

## Lecture 3

# Welcome to MAT 250!

MAT 250  
Lecture 3  
Definitions

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Brightspace

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**Brightspace** for MAT 250.01

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Midterm 2	25%
Final (5/14)	35%
HW	15%
Quizzes	5%

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The **final grade** is the **maximum** of the score for final exam and the total grade calculated according to the scheme described above.

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$P$	$Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

$P$	$\neg P$
T	F
F	T

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In Calculus their counterparts are **polynomials**.

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**Theorem.** Any Boolean function of finitely many variable which is not identically false has a full disjunctive normal form.

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**Theorem.** Any Boolean function of finitely many variable which is not identically false has a full disjunctive normal form.

Conjunction and disjunction are involved in a few simple relations, which allow to simplify a disjunctive normal form.

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In order to answer, we need to study  $\implies$  and  $\iff$  .

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**Conclusion.** the connective  $\implies$  is needed for **growth** of theories.

# Objectives

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- Why is it important to **remember** definitions?
- How can we **work with** definitions?
- How should we **read** a mathematical text?

# The nature of a mathematical definition

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Let us illustrate the nature of a mathematical definition

using the definition of a rational number.

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The definition is clear and unambiguous, but to fully understand it,  
we must know what an integer is and what a quotient means.

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Each time we say “rational number,”  
we must mean exactly what the definition says - no more, no less.

The definition is clear and unambiguous, but to fully understand it,  
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The definition also explains which number is not rational:

# Definition of a rational number

**Definition.** A number is called **rational number**  
if it can be presented as a quotient of two integers.

This definition contains three essential parts:

- The **term** (word or phrase) to be defined - “rational number”.
- The **class** it belongs to - “numbers”.
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The definition is clear and unambiguous, but to fully understand it,  
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Give a **definition**, not just an example!

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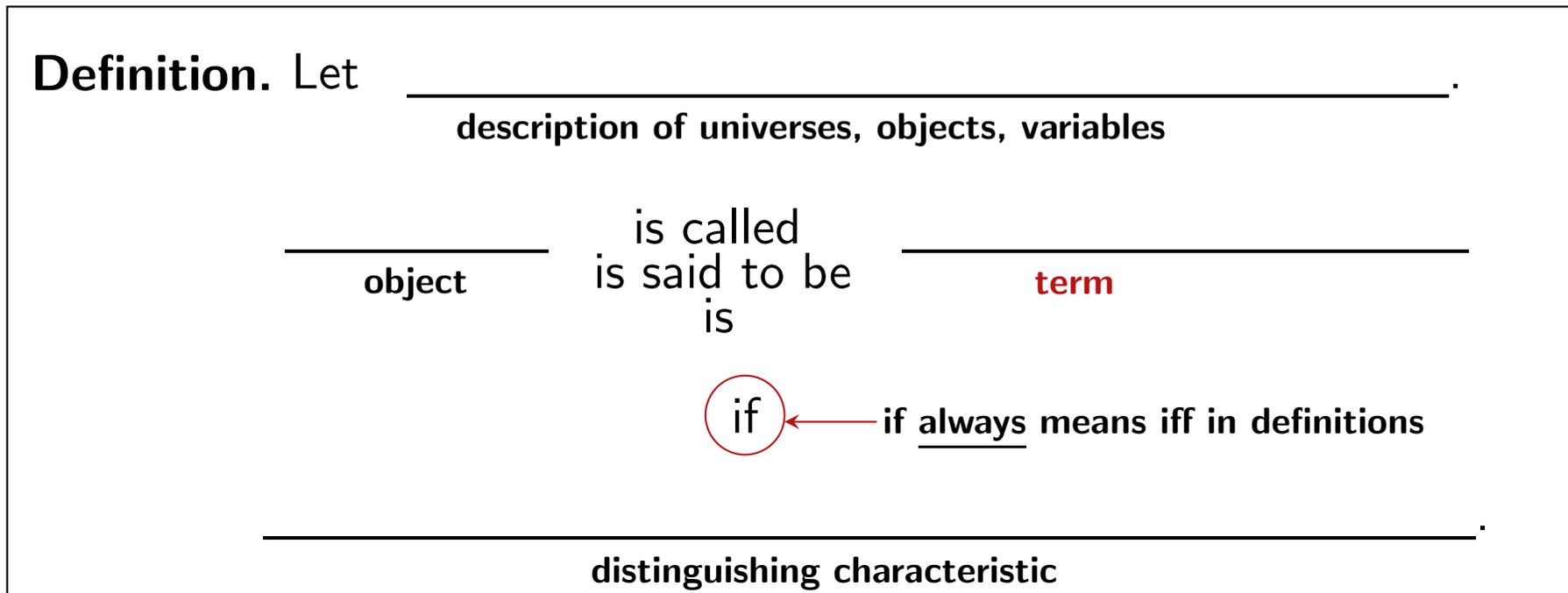
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$$d|n \iff \exists k \in \mathbb{Z} \quad n = d \cdot k.$$

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**Notation:**  $d|n$

**Remarks. 1.** Variables  $d$  and  $n$  are free.

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## Example.

Let  $X$ ,  $Y$  and  $Z$  be sets, and let  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  be maps.

Then the map  $g \circ f : X \rightarrow Z$  defined by formula  $g \circ f(x) = g(f(x))$  is called the **composition** of  $f$  and  $g$ .

Given a definition “ $X$  is called  $\langle \text{name} \rangle$  if  $P(X)$ ”, a statement

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There is no special notation for a prime number,

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**Answer:** Yes, it is true that if a integer greater than 1 is not prime, then it is composite.

# Increasing function

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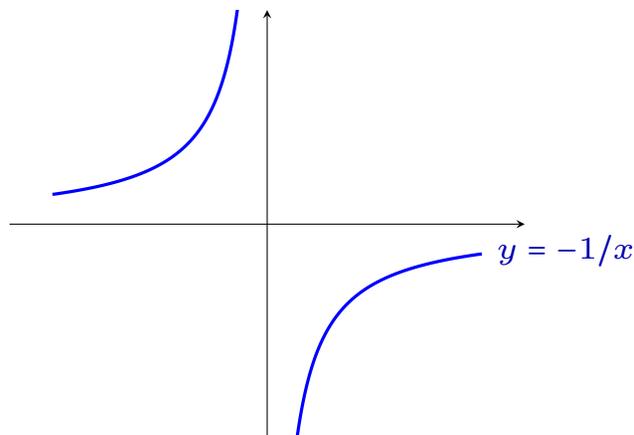
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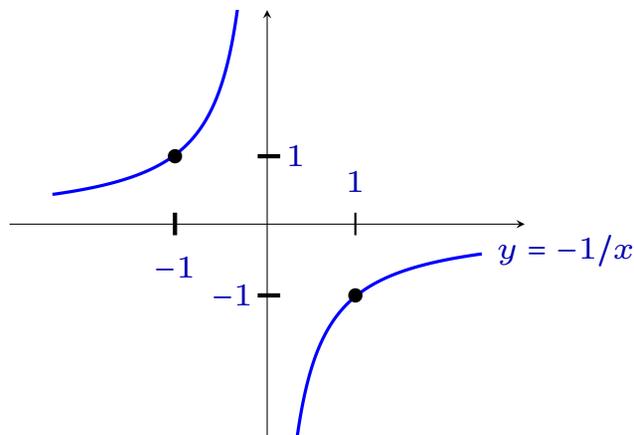
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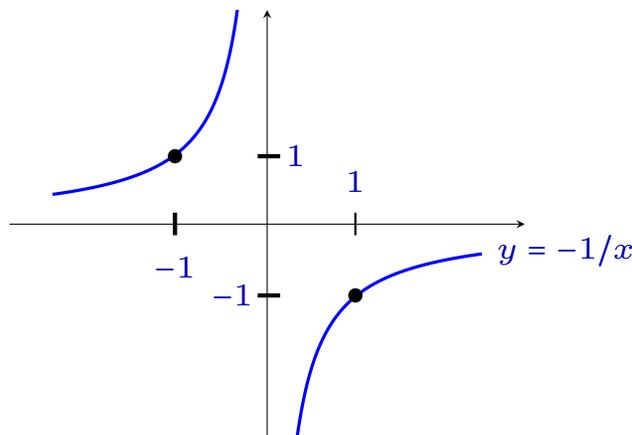
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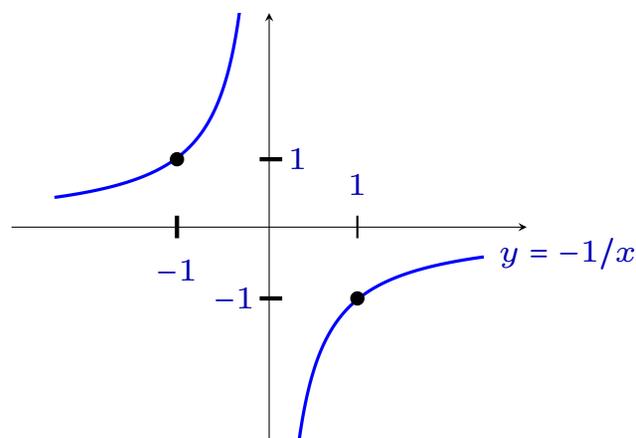
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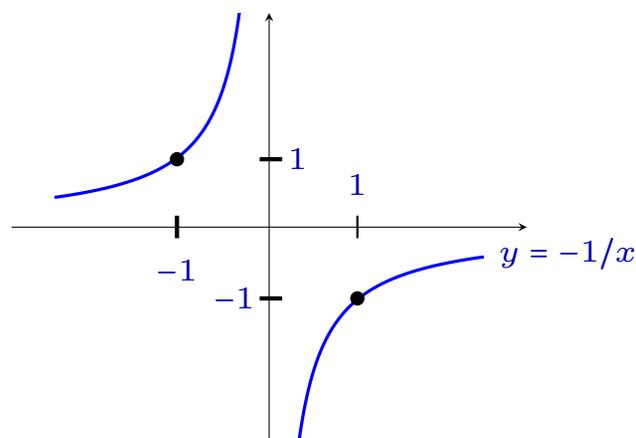
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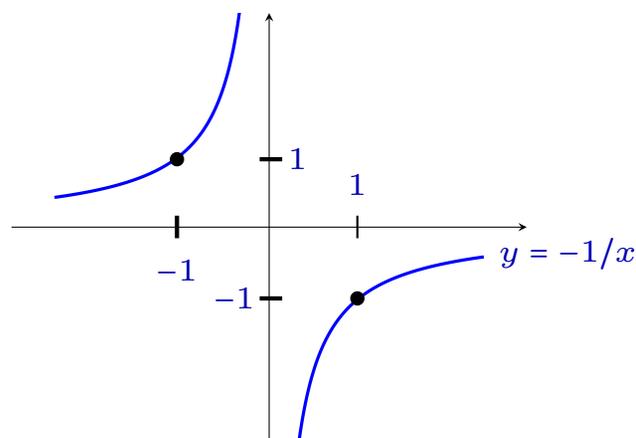
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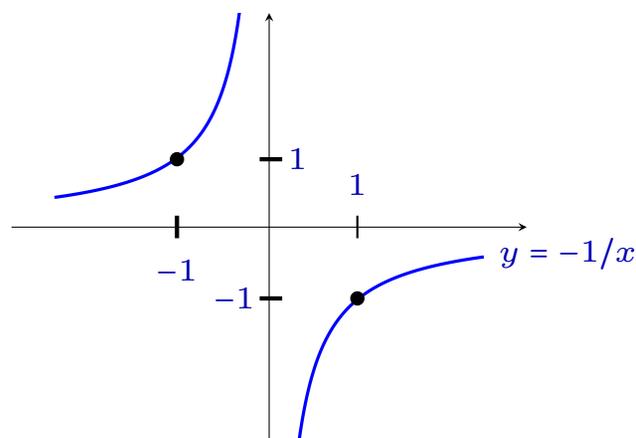
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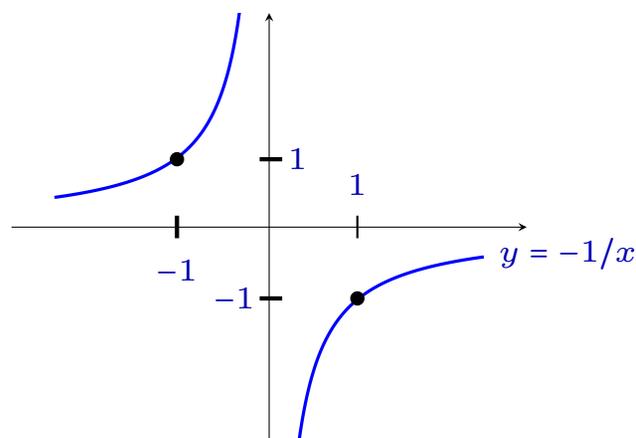
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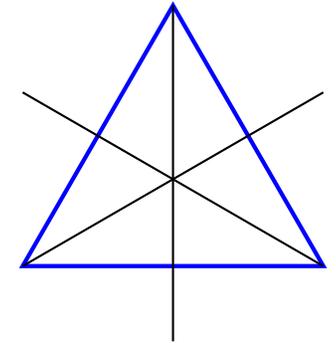
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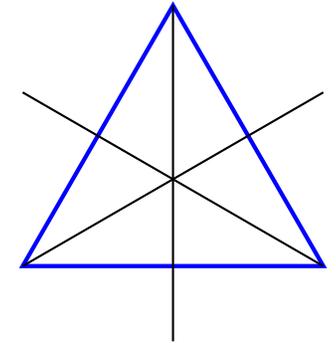


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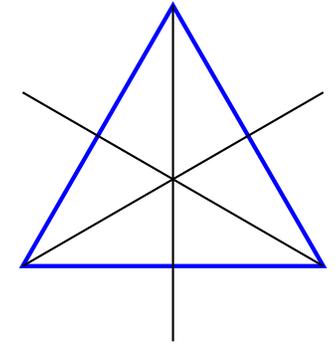


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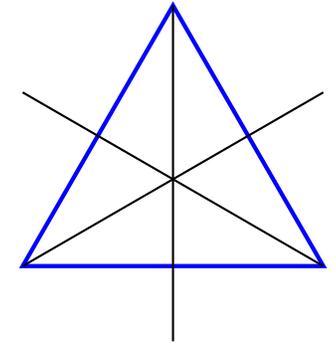
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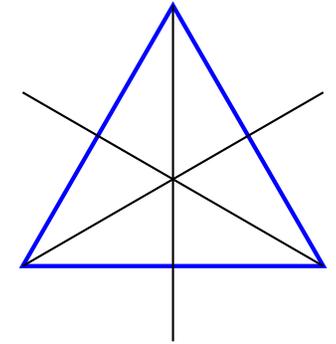
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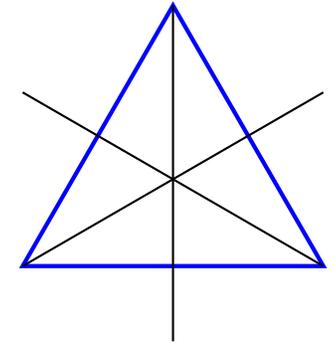
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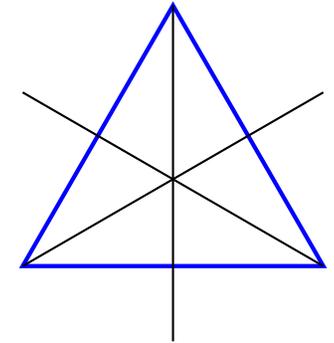
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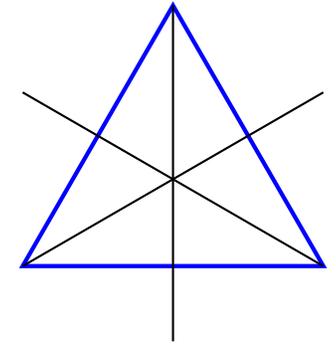
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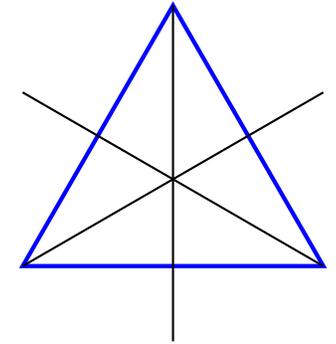
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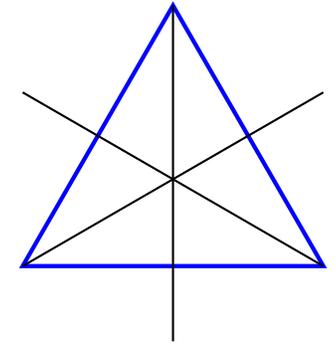
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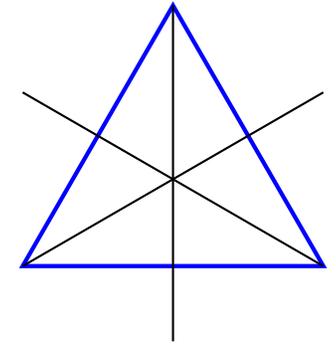
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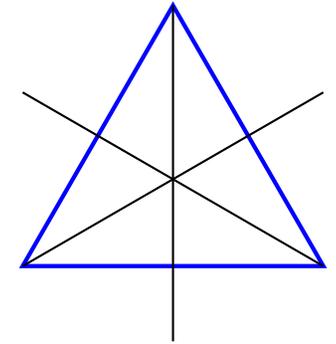
- **Associativity:** Composition of symmetries is associative, just like any composition of functions.
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# Symmetries of triangle

**Example 2.** Consider an equilateral triangle and all of its **symmetries**.

There are exactly six symmetries:

- three clockwise **rotations** by  $120^\circ$ ,  $240^\circ$ ,  $360^\circ$  (which is the identity transformation, or rotation by  $0^\circ$ ),
- and three **reflections**, each across a line of symmetry through a vertex and the midpoint of the opposite side.



The **composition** of any two of these symmetries is again a symmetry (you can verify this). We say that the set of symmetries is **closed** under composition.

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- **Associativity:** Composition of symmetries is associative, just like any composition of functions.
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- **Inverses:** Every symmetry has an inverse that undoes its effect.

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This leads to the definition of a **group**.

# Definition of a group

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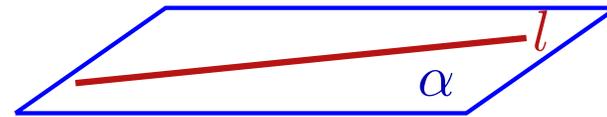
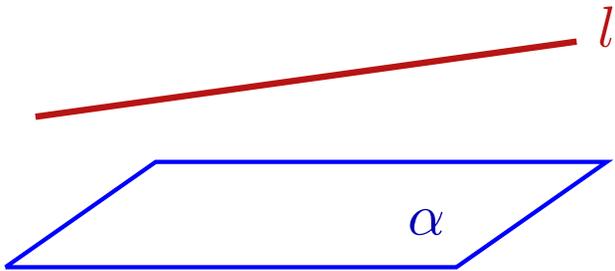
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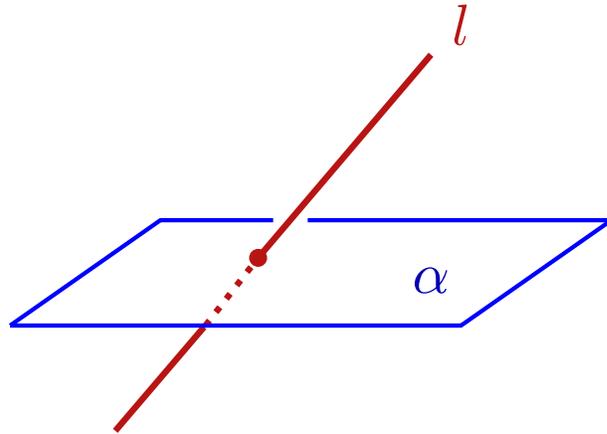
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# Understanding the definition of limit

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MAT 250  
Lecture 3  
Definitions

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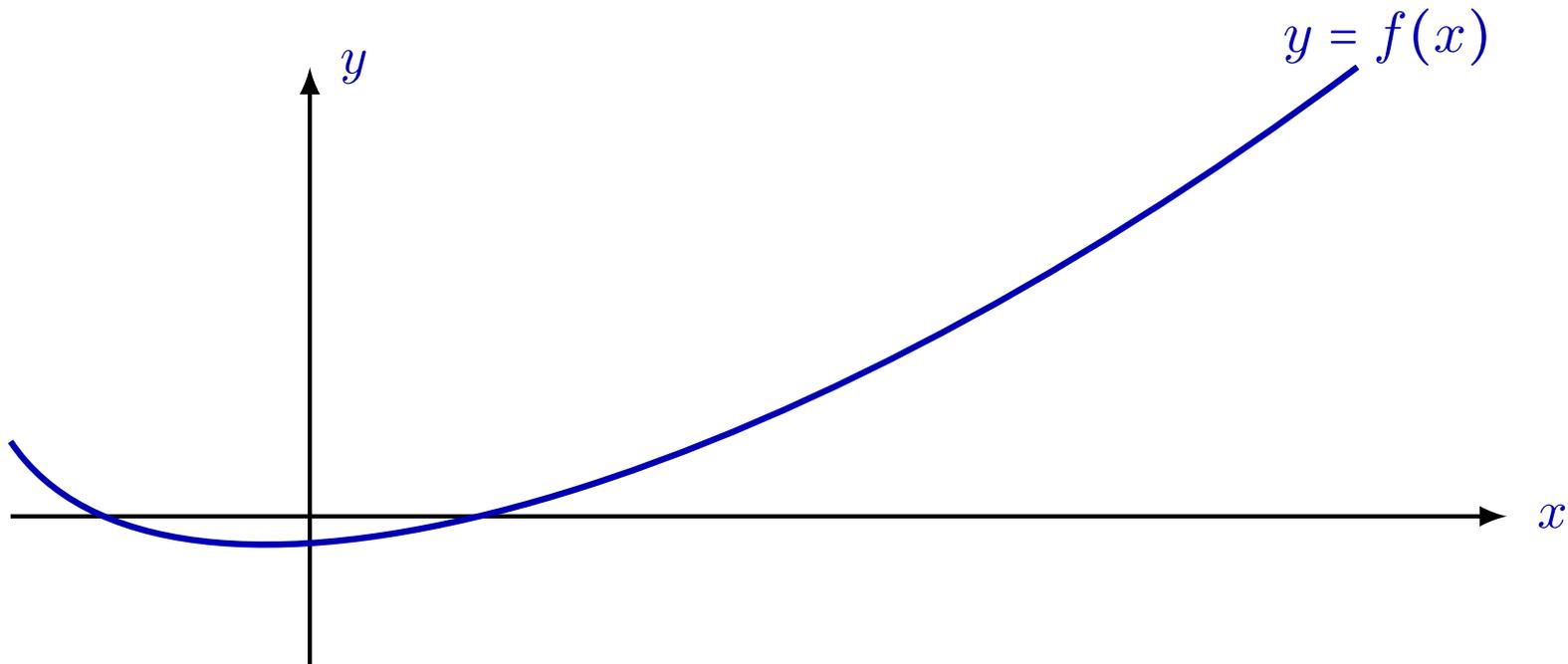
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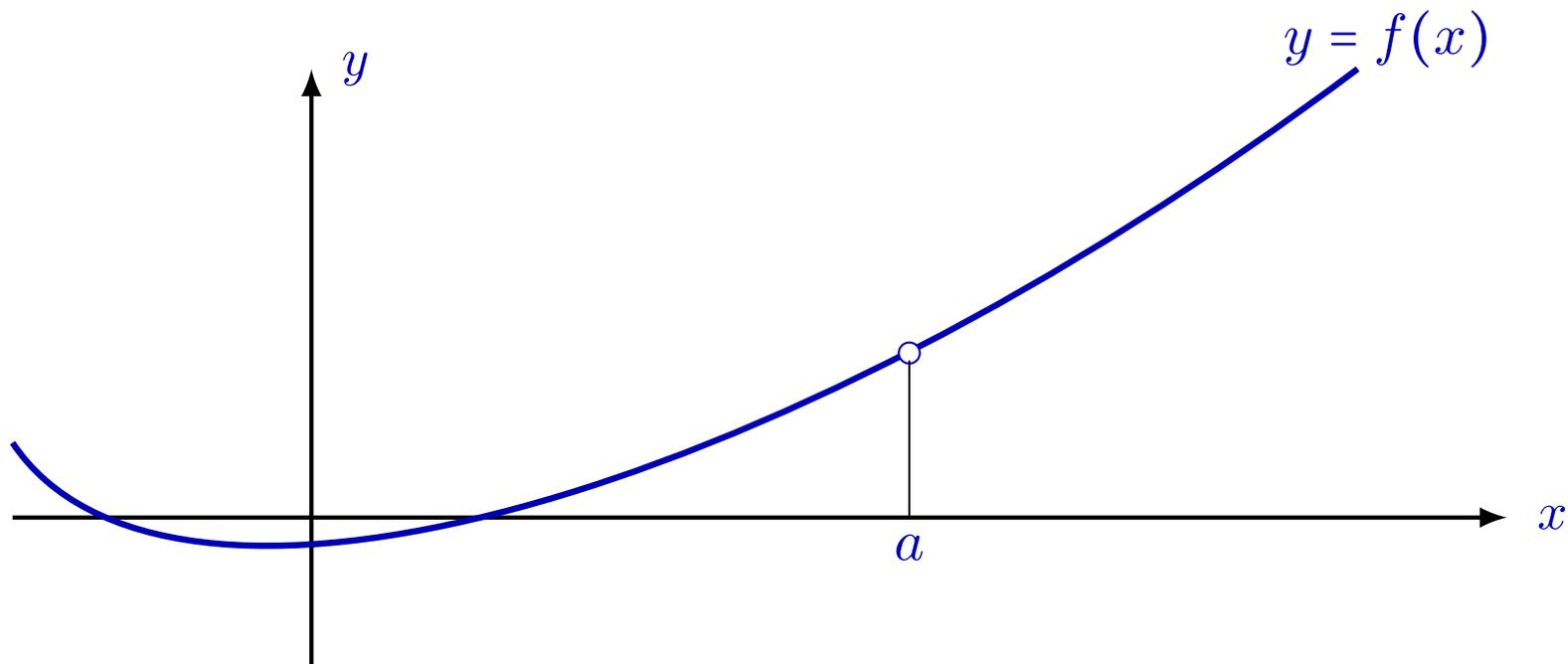
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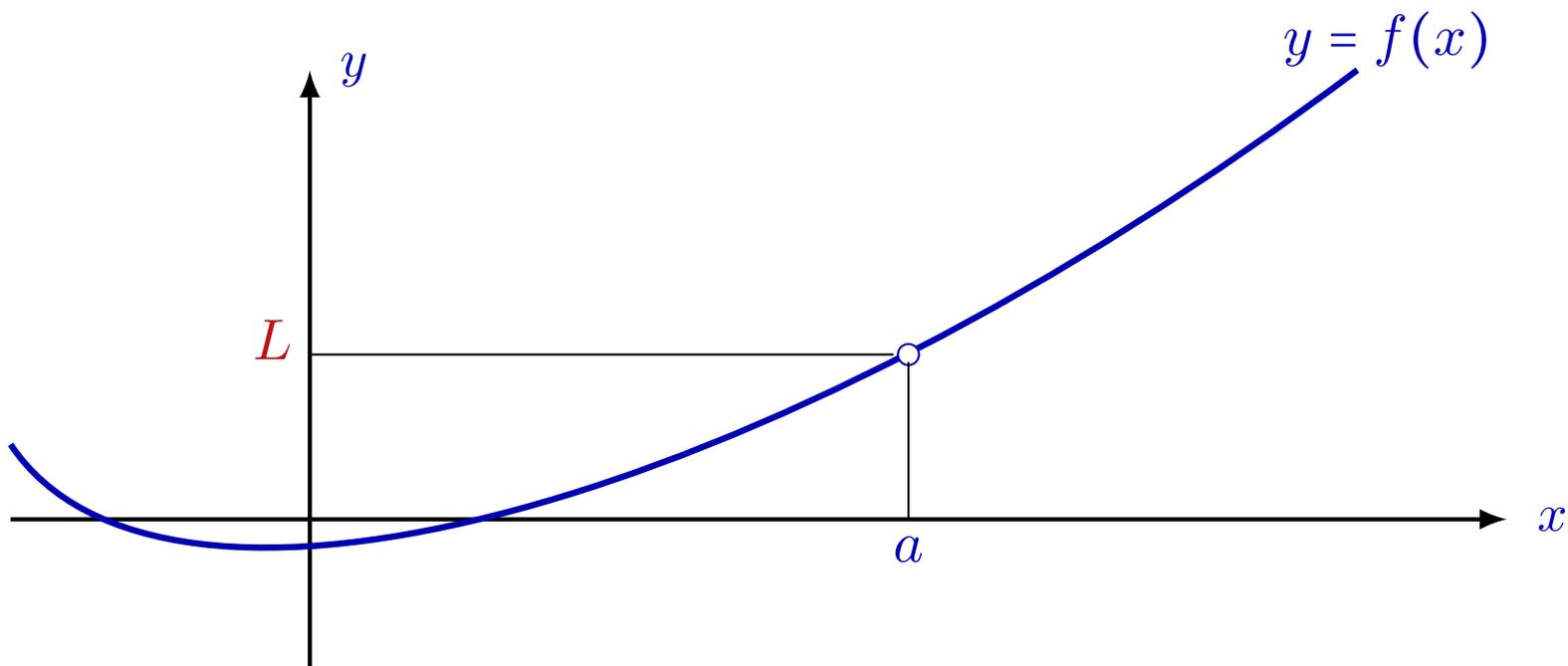
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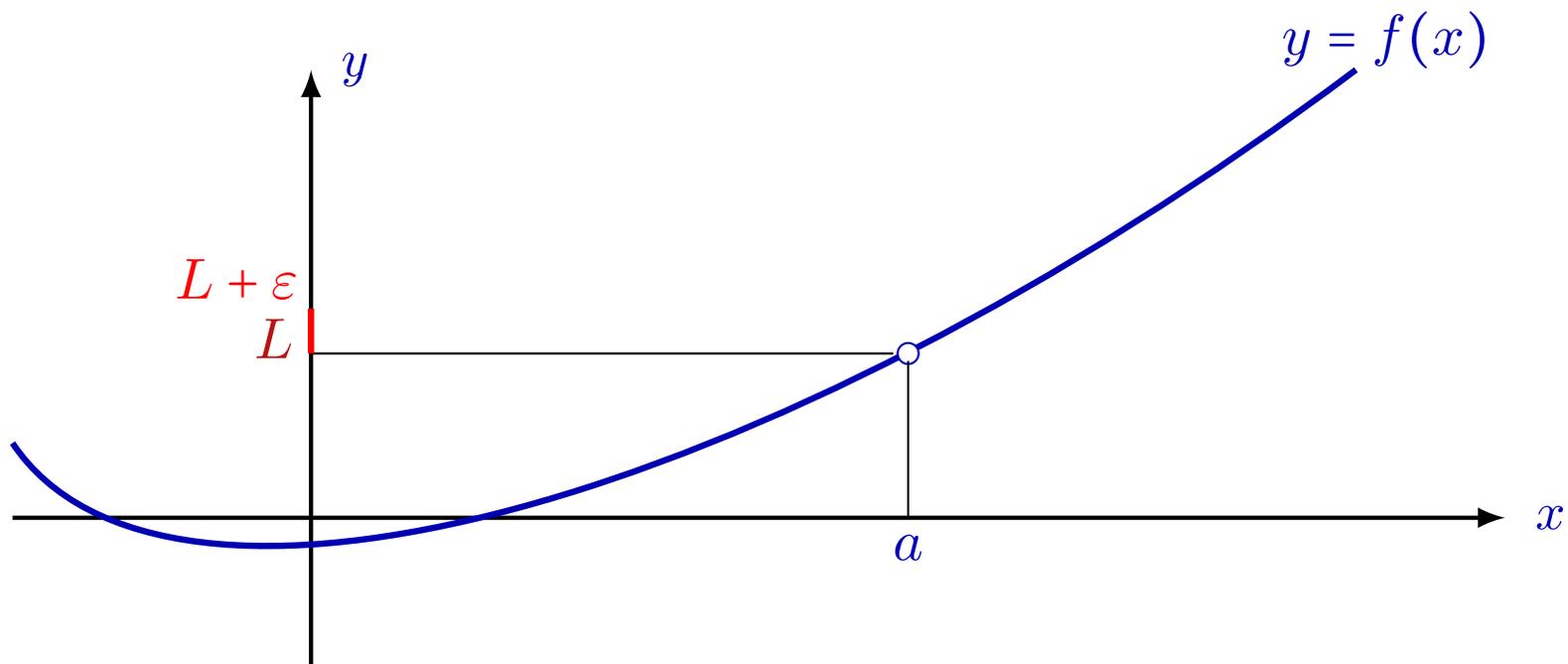
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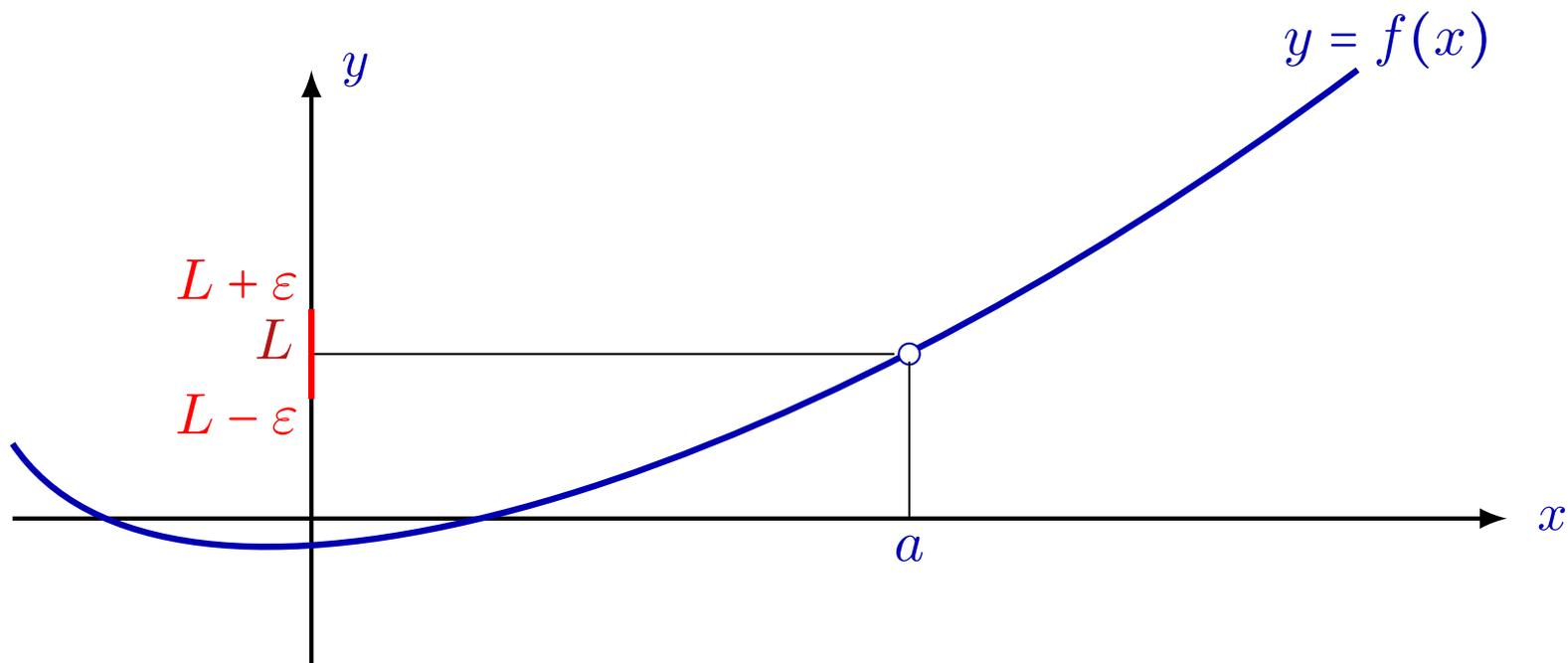
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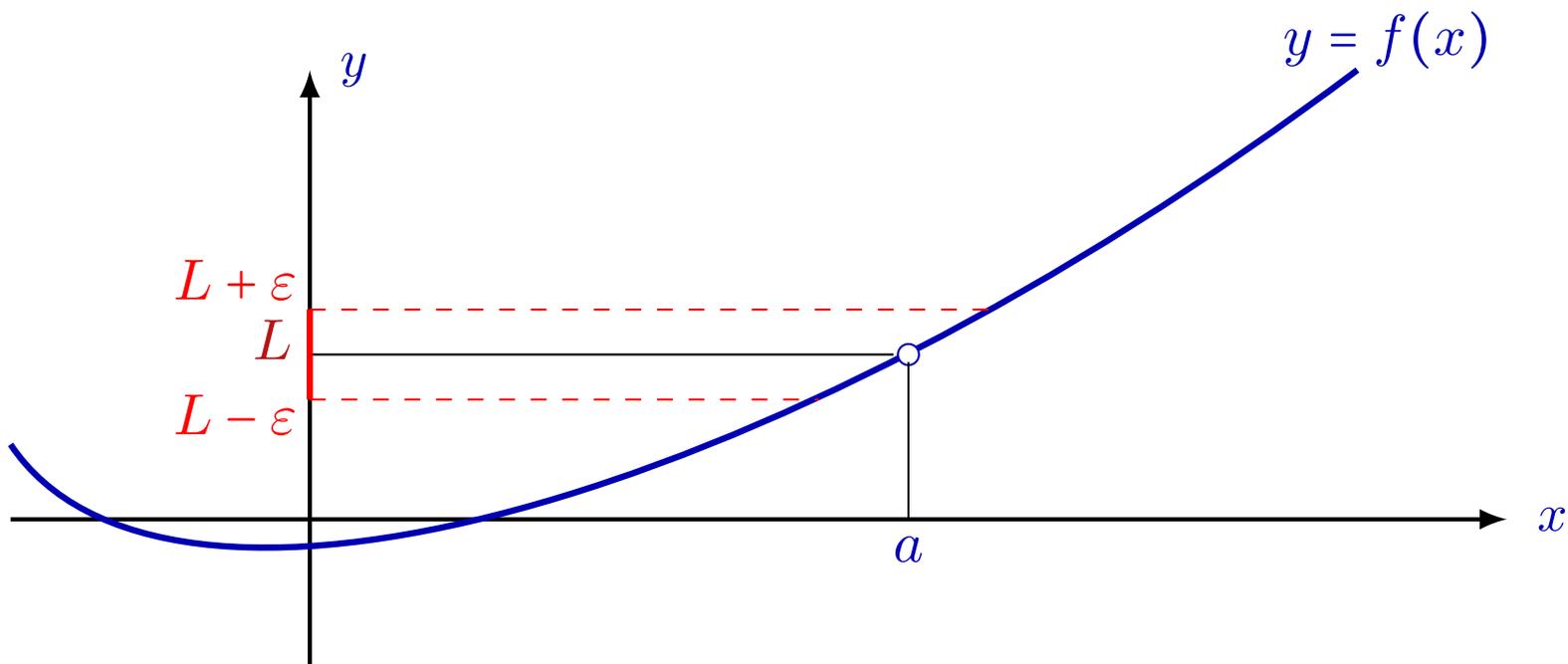
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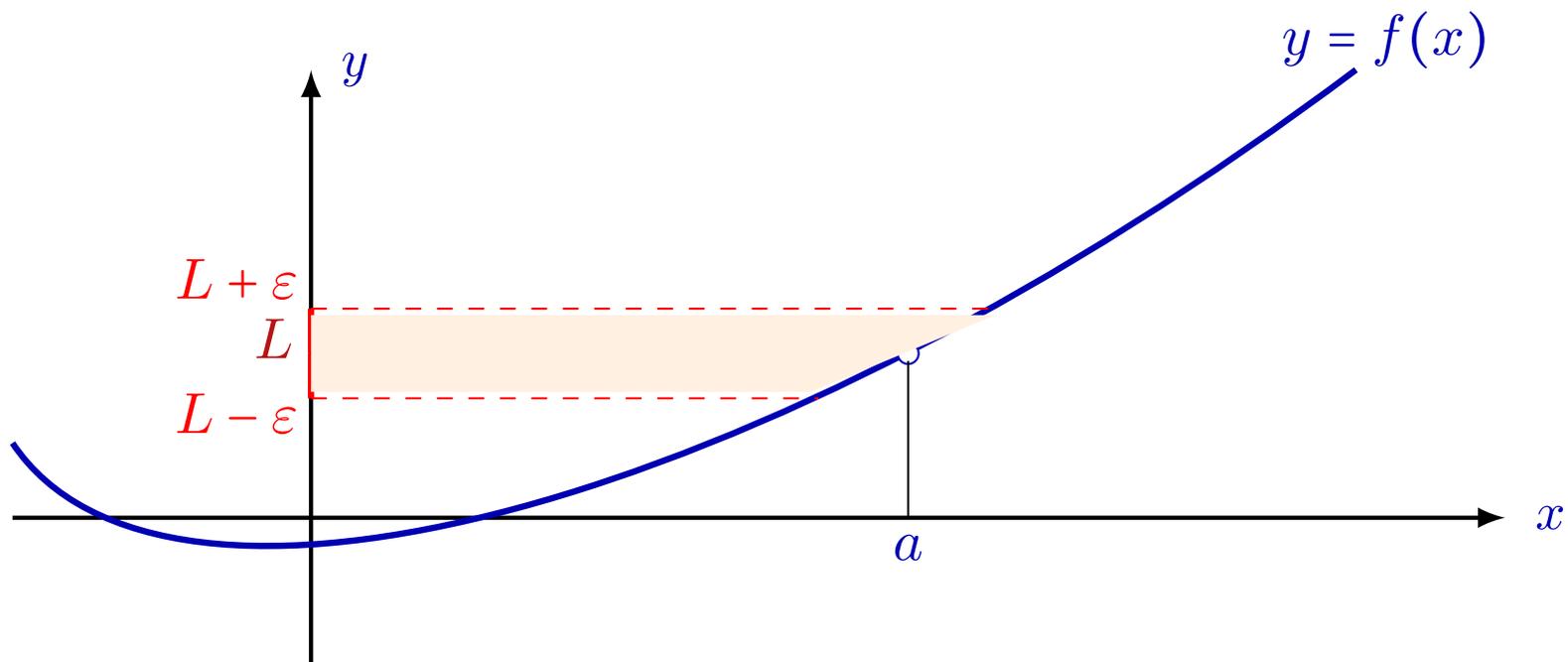
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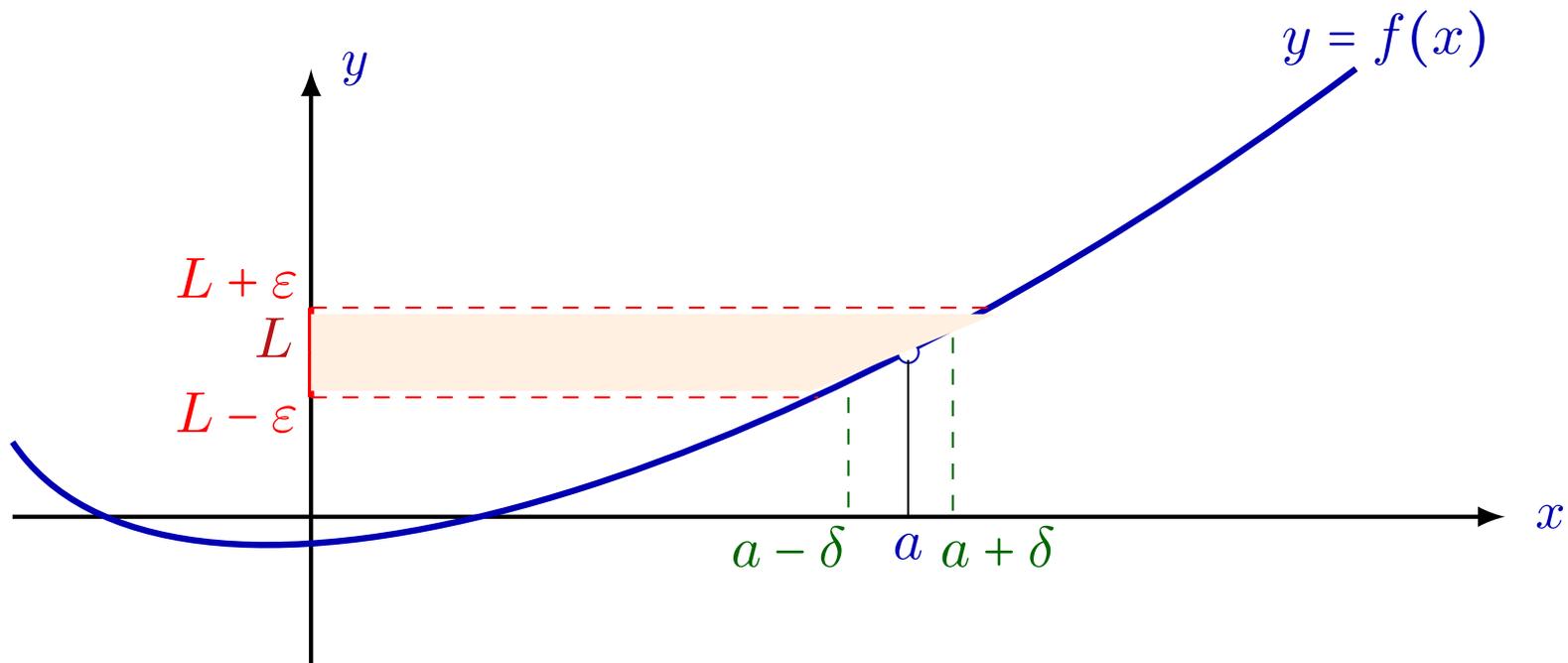
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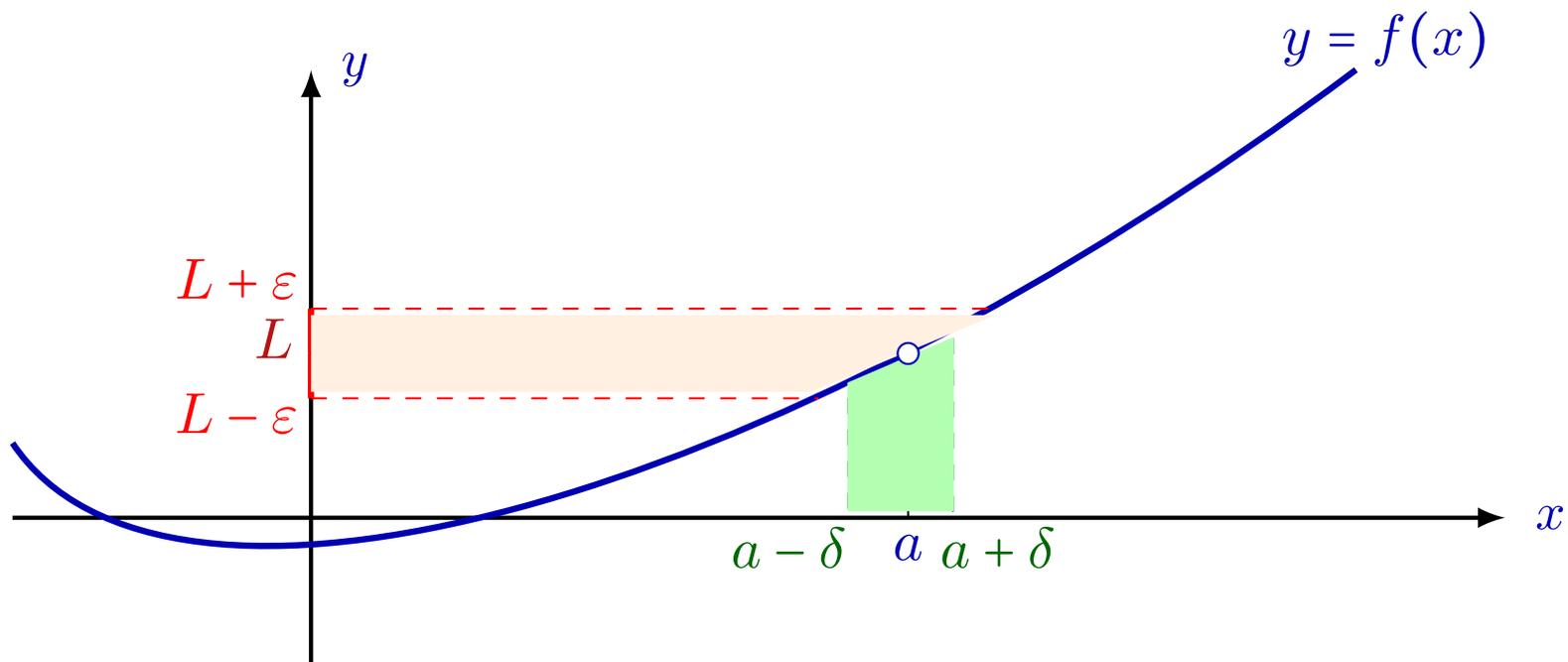
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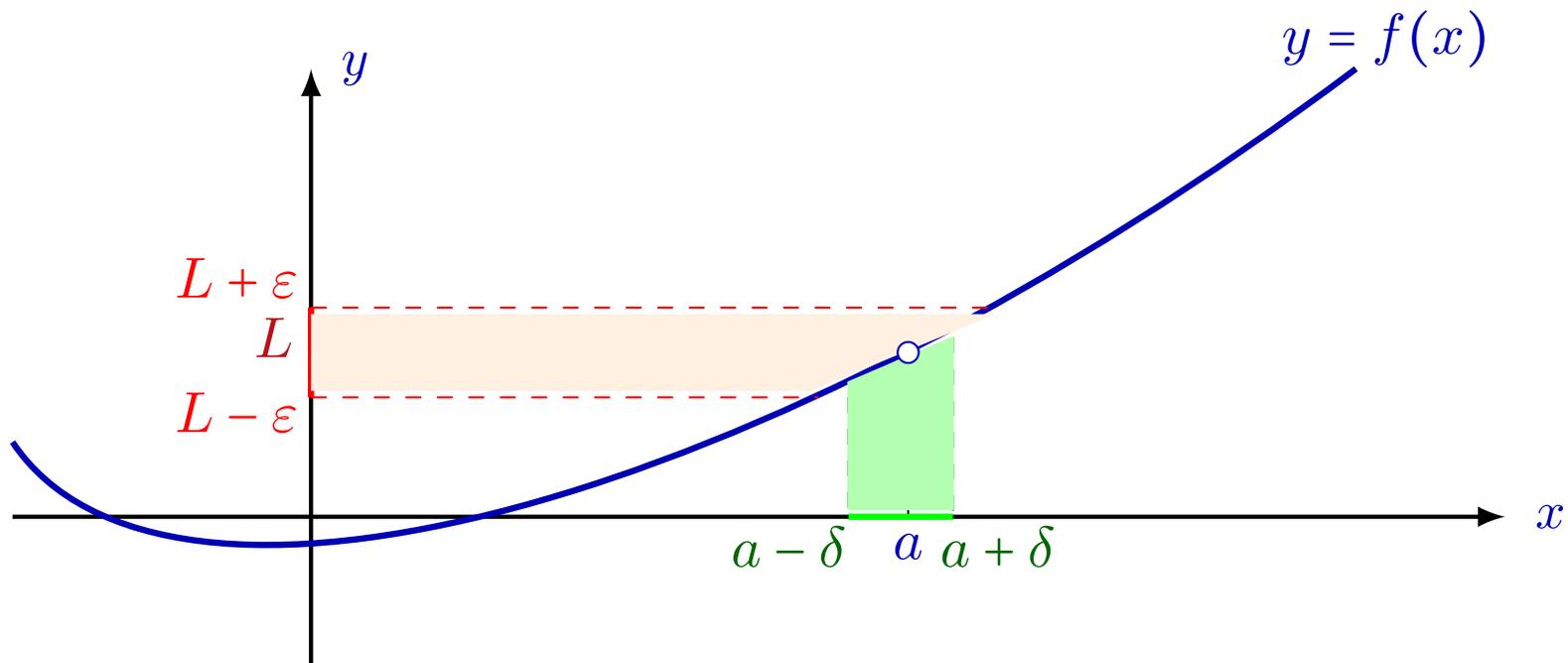
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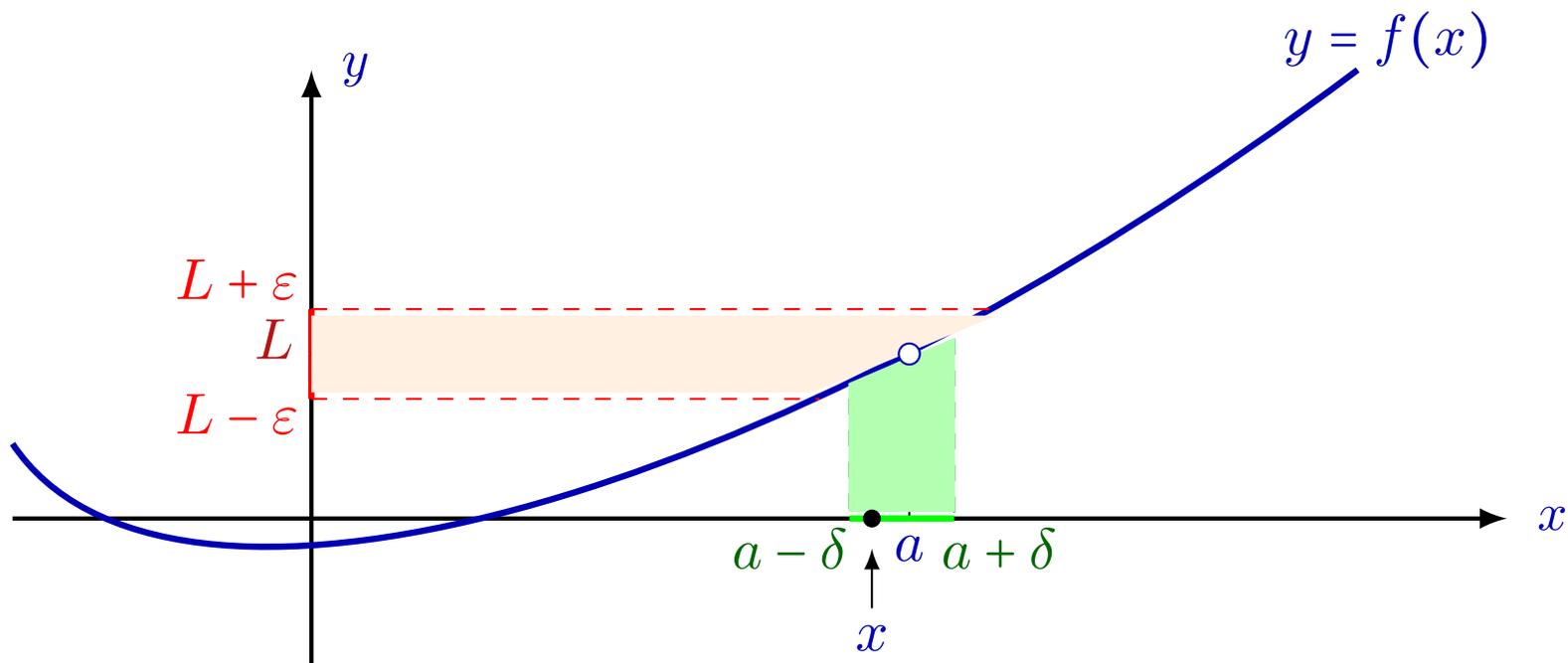
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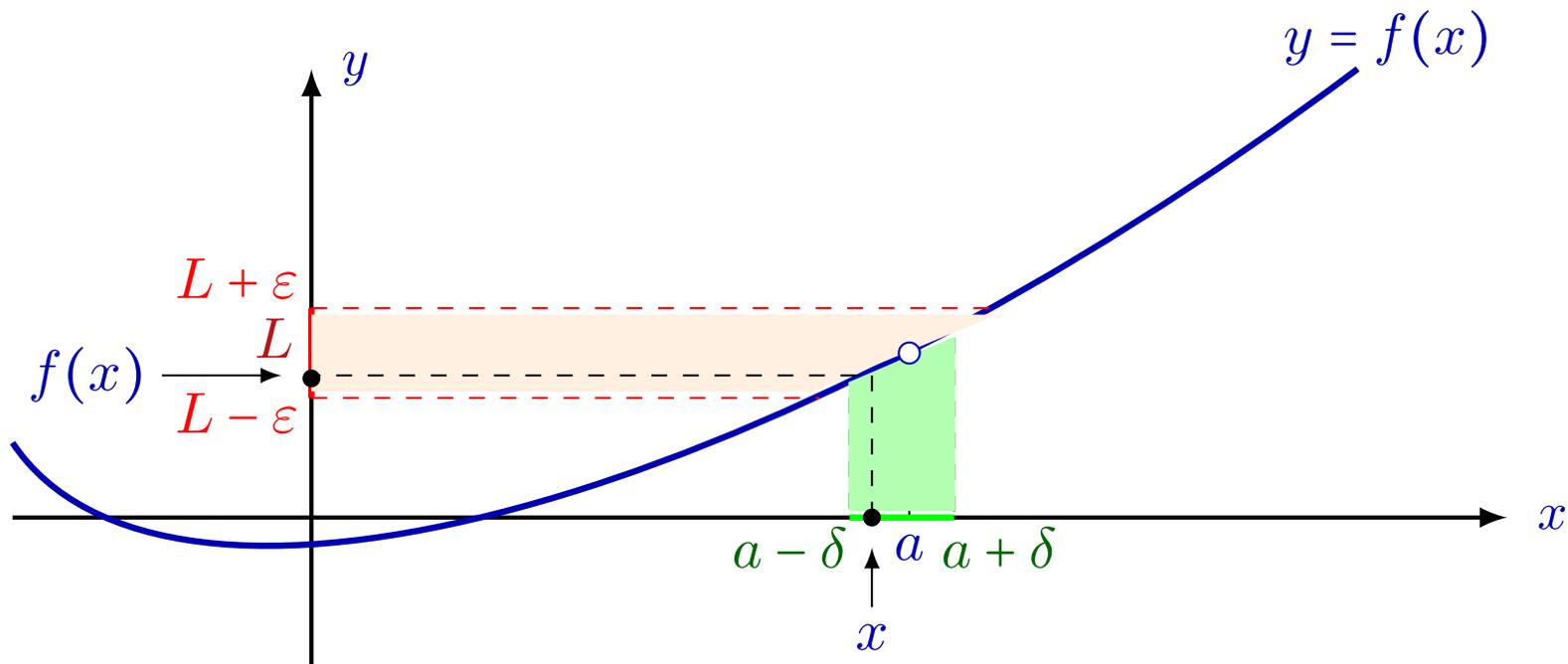
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For any  $x$  such that  $x \in (a - \delta, a + \delta)$ , we have  $f(x) \in (L - \varepsilon, L + \varepsilon)$ .



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$$|(2x + 1) - 7| < \varepsilon \iff |2x - 6| < \varepsilon \iff 2|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{2}.$$

Therefore, the condition  $|x - 3| < \frac{\varepsilon}{2}$  will ensure the inequality  $|(2x + 1) - 7| < \varepsilon$ .

A number  $\delta$  which we are looking for, is, therefore,  $\delta = \frac{\varepsilon}{2}$ .



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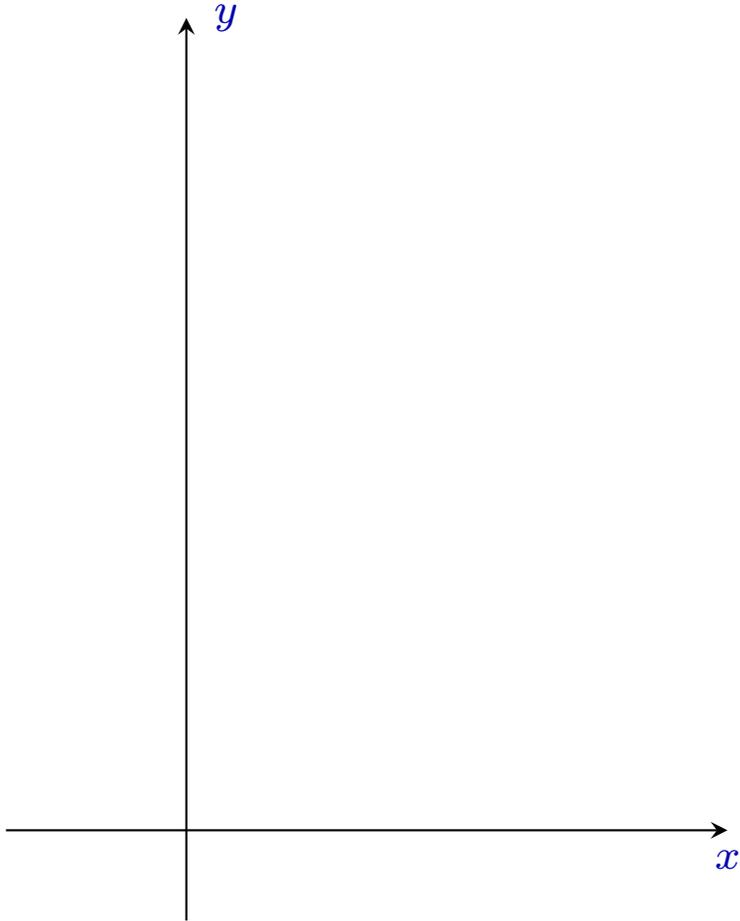
Therefore, by definition of limit,  $\lim_{x \rightarrow 3} (2x + 1) = 7$ .



Let us illustrate our proof of  $\lim_{x \rightarrow 3} (2x + 1) = 7$  geometrically.

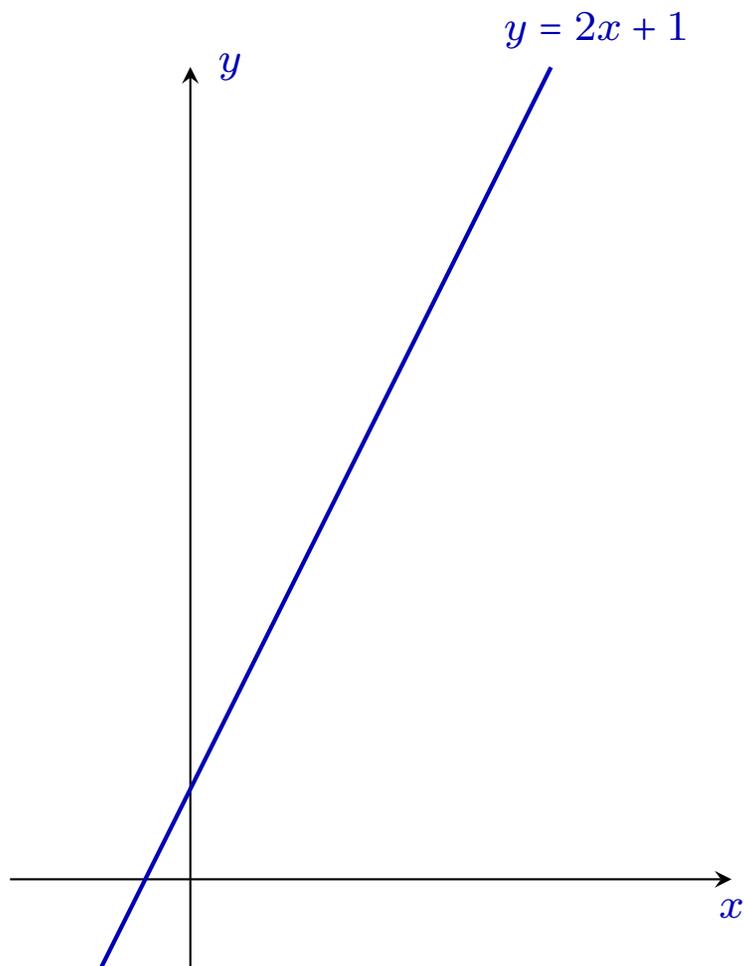
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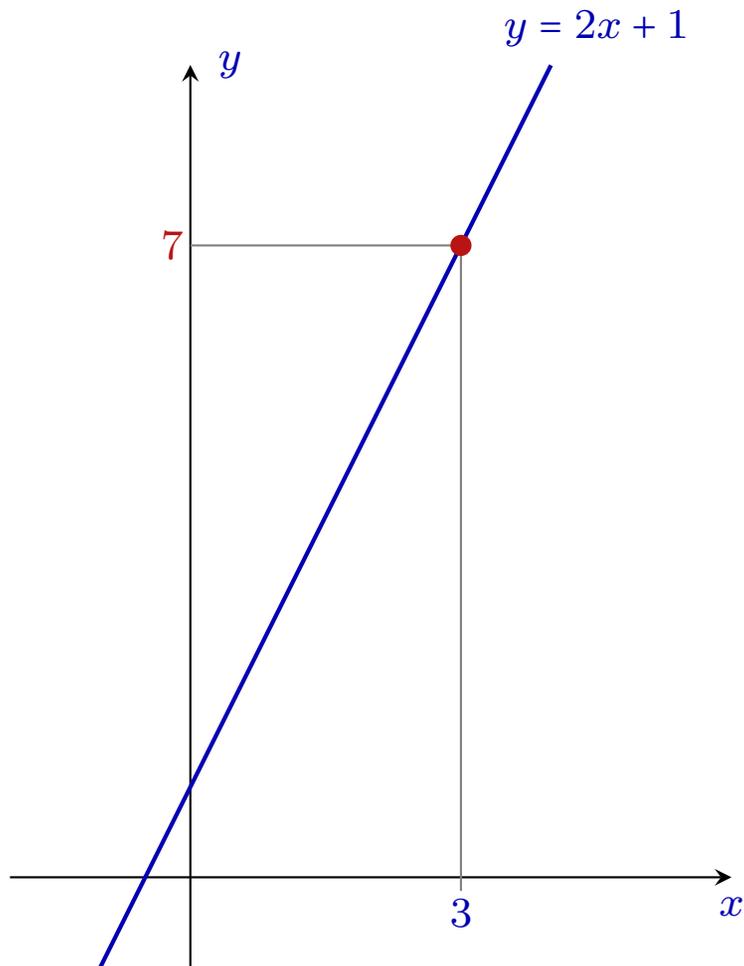
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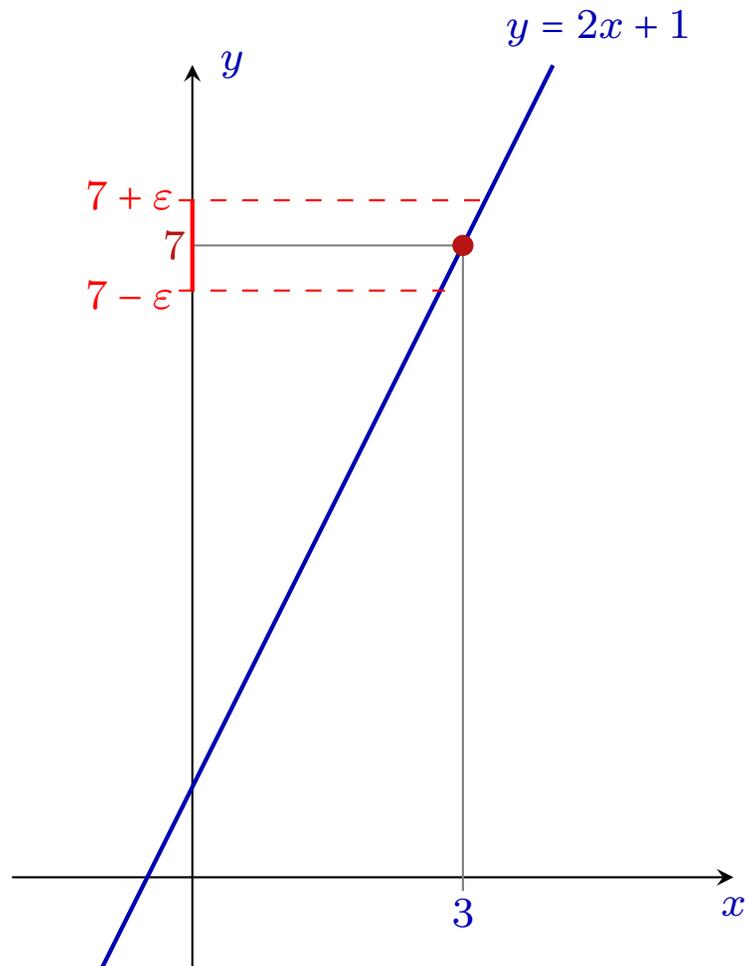
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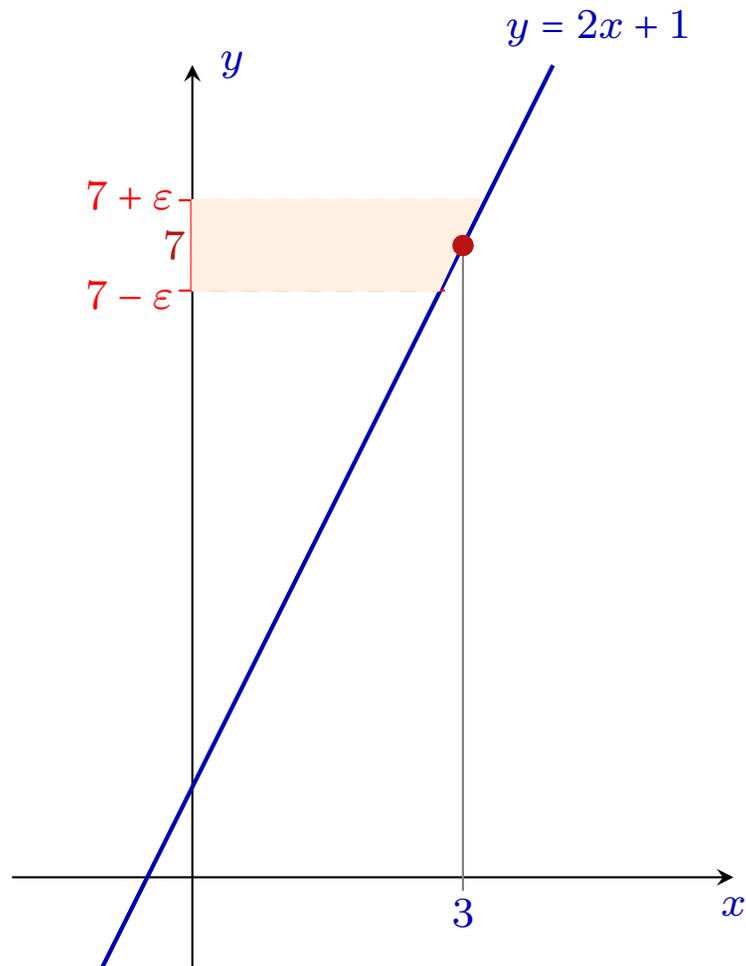
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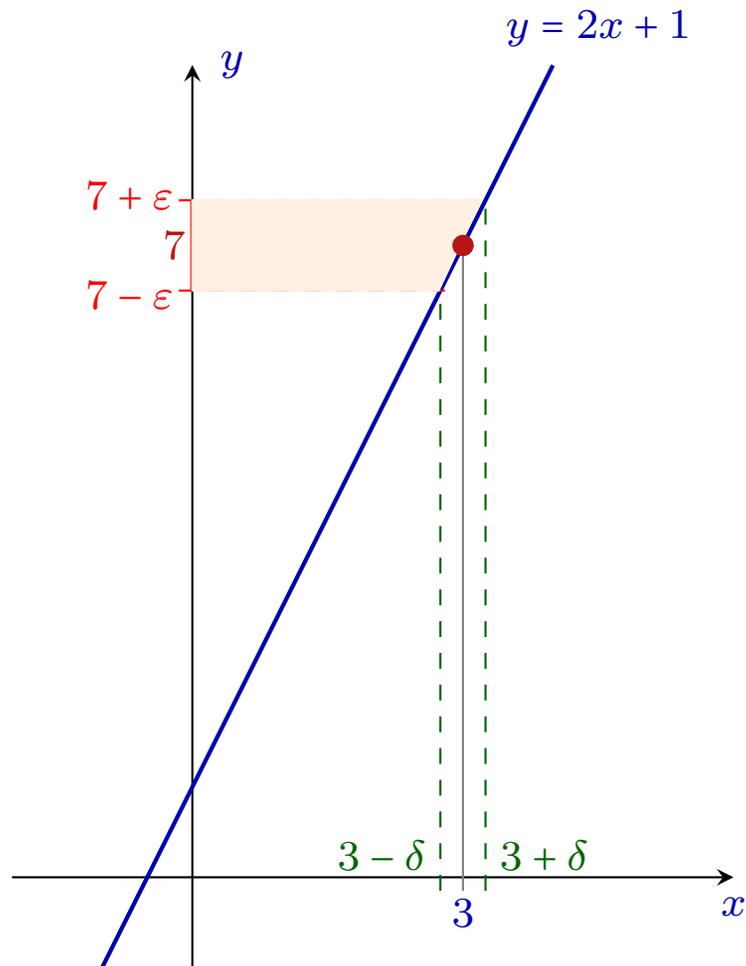
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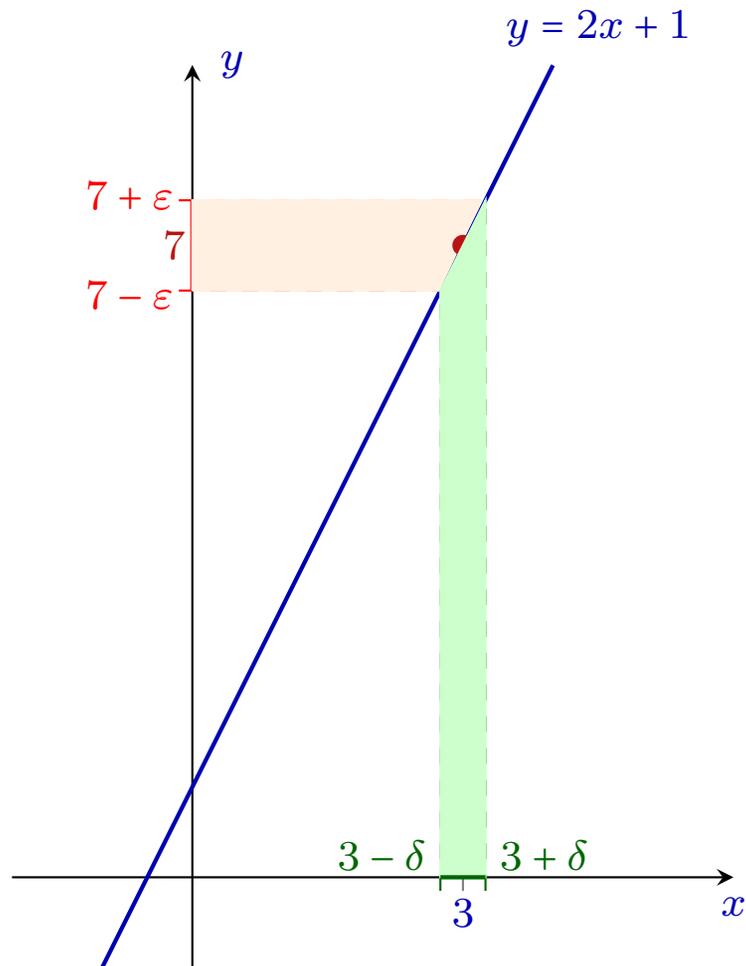
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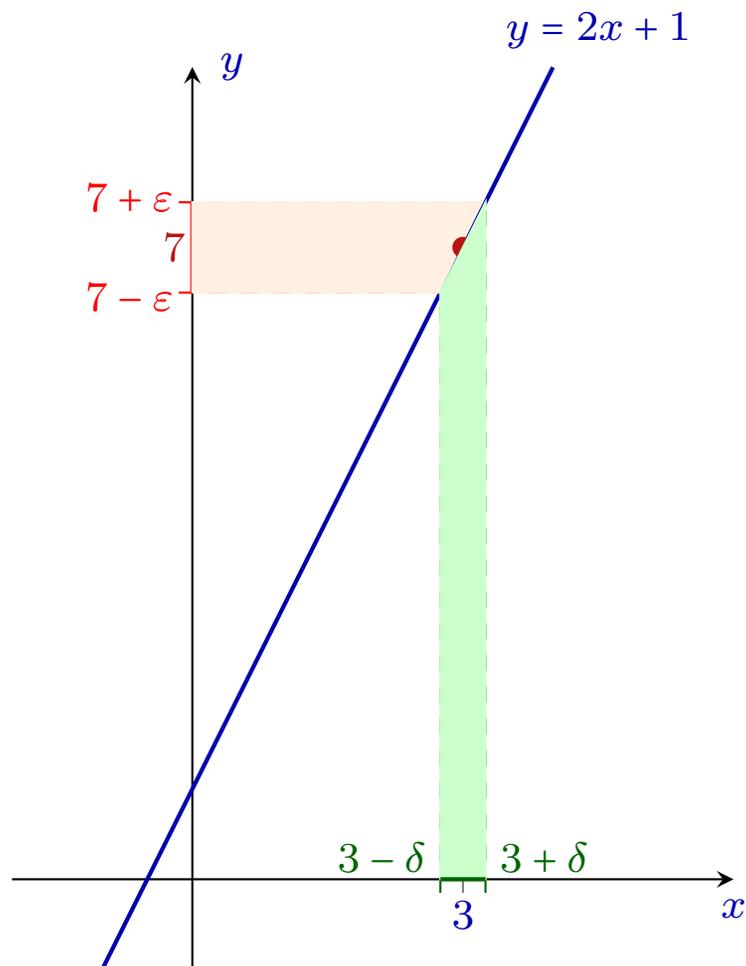
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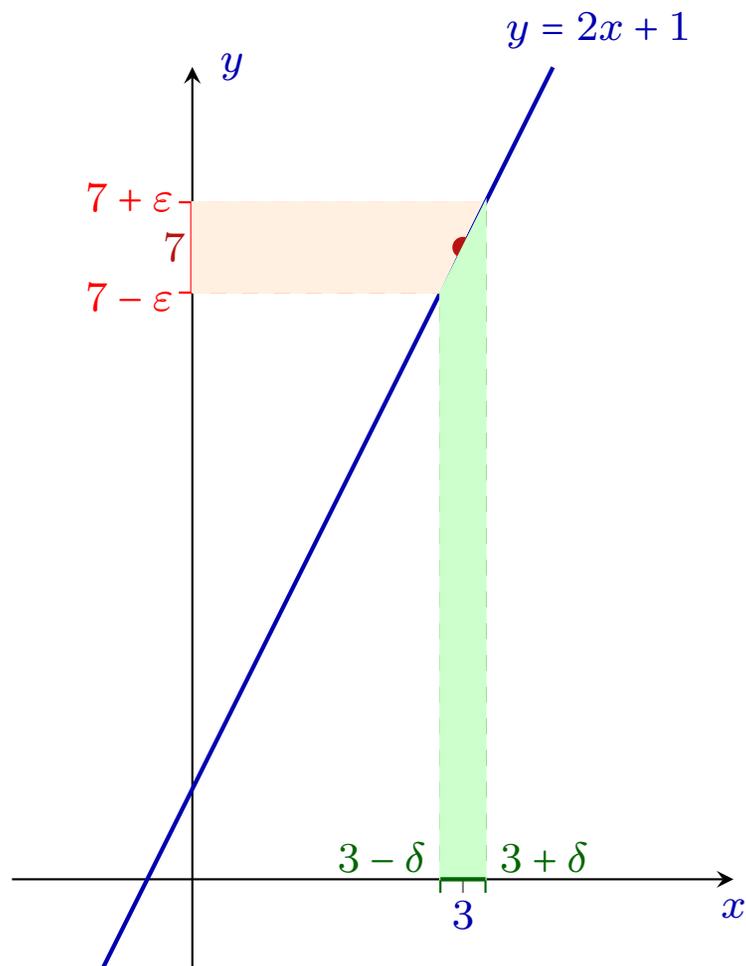


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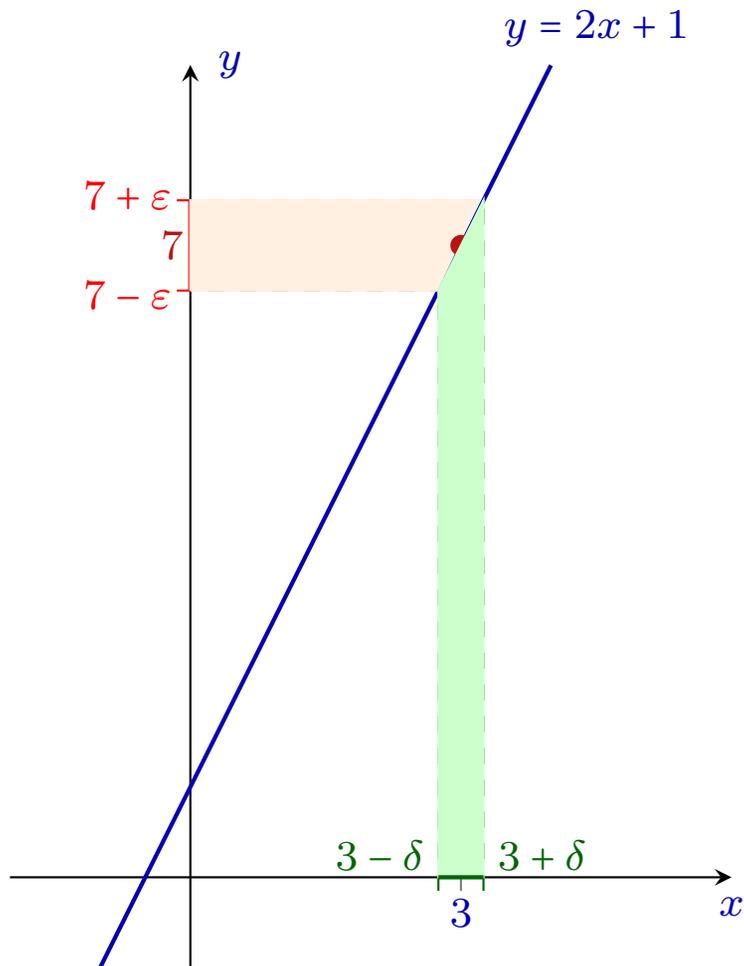


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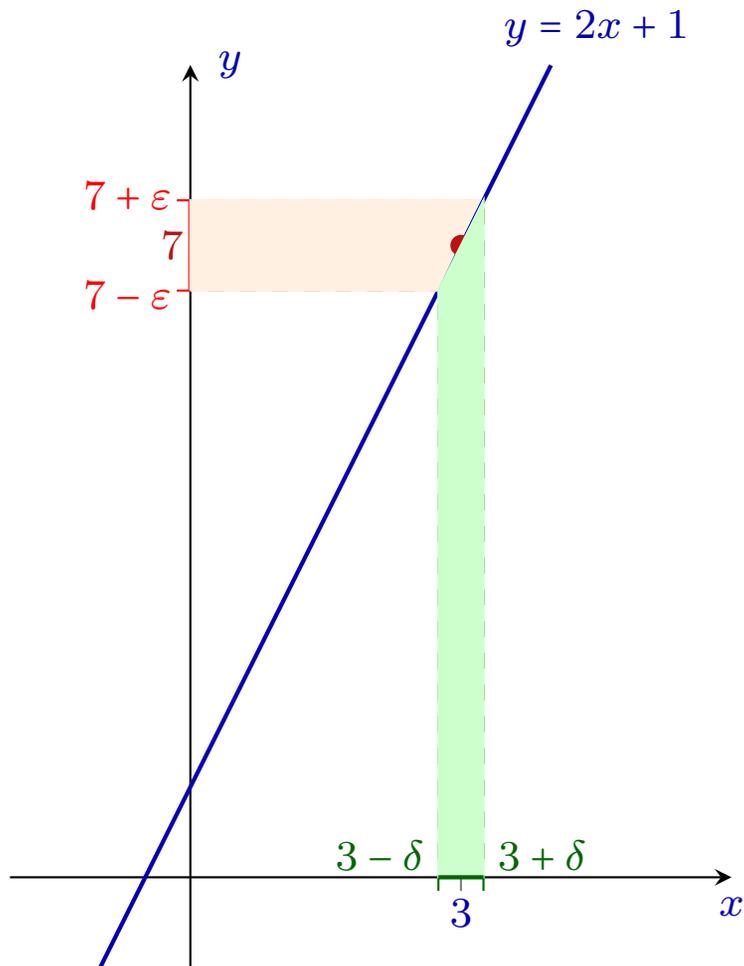
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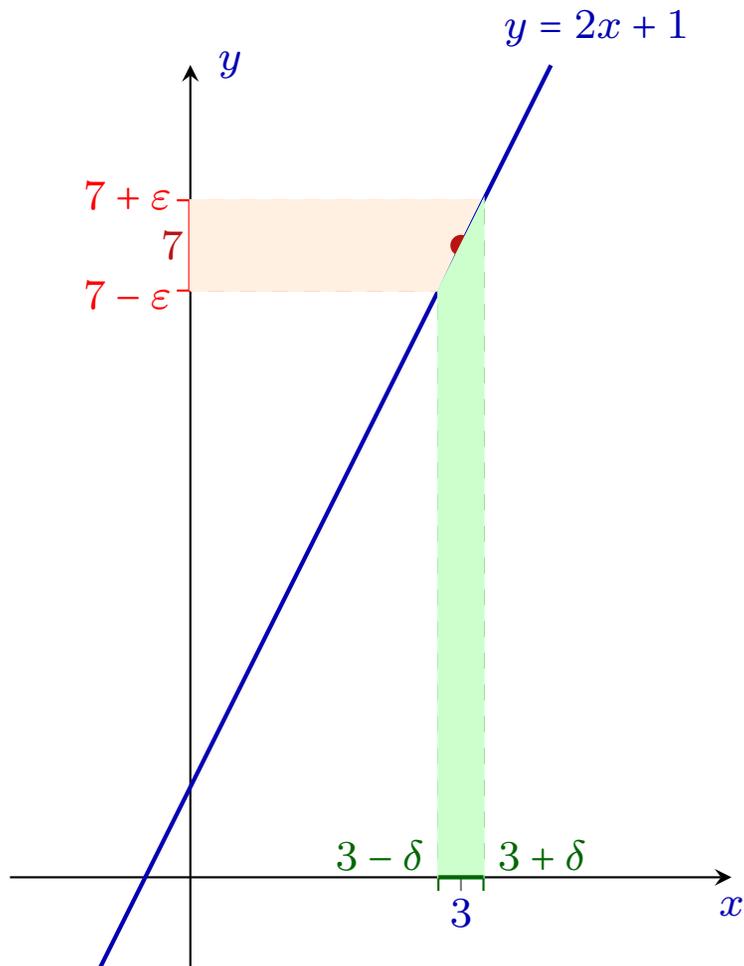
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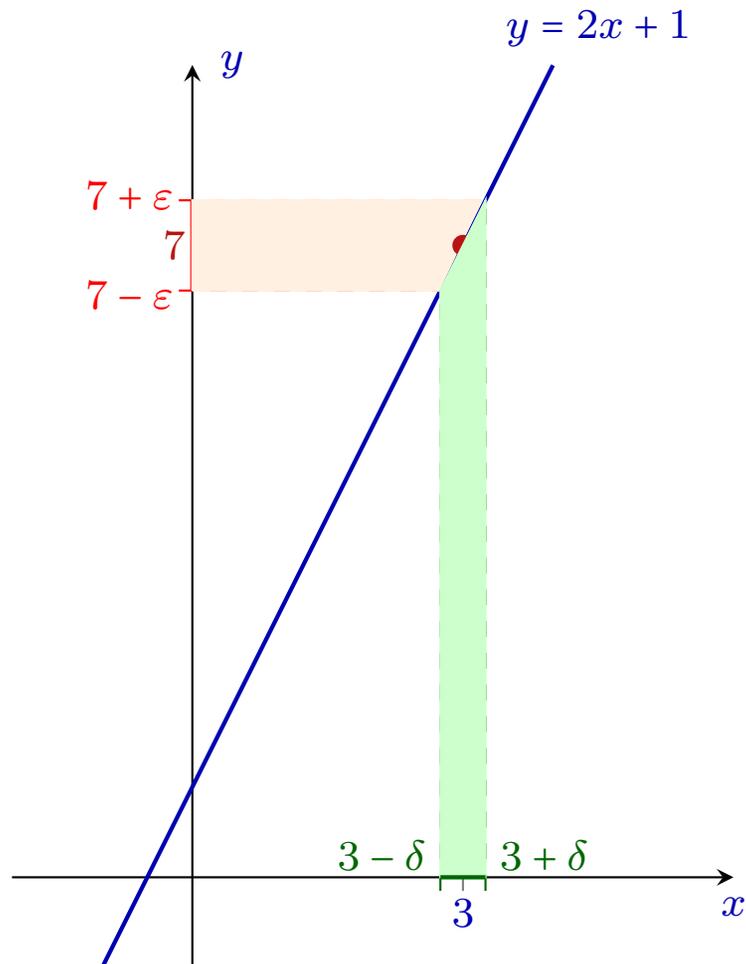
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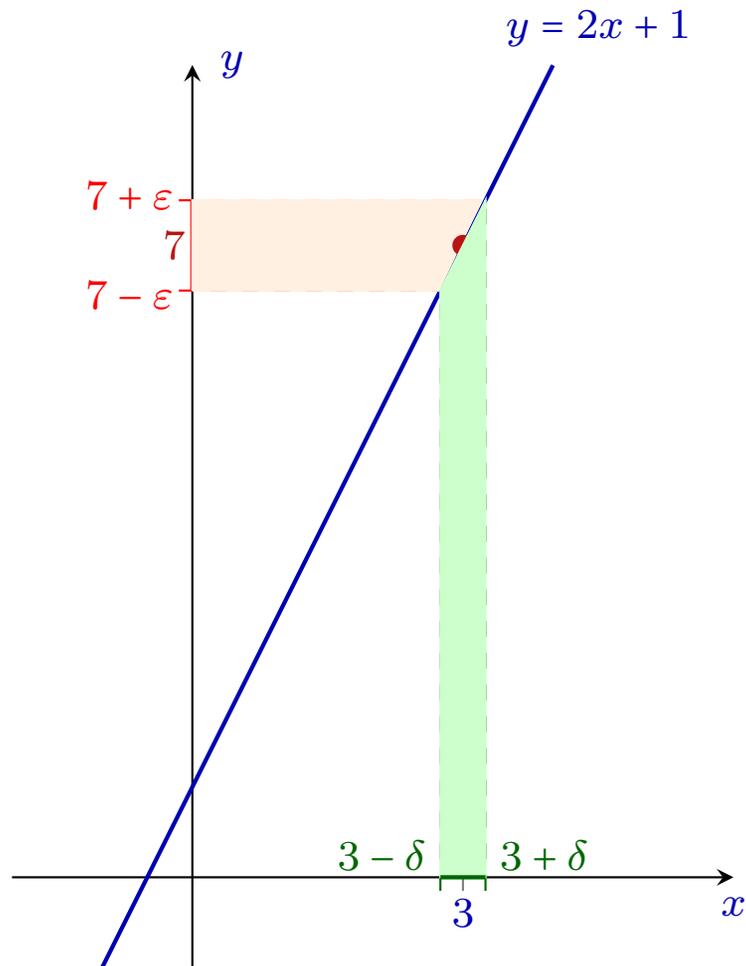
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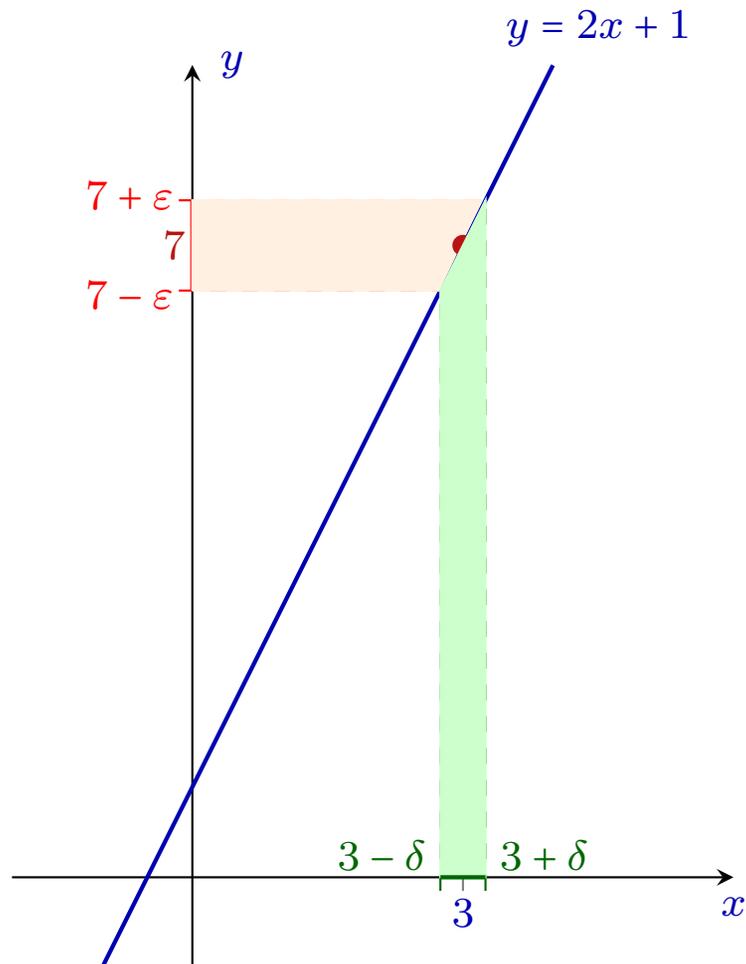
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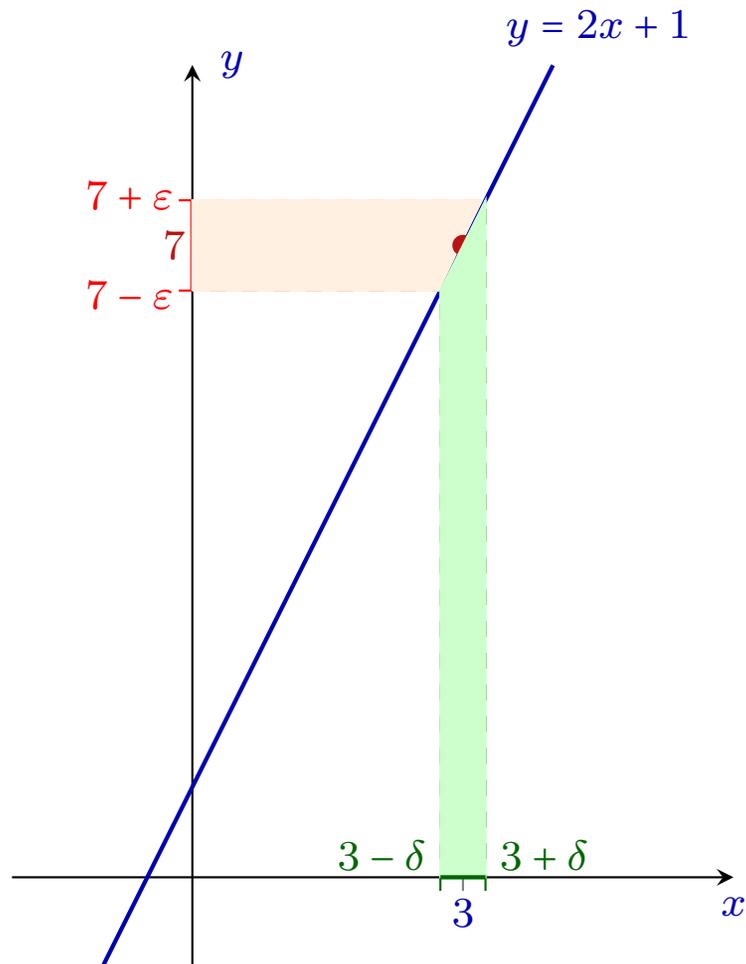
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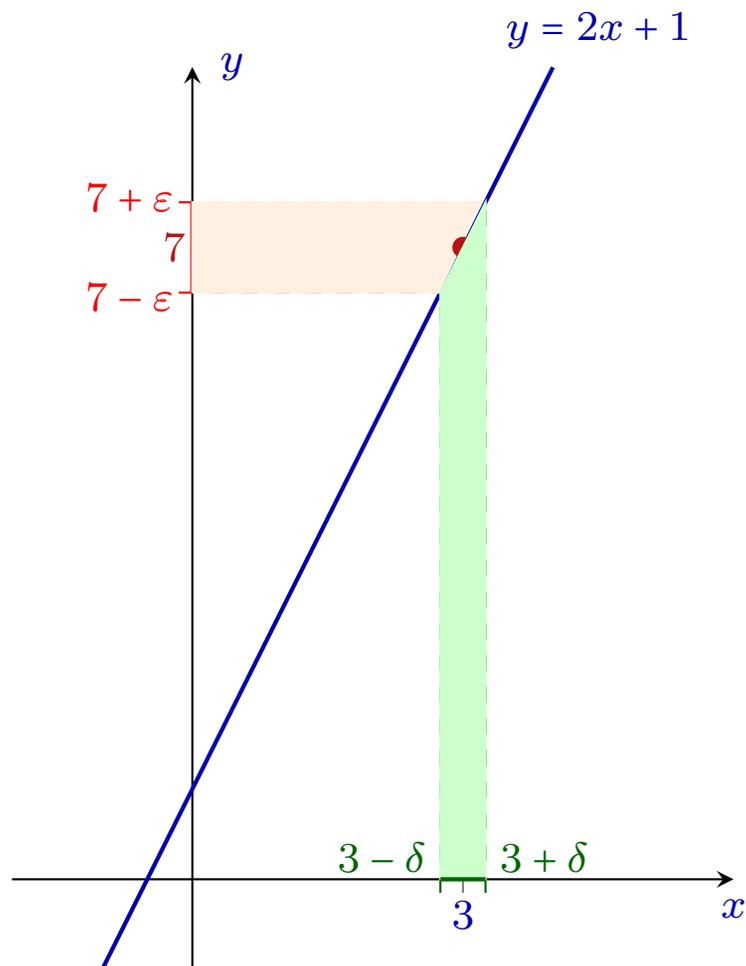
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It means, by definition of limit,

$$7 = \lim_{x \rightarrow 3} f(x), \text{ where } f(x) = 2x + 1.$$

# Working with the definition of limit

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**Exercise.** Use the definition of limit to prove that  $\lim_{x \rightarrow 0} \left( \sin \frac{1}{x} \right) \neq 0$ .