0}^n$ such that $x_0 = a$ and $x_n = b$. Define the *partial variation* of f on an interval $[a,b]$ as its variation along a chain on the interval.

The least number surpassing all partial variations function f over $[a,b]$ is called *the (ultimate) variation of a function* $f(x)$ on an interval $[a,b]$ and is denoted by $\operatorname{var}_f[a,b]$.

LEMMA 2.6.10. *For any function f one has the inequality* $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$. *If f is a monotone function on $[a,b]$, then* $\operatorname{var}_f[a,b] = |f(b) - f(a)|$.

PROOF. The inequality $\operatorname{var}_f[a,b] \geq |f(b) - f(a)|$ follows immediately from the definition because $\{a,b\}$ is a chain. For monotone f, all partial variations are telescopic sums equal to $|f(b) - f(a)|$ □

THEOREM 2.6.11 (additivity of variation). $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] = \operatorname{var}_f[a,c]$.

PROOF. Consider a chain $\{x_k\}_{k=0}^n$ of $[a,c]$, which contains b. In this case the variation of f along $\{x_k\}_{k=0}^n$ splits into sums of partial variations of f along $[a,b]$ and along $[b,c]$. As a partial variations does not exceed an ultimate, we get that in this case the variation of f along $\{x_k\}_{k=0}^n$ does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$.

If $\{x_k\}_{k=0}^n$ does not contain b, let us add b to the chain. Then in the sum expressing the partial variation of f, the summand $|f(x_{i+1}) - f(x_i)|$ changes by the sum $|f(b) - f(x_i)| + |f(x_{i+1} - f(b)|$ which is greater or equal. Hence the variation does not decrease after such modification. But the variation along the modified chain does not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$ as was proved above. As all partial variations of f over $[a,c]$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c]$, the same is true for the ultimate variation.

To prove the opposite inequality we consider a *relaxed* inequality $\operatorname{var}_f[a,b] + \operatorname{var}_f[b,c] \leq \operatorname{var}_f[a,c] + \varepsilon$ where ε is an positive number. Choose chains $\{x_k\}_{k=0}^n$ on $[a,b]$ and $\{y_k\}_{k=0}^m$ on $[b,c]$ such that corresponding partial variations of f are $\geq \operatorname{var}_f[a,b] + \varepsilon/2$ and $\geq \operatorname{var}_f[b,c] + \varepsilon/2$ respectively. As the union of these chains is a chain on $[a,c]$ the sum of these partial variations is a partial variation of f on $[a,c]$. Consequently this sum is less or equal to $\operatorname{var}_f[a,c]$. On the other hand it is greater or equal to $\operatorname{var}_f[a,b] + \varepsilon/2 + \operatorname{var}_f[b,c] + \varepsilon/2$. Comparing these results gives just the relaxed inequality. As the relaxed inequality is proved for all $\varepsilon > 0$ it also holds for $\varepsilon = 0$. □

LEMMA 2.6.12. *For any functions f, g one has the inequality* $\operatorname{var}_{f+g}[a,b] \leq \operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$.

PROOF. Since $|f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \leq |f(x_{k+1}) - f(x_k)| + |g(x_{k+1}) - g(x_k)|$, the variation of $f + g$ along any sequence does not exceed the sum of the variations of f and g along the sequence. Hence all partial variations of $f + g$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$, and so the same is true for the ultimate variation. □

LEMMA 2.6.13. *For any function of finite variation on $[a,b]$, the functions* $\operatorname{var}_f[a,x]$ *and* $\operatorname{var}_f[a,x] - f(x)$ *are both nondecreasing functions of x.*

PROOF. That $\operatorname{var}_f[a,x]$ is nondecreasing follows from nonnegativity and additivity of variation. If $x > y$ then the inequality $\operatorname{var}_f[a,x] - f(x) \geq \operatorname{var}_f[a,y] - f(y)$

is equivalent to $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] \geq f(x)-f(y)$. This is true because $\operatorname{var}_f[a,x]-\operatorname{var}_f[a,y] = \operatorname{var}_f[x,y] \geq |f(x)-f(y)|$. □

LEMMA 2.6.14. $\operatorname{var}_{f^2}[a,b] \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b]$.

PROOF. For all $x, y \in [a,b]$ one has

$$\begin{aligned} |f(x)+f(y)| &= |2f(a)+f(x)-f(a)+f(y)-f(a)| \\ &\leq 2|f(a)| + \operatorname{var}_f[a,x] + \operatorname{var}_f[a,y] \\ &\leq 2|f(a)| + 2\operatorname{var}_f[a,b]. \end{aligned}$$

Hence

$$\begin{aligned} \textstyle\sum_{k=0}^{n-1} |f^2(x_{k+1}) - f^2(x_k)| &= \textstyle\sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)||f(x_{k+1})+f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \textstyle\sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)| \\ &\leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b] \end{aligned}$$

□

LEMMA 2.6.15. *If* $\operatorname{var}_f[a,b] < \infty$ *and* $\operatorname{var}_g[a,b] < \infty$, *then* $\operatorname{var}_{fg}[a,b] < \infty$.

PROOF. $4fg = (f+g)^2 - (f-g)^2$. □

THEOREM 2.6.16. *The function f is virtually monotone on $[a,b]$ if and only if it has a finite variation.*

PROOF. Since monotone functions have finite variation on finite intervals, and the variation of a sum does not exceed the sum of variations, one gets that all virtually monotone functions have finite variation. On the other hand, if f has finite variation then $f = (\operatorname{var}_f[a,x] + c) + (f(x) - \operatorname{var}_f[a,x] - c)$, the functions in the brackets are monotone due to Lemma 2.6.13, and by choosing a constant c sufficiently large, one obtains that the second bracket is negative. □

Problems.

1. Evaluate $\int_1^i z^2\,dz$.

2. Prove that $1/f(x)$ has finite variation if it is bounded.

3. Prove $\int_a^b f(x)\,dg(x) \leq \max_{[a,b]} f \operatorname{var}_g[a,b]$.