

## Lecture 14

### ▷ Cardinalities

Finite, countable  
and uncountable

$\mathbb{R}$  is uncountable  
Proof of Cantor  
theorem

The set of irrational  
numbers is  
uncountable

The interval  
 $(-1, 1)$  is  
uncountable

Any interval  $(a, b)$   
is uncountable

Applications of  
Cantor-Schröder-  
Bernstein theorem

Cardinality of power  
set

Huge sets

$$c \leq 2^{\aleph_0}$$

$$2^{\aleph_0} \leq c$$

$$c^2 = c$$

Constructing new  
categories

Classification of  
maps

Category of  
endomorphisms

Categorical  
generalizations of  
injective and  
surjective

# Lecture 14

# Cardinalities

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

# Finite, countable and uncountable

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

So far we had many examples of infinite sets of cardinality  $\aleph_0$ ,

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

So far we had many examples of infinite sets of cardinality  $\aleph_0$ ,  
that is **denumerable** sets

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

So far we had many examples of infinite sets of cardinality  $\aleph_0$ ,  
but **no other infinite** cardinalities have **appeared**.

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

So far we had many examples of infinite sets of cardinality  $\aleph_0$ ,  
but **no other infinite** cardinalities have **appeared**.

Are there uncountable sets?

A set  $X$  is called **finite** if  $|X| < \aleph_0$ .

A set  $X$  is called **countable** if  $|X| \leq \aleph_0$ .

Any finite set is countable.

A set  $X$  is called **denumerable** if  $|X| = \aleph_0$ .

A set  $X$  is called **uncountable** if it is not countable.

So far we had many examples of infinite sets of cardinality  $\aleph_0$ ,  
but **no other infinite** cardinalities have **appeared**.

Are there uncountable sets?

Are there infinite cardinalities beyond  $\aleph_0$ ?

# $\mathbb{R}$ is uncountable

**Theorem** (Cantor).  $\mathbb{R}$  is uncountable.

**Theorem** (Cantor).  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$$a = a_0.a_1a_2a_3\dots \quad \text{for} \quad a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots ,$$

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's,

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ ,

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ , that is  $xy9999\dots$  equals  $x(y + 1)0000\dots$ .

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ , that is  $xy9999\dots$  equals  $x(y + 1)0000\dots$ .

For example,  $33.14999\dots = 33.15000\dots$

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ , that is  $xy9999\dots$  equals  $x(y + 1)0000\dots$ .

For example,  $33.14999\dots = 33.15000\dots$

**Lemma.** If a real number admits two different decimal presentations, then these presentations can be obtained from each other as above.

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ , that is  $xy9999\dots$  equals  $x(y + 1)0000\dots$ .

For example,  $33.14999\dots = 33.15000\dots$

**Lemma.** If a real number admits two different decimal presentations, then these presentations can be obtained from each other as above.

In what follows we will not use decimal presentations of real numbers which end with infinitely many 9's.

**Theorem (Cantor).**  $\mathbb{R}$  is uncountable.

**Digression.** In the proof we will use **decimal presentations** of real numbers:

$a = a_0.a_1a_2a_3\dots$  for  $a = a_0 + a_1/10 + a_2/10^2 + a_3/10^3 + \dots$ , where  $a_0$  is the greatest integer, which is  $\leq a$ , and  $a_i \in \{0, 1, 2, \dots, 9\}$  for any  $i > 0$ .

A decimal presentation may be **ambiguous**:

a decimal presentation, which ends with infinitely many 9's, defines the same number as the decimal presentation,

in which these 9's are replaced by 0's and the last digit  $x \neq 9$  is replaced by the next digit  $x + 1$ , that is  $xy9999\dots$  equals  $x(y + 1)0000\dots$ .

For example,  $33.14999\dots = 33.15000\dots$

**Lemma.** If a real number admits two different decimal presentations, then these presentations can be obtained from each other as above.

In what follows we will not use decimal presentations of real numbers

which end with infinitely many 9's.

Due to Lemma, such presentation of a real number is unique.

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ .

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} . a_{11} a_{12} a_{13} \dots$$

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} \cdot a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} \cdot a_{21} a_{22} a_{23} \dots$$

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} \cdot a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} \cdot a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30} \cdot a_{31} a_{32} a_{33} \dots$$

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} \cdot a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} \cdot a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30} \cdot a_{31} a_{32} a_{33} \dots$$

.....

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10}. a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20}. a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30}. a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0. b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ ,  $\dots$

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} . a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} . a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30} . a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0 . b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ ,  $\dots$   
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} . a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} . a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30} . a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0 . b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10} . a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20} . a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30} . a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0 . b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

Hence,  $|\mathbb{N}| < |\mathbb{R}|$ .

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10}.a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20}.a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30}.a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0.b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

Hence,  $|\mathbb{N}| < |\mathbb{R}|$ . In particular,  $\mathbb{R} \not\approx \mathbb{N}$ .

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10}.a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20}.a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30}.a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0.b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

Hence,  $|\mathbb{N}| < |\mathbb{R}|$ . In particular,  $\mathbb{R} \not\approx \mathbb{N}$ .

**Notation:**

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10}.a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20}.a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30}.a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0.b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

Hence,  $|\mathbb{N}| < |\mathbb{R}|$ . In particular,  $\mathbb{R} \not\approx \mathbb{N}$ .

**Notation:**  $|\mathbb{R}| = \mathbf{c}$  (**continuum**).

# Proof of Cantor theorem

Let us show that there is no surjection  $\mathbb{N} \rightarrow \mathbb{R}$ .

Assume the contrary: there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Write all the images in their **decimal** presentations, which do not end with sequence of 9's:

$$f(1) = a_{10}.a_{11} a_{12} a_{13} \dots$$

$$f(2) = a_{20}.a_{21} a_{22} a_{23} \dots$$

$$f(3) = a_{30}.a_{31} a_{32} a_{33} \dots$$

.....

Consider a real number  $b = b_0.b_1 b_2 b_3 \dots$   
with  $b_0 \neq a_{10}$ ,  $b_1 \neq a_{21}$ ,  $b_3 \neq a_{32}$ , ...  
making sure that none of  $b_i$  is 9.

Obviously,  $b \notin \text{Im } f$ .

So,  $f$  is not a **surjection**.

Therefore, the assumption that there exists a surjection  $f : \mathbb{N} \rightarrow \mathbb{R}$

is wrong, □

Hence,  $|\mathbb{N}| < |\mathbb{R}|$ . In particular,  $\mathbb{R} \not\approx \mathbb{N}$ .

**Notation:**  $|\mathbb{R}| = \mathbf{c}$  (**continuum**). Since  $|\mathbb{N}| < |\mathbb{R}|$ ,  $\aleph_0 < \mathbf{c}$ .

# The set of irrational numbers is uncountable

---

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.**

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.** Assume, to the contrary, that  $\mathbb{I}$  is countable.

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.** Assume, to the contrary, that  $\mathbb{I}$  is countable.

Then  $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.** Assume, to the contrary, that  $\mathbb{I}$  is countable.

Then  $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$  is countable as a union of two countable sets.

# The set of irrational numbers is uncountable

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.** Assume, to the contrary, that  $\mathbb{I}$  is countable.

Then  $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$  is countable as a union of two countable sets.

This contradicts to the fact that  $\mathbb{R}$  is uncountable.

**Theorem.** *The set  $\mathbb{I}$  of irrational numbers is uncountable.*

**Proof.** Assume, to the contrary, that  $\mathbb{I}$  is countable.

Then  $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$  is countable as a union of two countable sets.

This contradicts to the fact that  $\mathbb{R}$  is uncountable.

Therefore,  $\mathbb{I}$  is uncountable. □

# The interval $(-1, 1)$ is uncountable

---

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.**

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$x \mapsto \frac{x}{1 - x^2}$$

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$x \mapsto \frac{x}{1 - x^2} \quad \text{is a bijection.}$$

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

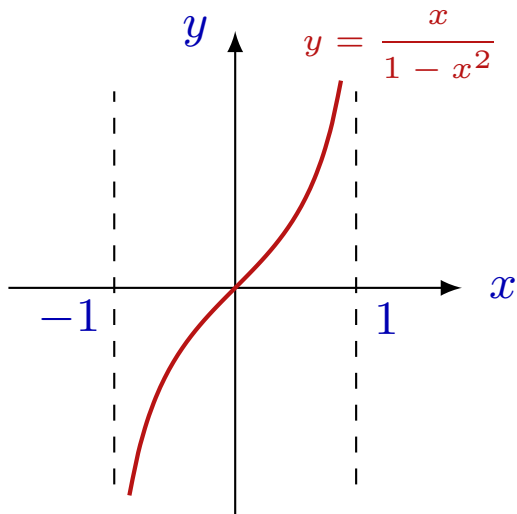
$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$x \mapsto \frac{x}{1 - x^2} \quad \text{is a bijection. Why is } f \text{ a bijection?}$$



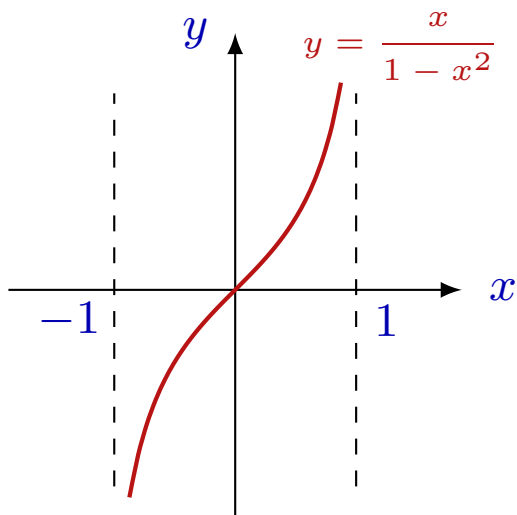
# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$x \mapsto \frac{x}{1 - x^2} \quad \text{is a bijection. Why is } f \text{ a bijection?}$$

$f$  is strictly increasing,



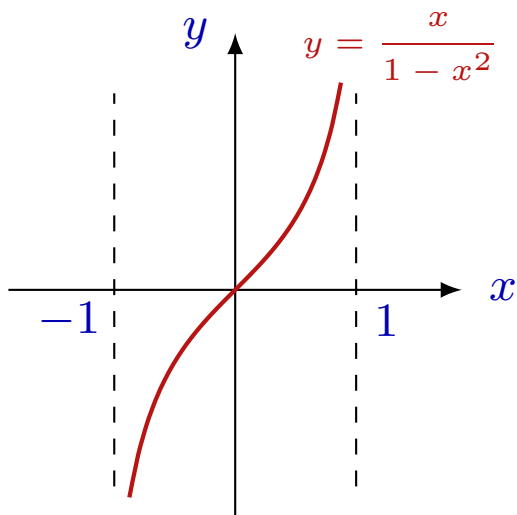
# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$x \mapsto \frac{x}{1 - x^2} \quad \text{is a bijection. Why is } f \text{ a bijection?}$$

$f$  is strictly increasing, so it is injective.



# The interval $(-1, 1)$ is uncountable

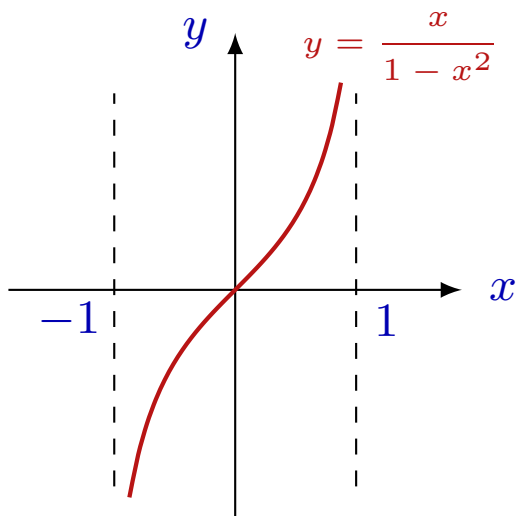
**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty,$$



# The interval $(-1, 1)$ is uncountable

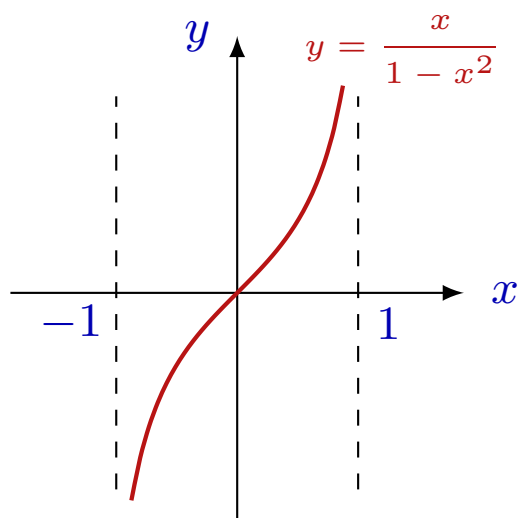
**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$



# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

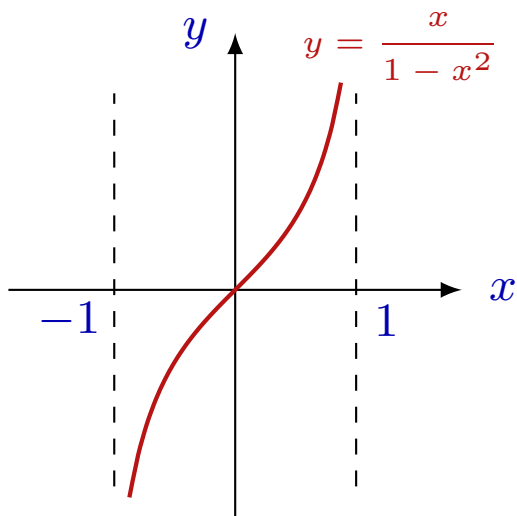
$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$

Hence  $\forall y_0 \in \mathbb{R} \exists a, b \in (-1, 1)$

$$\frac{a}{1-a^2} < y_0 < \frac{b}{1-b^2}.$$



# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

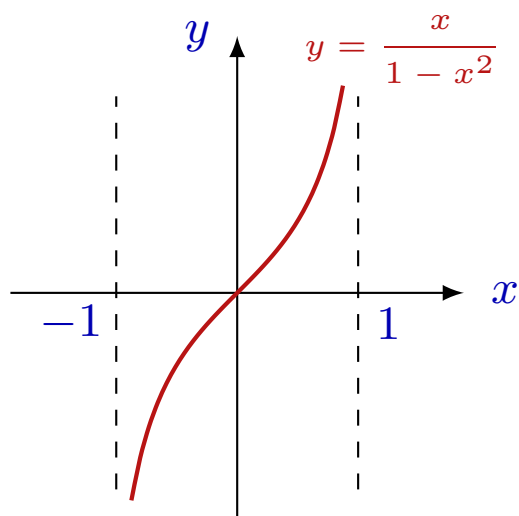
$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$

Hence  $\forall y_0 \in \mathbb{R} \exists a, b \in (-1, 1)$

$$\frac{a}{1-a^2} < y_0 < \frac{b}{1-b^2}.$$

By intermediate value theorem,

$$\exists c \in (a, b) \frac{c}{1-c^2} = y_0.$$



# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$

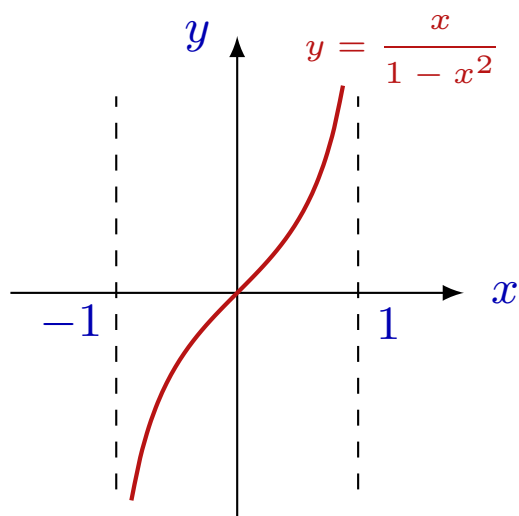
Hence  $\forall y_0 \in \mathbb{R} \exists a, b \in (-1, 1)$

$$\frac{a}{1-a^2} < y_0 < \frac{b}{1-b^2}.$$

By intermediate value theorem,

$$\exists c \in (a, b) \frac{c}{1-c^2} = y_0.$$

Hence  $f$  is surjective.



# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$

Hence  $\forall y_0 \in \mathbb{R} \exists a, b \in (-1, 1)$

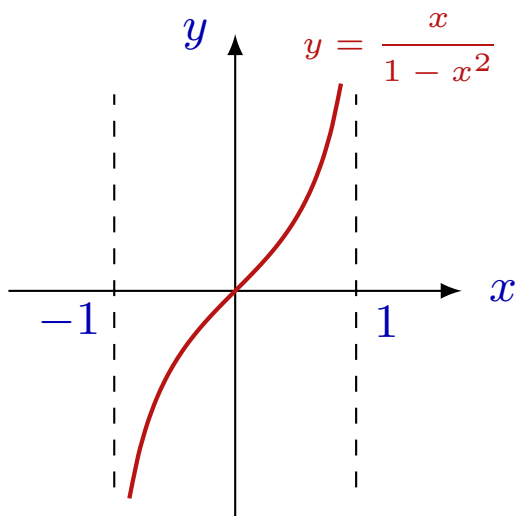
$$\frac{a}{1-a^2} < y_0 < \frac{b}{1-b^2}.$$

By intermediate value theorem,

$$\exists c \in (a, b) \frac{c}{1-c^2} = y_0.$$

Hence  $f$  is surjective.

Therefore,  $(-1, 1) \approx \mathbb{R}$ , and  $|(-1, 1)| = \mathfrak{c}$ .



# The interval $(-1, 1)$ is uncountable

**Theorem.**  $(-1, 1) \approx \mathbb{R}$ .

**Proof.** A map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by

$x \mapsto \frac{x}{1-x^2}$  is a bijection. Why is  $f$  a bijection?

$f$  is strictly increasing, so it is injective.

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x^2} = +\infty, \quad \lim_{x \rightarrow -1^+} \frac{x}{1-x^2} = -\infty$$

Hence  $\forall y_0 \in \mathbb{R} \exists a, b \in (-1, 1)$

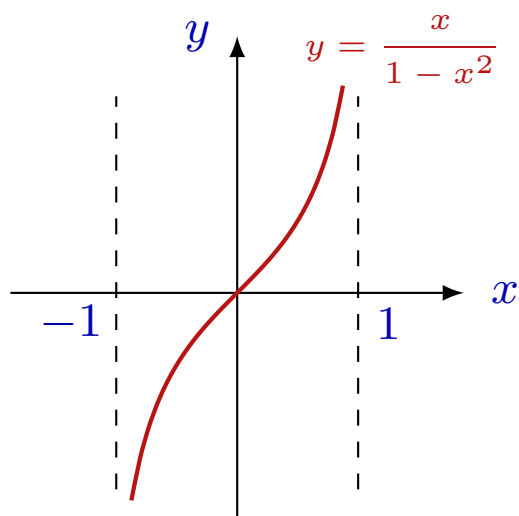
$$\frac{a}{1-a^2} < y_0 < \frac{b}{1-b^2}.$$

By intermediate value theorem,

$$\exists c \in (a, b) \frac{c}{1-c^2} = y_0.$$

Hence  $f$  is surjective.

Therefore,  $(-1, 1) \approx \mathbb{R}$ , and  $|(-1, 1)| = \mathfrak{c}$ . □



# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a}$$

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \quad \text{is a bijection.}$$

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

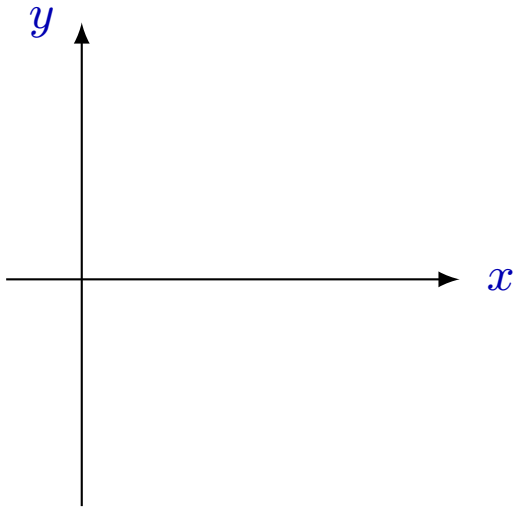
$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \quad \text{is a bijection.}$$

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$

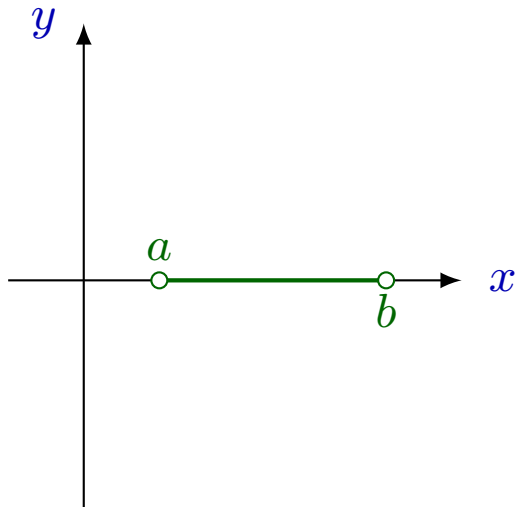


# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$

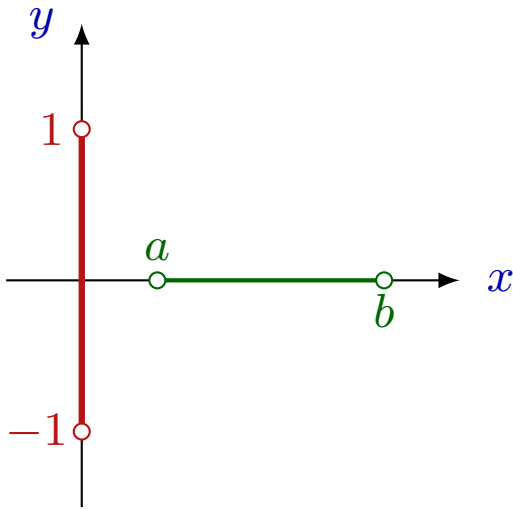


# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$

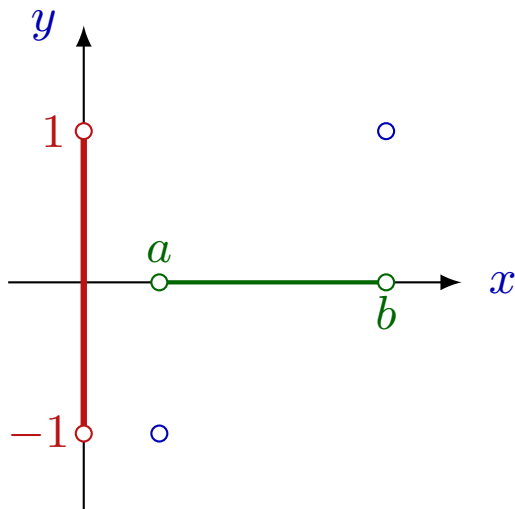


# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$

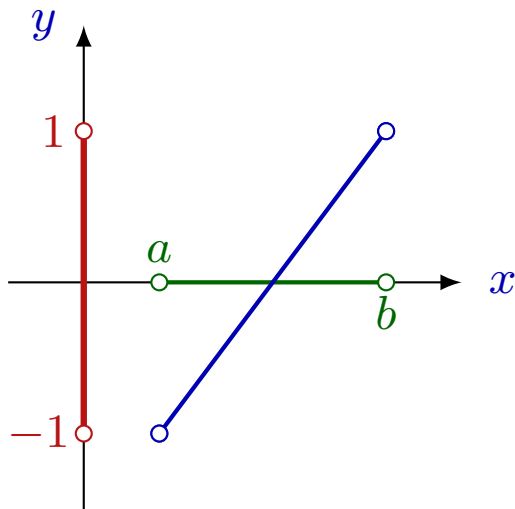


# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$

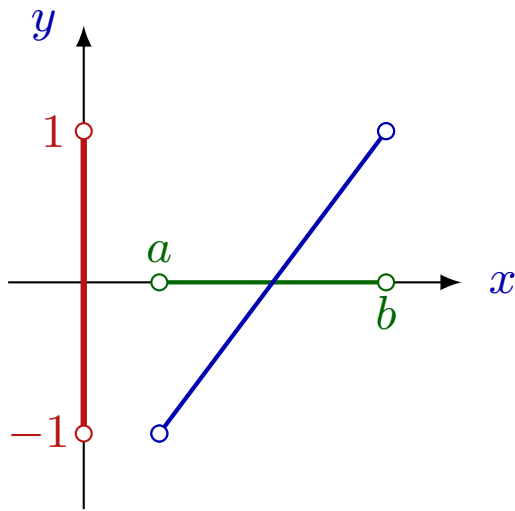


# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$



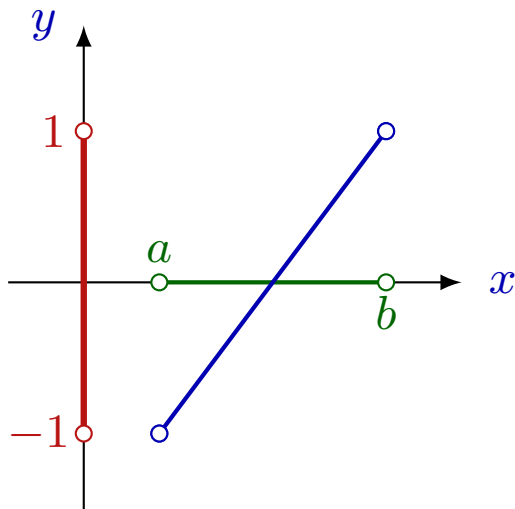
Therefore,  $(a, b) \approx (-1, 1)$ ,

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$



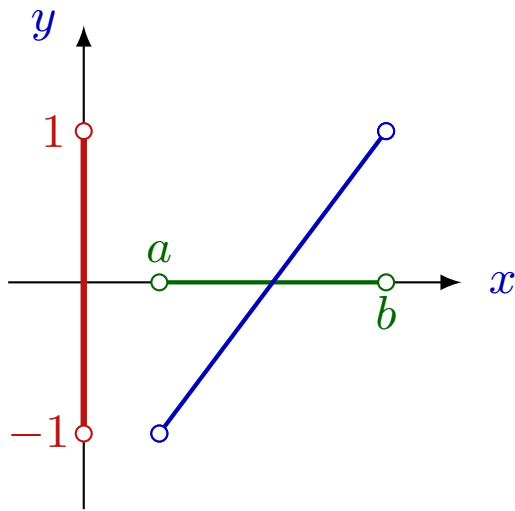
Therefore,  $(a, b) \approx (-1, 1)$ , and since  $(-1, 1) \approx \mathbb{R}$ ,

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$



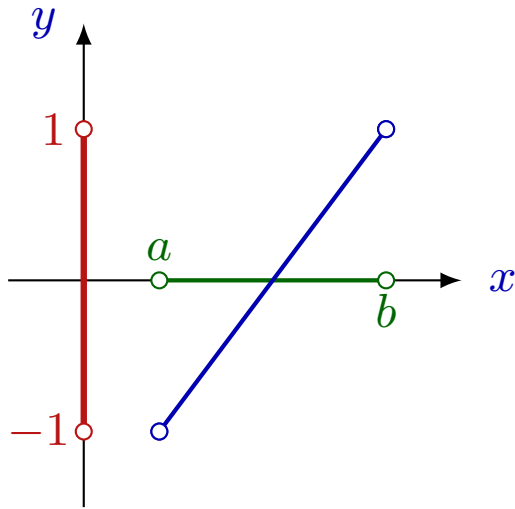
Therefore,  $(a, b) \approx (-1, 1)$ , and since  $(-1, 1) \approx \mathbb{R}$ , we get  $(a, b) \approx \mathbb{R}$ . □

# Any interval $(a, b)$ is uncountable

**Theorem.** For any interval  $(a, b)$  with  $a < b$ ,  $(a, b) \approx \mathbb{R}$ .

**Proof.** A linear function  $f : (a, b) \rightarrow (-1, 1)$  defined by

$$f(x) = \frac{2}{b-a}x - \frac{a+b}{b-a} \text{ is a bijection.}$$



Therefore,  $(a, b) \approx (-1, 1)$ , and since  $(-1, 1) \approx \mathbb{R}$ , we get  $(a, b) \approx \mathbb{R}$ . □

By this,  $|(a, b)| = \mathfrak{c}$ .

# Applications of Cantor-Schröder-Bernstein theorem

---

MAT 250  
Lecture 14  
Cardinalities

**Example 1.**

**Example 1.**  $(0, 1) \approx [0, 1]$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1]$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2)$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)|$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R}$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1)$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1) \implies |(2, 3)| \leq |\mathbb{R} \setminus (0, 1)|$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1) \implies |(2, 3)| \leq |\mathbb{R} \setminus (0, 1)|$ .

$\parallel$   
 $|\mathbb{R}|$

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1) \implies |(2, 3)| \leq |\mathbb{R} \setminus (0, 1)|$ .

$\parallel$   
 $|\mathbb{R}|$

So  $|\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$  and  $|\mathbb{R}| \leq |\mathbb{R} \setminus (0, 1)|$ ,

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1) \implies |(2, 3)| \leq |\mathbb{R} \setminus (0, 1)|$ .

$\parallel$   
 $|\mathbb{R}|$

So  $|\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$  and  $|\mathbb{R}| \leq |\mathbb{R} \setminus (0, 1)|$ ,

therefore, by **C-S-B** theorem,  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Example 1.**  $(0, 1) \approx [0, 1]$ .

Indeed,  $(0, 1) \subseteq [0, 1] \implies |(0, 1)| \leq |[0, 1]|$ .

$[0, 1] \subseteq (-1, 2) \implies |[0, 1]| \leq |(-1, 2)| = |(0, 1)|$ .

So  $|(0, 1)| \leq |[0, 1]|$  and  $|[0, 1]| \leq |(0, 1)|$ ,

therefore, by **C-S-B** theorem,  $(0, 1) \approx [0, 1]$ .

**Example 2.**  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Proof.**  $\mathbb{R} \setminus (0, 1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$ .

$(2, 3) \subseteq \mathbb{R} \setminus (0, 1) \implies |(2, 3)| \leq |\mathbb{R} \setminus (0, 1)|$ .

$\parallel$   
 $|\mathbb{R}|$

So  $|\mathbb{R} \setminus (0, 1)| \leq |\mathbb{R}|$  and  $|\mathbb{R}| \leq |\mathbb{R} \setminus (0, 1)|$ ,

therefore, by **C-S-B** theorem,  $\mathbb{R} \setminus (0, 1) \approx \mathbb{R}$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .  
We have to prove that  $|X| \neq |\mathcal{P}(X)|$ ,

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

**Theorem** (Cantor). *For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .*

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**,

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ?

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ? If  $a \in A$ , then  $a \notin F(a)$  by definition of  $A$ .

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ? If  $a \in A$ , then  $a \notin F(a)$  by definition of  $A$ . But  $A = F(a)$ , so  $a \notin A$ . Contradiction!

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ? If  $a \in A$ , then  $a \notin F(a)$  by definition of  $A$ . But  $A = F(a)$ , so  $a \notin A$ . Contradiction!

If  $a \notin A$ , then it is not true that  $a \notin F(a)$ ,

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ? If  $a \in A$ , then  $a \notin F(a)$  by definition of  $A$ . But  $A = F(a)$ , so  $a \notin A$ . Contradiction!

If  $a \notin A$ , then it is not true that  $a \notin F(a)$ , that is,  $a \in F(a) = A$ .

It's a contradiction.

**Theorem** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

**Proof.**  $|X| \leq |\mathcal{P}(X)|$ , since there is an injection  $X \rightarrow \mathcal{P}(X)$ ,  $x \mapsto \{x\}$ .

We have to prove that  $|X| \neq |\mathcal{P}(X)|$ , that is, there is **no** bijection  $X \rightarrow \mathcal{P}(X)$ .

Assume that there exists a bijection  $F : X \rightarrow \mathcal{P}(X)$ .  $F(x) \subset X$  for  $\forall x \in X$ .

Consider a set  $A = \{x \in X \mid x \notin F(x)\}$ .  $A \subseteq X$ , so  $A \in \mathcal{P}(X)$ .

Since  $F$  is a **surjection**, there exists  $a \in X$  such that  $F(a) = A$ .

Does  $a \in A$ ? If  $a \in A$ , then  $a \notin F(a)$  by definition of  $A$ . But  $A = F(a)$ , so  $a \notin A$ . Contradiction!

If  $a \notin A$ , then it is not true that  $a \notin F(a)$ , that is,  $a \in F(a) = A$ .

It's a contradiction.

Therefore, the assumption about existing a bijection  $X \rightarrow \mathcal{P}(X)$  was wrong,

and there is no such a bijection.  $\square$



$$\aleph_0 = |\mathbb{N}|$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}|$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ .

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $|\mathcal{P}(X)| \approx |\text{Map}(X, \{0, 1\})|$ .

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.**

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ ,

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ ,

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ , define  $2^{|X|}$

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ , define  $2^{|X|} = |\text{Map}(X, \{0, 1\})|$ .

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ , define  $2^{|X|} = |\text{Map}(X, \{0, 1\})|$ .

Since, by the Corollary,  $|\mathcal{P}(\mathbb{N})| = |\text{Map}(\mathbb{N}, \{0, 1\})|$ ,

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ , define  $2^{|X|} = |\text{Map}(X, \{0, 1\})|$ .

Since, by the Corollary,  $|\mathcal{P}(\mathbb{N})| = |\text{Map}(\mathbb{N}, \{0, 1\})|$ , and  $|\text{Map}(\mathbb{N}, \{0, 1\})| = 2^{\aleph_0}$ ,

# Huge sets

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

We know that  $\aleph_0 < \mathfrak{c}$  and  $\aleph_0 < |\mathcal{P}(\mathbb{N})|$ . How are  $\mathfrak{c}$  and  $|\mathcal{P}(\mathbb{N})|$  related?

(Spoiler: they are equal!)

**Theorem.** For any set  $X$ ,  $\mathcal{P}(X) \approx \text{Map}(X, \{0, 1\})$ .

**Proof.** A bijection  $\mathcal{P}(X) \rightarrow \text{Map}(X, \{0, 1\})$  is given by  $A \mapsto \chi_A$ . □

**Corollary.**  $\mathcal{P}(\mathbb{N}) \approx \text{Map}(\mathbb{N}, \{0, 1\})$ .

We remember that for **finite** sets  $X, Y$ , we have  $|\text{Map}(X, Y)| = |Y|^{|X|}$ .

In particular, when  $Y = \{0, 1\}$ ,  $|\text{Map}(X, \{0, 1\})| = 2^{|X|}$ .

For **infinite**  $X$ , define  $2^{|X|} = |\text{Map}(X, \{0, 1\})|$ .

Since, by the Corollary,  $|\mathcal{P}(\mathbb{N})| = |\text{Map}(\mathbb{N}, \{0, 1\})|$ , and  $|\text{Map}(\mathbb{N}, \{0, 1\})| = 2^{\aleph_0}$ ,

we get  $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$ .

$$c \leq 2^{\aleph_0}$$

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.**

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written  
in **binary** representation

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written  
in **binary** representation  $0.a_1a_2a_3 \dots$ ,

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

$$c \leq 2^{\aleph_0}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3}$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8.$$

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8.$$

Define a map  $f : (0, 1) \rightarrow \mathcal{P}(\mathbb{N})$  by  $f(0.a_1a_2a_3\dots) = \{n \in \mathbb{N} \mid a_n = 1\}$ .

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8.$$

Define a map  $f : (0, 1) \rightarrow \mathcal{P}(\mathbb{N})$  by  $f(0.a_1a_2a_3\dots) = \{n \in \mathbb{N} \mid a_n = 1\}$ .

For example,  $f(0.011) = \{2, 3\}$ .

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8.$$

Define a map  $f : (0, 1) \rightarrow \mathcal{P}(\mathbb{N})$  by  $f(0.a_1a_2a_3\dots) = \{n \in \mathbb{N} \mid a_n = 1\}$ .

For example,  $f(0.011) = \{2, 3\}$ .

$f$  is **injective** since the binary representation of a number is unique.

**Theorem.**  $\mathcal{P}(\mathbb{N}) \approx (0, 1)$ .

**Proof.** We will prove that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Any number in the interval  $(0, 1)$  can be written

in **binary** representation  $0.a_1a_2a_3\dots$ , where each  $a_i$  is either 0 or 1.

We exclude the sequences that terminate with infinitely many 1's

to guarantee the uniqueness of the representation

(so  $0.0111\dots$  should be written as  $0.100\dots$ ).

For example, the binary presentation  $0.011$  stands for the number

$$0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8.$$

Define a map  $f : (0, 1) \rightarrow \mathcal{P}(\mathbb{N})$  by  $f(0.a_1a_2a_3\dots) = \{n \in \mathbb{N} \mid a_n = 1\}$ .

For example,  $f(0.011) = \{2, 3\}$ .

$f$  is **injective** since the binary representation of a number is unique.

Therefore,  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$ , that is  $c \leq 2^{\aleph_0}$ .

$$2^{\aleph_0} \leq \mathfrak{c}$$

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3 \dots$ ,

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

$$2^{\aleph_0} \leq \mathfrak{c}$$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**,

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Overall, we have got that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

which **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Overall, we have got that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Therefore, by **C-S-B** theorem,  $|\mathcal{P}(\mathbb{N})| = |(0, 1)|$ .

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Overall, we have got that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Therefore, by **C-S-B** theorem,  $|\mathcal{P}(\mathbb{N})| = |(0, 1)|$ .

**Remark.** Since  $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Overall, we have got that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Therefore, by **C-S-B** theorem,  $|\mathcal{P}(\mathbb{N})| = |(0, 1)|$ .

**Remark.** Since  $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$  and  $|(0, 1)| = \mathfrak{c}$ ,

Prove now that  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Define a map  $g : \mathcal{P}(\mathbb{N}) \rightarrow (0, 1)$  by  $g(A) = 0.a_1a_2a_3\dots$ , where

$$a_n = \begin{cases} 3, & \text{if } n \in A \\ 5, & \text{if } n \notin A. \end{cases}$$

For any set  $A \in \mathcal{P}(\mathbb{N})$ ,  $g(A)$  is a number

whose **decimal** representation contains only 3's and 5's.

For example,  $g(\{2, 4\}) = 0.5353555\dots$

(Any pair of digits not including 9 will work.)

$g$  is **injective**, therefore  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Overall, we have got that  $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$  and  $|\mathcal{P}(\mathbb{N})| \leq |(0, 1)|$ .

Therefore, by **C-S-B** theorem,  $|\mathcal{P}(\mathbb{N})| = |(0, 1)|$ .

**Remark.** Since  $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$  and  $|(0, 1)| = \mathfrak{c}$ , we get  $2^{\aleph_0} = \mathfrak{c}$ .

$$\mathfrak{c}^2 = \mathfrak{c}$$

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection.

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

$$c^2 = c$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

By the **C-S-B** theorem,  $|(0, 1)| = |(0, 1) \times (0, 1)|$ .

$$c^2 = c$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

By the **C-S-B** theorem,  $|(0, 1)| = |(0, 1) \times (0, 1)|$ .

G. Cantor:

$$c^2 = c$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

By the **C-S-B** theorem,  $|(0, 1)| = |(0, 1) \times (0, 1)|$ .

G. Cantor: "I see it, but I don't believe it!"

$$c^2 = c$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

By the **C-S-B** theorem,  $|(0, 1)| = |(0, 1) \times (0, 1)|$ .

G. Cantor: "I see it, but I don't believe it!"

**Corollary.**  $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$ .

Or, equivalently,  $|\mathbb{R}^2| = |\mathbb{R}|$ ,

$$\mathfrak{c}^2 = \mathfrak{c}$$

**Theorem.** *A square is equipotent to an interval:  $(0, 1) \times (0, 1) \approx (0, 1)$ .*

**Proof.** The map  $(0, 1) \rightarrow (0, 1) \times (0, 1)$  defined by  $x \mapsto (x, 1/2)$  is an injection.

Therefore  $|(0, 1)| \leq |(0, 1) \times (0, 1)|$ .

The map  $(0, 1) \times (0, 1) \rightarrow (0, 1)$  given by

$$(0.a_1a_2a_3 \dots, 0.b_1b_2b_3 \dots) \mapsto (0.a_1b_1a_2b_2a_3b_3 \dots)$$

is an injection. Therefore,  $|(0, 1) \times (0, 1)| \leq |(0, 1)|$ .

By the **C-S-B** theorem,  $|(0, 1)| = |(0, 1) \times (0, 1)|$ .

G. Cantor: "I see it, but I don't believe it!"

**Corollary.**  $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$ .

Or, equivalently,  $|\mathbb{R}^2| = |\mathbb{R}|$ , that is,  $\mathfrak{c}^2 = \mathfrak{c}$ .

Let  $\mathcal{C}$  be a category.

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction.

Let  $\mathcal{C}$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ .

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ .  
Verify that this is a category, indeed.

Let  $\mathcal{C}$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ .  
Verify that this is a category, indeed.

**Category of morphisms.**

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ .

Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ :

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ .  
Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ :  
Objects are morphisms of  $C$ .

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**.

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$

which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right)$$

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{\gamma \circ \alpha} & V \\ \downarrow h & & \downarrow f \\ B & \xleftarrow{\delta \circ \beta} & W \end{array} \right)$$

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{\gamma \circ \alpha} & V \\ \downarrow h & & \downarrow f \\ B & \xleftarrow{\delta \circ \beta} & W \end{array} \right)$$

Identity morphism: 
$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{\gamma \circ \alpha} & V \\ \downarrow h & & \downarrow f \\ B & \xleftarrow{\delta \circ \beta} & W \end{array} \right)$$

Identity morphism: 
$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$
 Which morphisms are inverse?

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{\gamma \circ \alpha} & V \\ \downarrow h & & \downarrow f \\ B & \xleftarrow{\delta \circ \beta} & W \end{array} \right)$$

Identity morphism: 
$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$
 Which morphisms are inverse? Invertible?

# Constructing new categories

Let  $C$  be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism  $f : A \rightarrow B$  is considered as a morphism  $B \rightarrow A$  of the **opposite** direction. Composition  $A \rightarrow B \rightarrow C$  turns to  $C \rightarrow B \rightarrow A$ . Verify that this is a category, indeed.

**Category of morphisms.** Here is another way to cook up a new category from  $C$ : Objects are morphisms of  $C$ .

A morphism  $(V \xrightarrow{g} W) \rightarrow (X \xrightarrow{f} Y)$  is a pair  $(V \xrightarrow{\alpha} X, W \xrightarrow{\beta} Y)$  of maps such that  $\beta \circ g = f \circ \alpha$ .

It is presented by a diagram: 
$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \downarrow g & & \downarrow f \\ W & \xrightarrow{\beta} & Y \end{array}$$
 which is **commutative**:  $\beta \circ g = f \circ \alpha$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{\gamma} & X \\ \downarrow h & & \downarrow g \\ B & \xleftarrow{\delta} & Y \end{array} \right) \circ \left( \begin{array}{ccc} X & \xleftarrow{\alpha} & V \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{\beta} & W \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{\gamma \circ \alpha} & V \\ \downarrow h & & \downarrow f \\ B & \xleftarrow{\delta \circ \beta} & W \end{array} \right)$$

Identity morphism: 
$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$

Which morphisms are inverse? Invertible?  
Isomorphisms?

Two special kinds of objects:

Two special kinds of objects: injections and surjections.

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target,

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y$$

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Classification problem for **surjections**.

# Classification of maps

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Classification problem for **surjections**.

A surjection  $X \xrightarrow{f} Y$  is isomorphic to the projection  $X \rightarrow X/\sim_f$ .

# Classification of maps

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Classification problem for **surjections**.

A surjection  $X \xrightarrow{f} Y$  is isomorphic to the projection  $X \rightarrow X/\sim_f$ .

Invariant: a family of sets,

which form the partition of  $X$  to  $f^{-1}(b)$  with  $b \in Y$  (or equivalence relation  $\sim_f$ ).

# Classification of maps

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Classification problem for **surjections**.

A surjection  $X \xrightarrow{f} Y$  is isomorphic to the projection  $X \rightarrow X/\sim_f$ .

Invariant: a family of sets,

which form the partition of  $X$  to  $f^{-1}(b)$  with  $b \in Y$  (or equivalence relation  $\sim_f$ ).

Families of sets form a category.

# Classification of maps

Two special kinds of objects: injections and surjections.

Classification problem for **injections**.

Invariants:

Cardinalities of source, target, and the complement of the image.

A complete system of invariants:

the cardinality of source, the cardinality of the complement of the image.

$$X \xrightarrow{f} Y = f(X) \cup (Y \setminus f(X)).$$

$$X \xrightarrow{f|} f(X) \amalg (Y \setminus f(X))$$

Classification problem for **surjections**.

A surjection  $X \xrightarrow{f} Y$  is isomorphic to the projection  $X \rightarrow X/\sim_f$ .

Invariant: a family of sets,

which form the partition of  $X$  to  $f^{-1}(b)$  with  $b \in Y$  (or equivalence relation  $\sim_f$ ).

Families of sets form a category.

Families  $\Gamma$  and  $\Delta$  of sets are equivalent

if there exists a bijection  $\Gamma \rightarrow \Delta$  formed of bijections.



Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 \downarrow f & & g \downarrow \\
 X & \xrightarrow{h} & Y
 \end{array}$$

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 \downarrow f & & g \downarrow \\
 X & \xrightarrow{h} & Y
 \end{array}$$

which is **commutative**.

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 \downarrow f & & g \downarrow \\
 X & \xrightarrow{h} & Y
 \end{array}$$

which is **commutative**:  $h \circ f = g \circ h$ .

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram: 
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$
 which is **commutative**:  $h \circ f = g \circ h$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{u} & Y \\ \downarrow a & & \downarrow g \\ A & \xleftarrow{u} & Y \end{array} \right) \circ \left( \begin{array}{ccc} Y & \xleftarrow{h} & X \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{h} & X \end{array} \right)$$

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram: 
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$
 which is **commutative**:  $h \circ f = g \circ h$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{u} & Y \\ \downarrow a & & \downarrow g \\ A & \xleftarrow{u} & Y \end{array} \right) \circ \left( \begin{array}{ccc} Y & \xleftarrow{h} & X \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{h} & X \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{u \circ h} & X \\ \downarrow a & & \downarrow f \\ A & \xleftarrow{u \circ h} & X \end{array} \right)$$

# Category of endomorphisms

Objects are maps  $f : X \rightarrow X$ , where  $X$  is a set.

A morphism  $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$  is a map  $X \xrightarrow{h} Y$  such that  $h \circ f = g \circ h$ .

It is presented by a diagram: 
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$
 which is **commutative**:  $h \circ f = g \circ h$ .

Composition: 
$$\left( \begin{array}{ccc} A & \xleftarrow{u} & Y \\ \downarrow a & & \downarrow g \\ A & \xleftarrow{u} & Y \end{array} \right) \circ \left( \begin{array}{ccc} Y & \xleftarrow{h} & X \\ \downarrow g & & \downarrow f \\ Y & \xleftarrow{h} & X \end{array} \right) = \left( \begin{array}{ccc} A & \xleftarrow{u \circ h} & X \\ \downarrow a & & \downarrow f \\ A & \xleftarrow{u \circ h} & X \end{array} \right)$$

Categories in Linear Algebra.

Working with maps, we often consider elements of the source and target sets. However often elements may be avoided and everything can be done on the categorical level, replacing elements by compositions of maps.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **epi = surjective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **epi = surjective**.

One might expect that **monic + epi = isomorphism**

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **epi = surjective**.

One might expect that **monic + epi = isomorphism** (like in the category of sets)

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **epi = surjective**.

One might expect that **monic + epi = isomorphism**,  
but in some categories this is not true.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **monic** if for any morphisms  $\alpha_1, \alpha_2 : X \rightarrow A$  the equality  $f \circ \alpha_1 = f \circ \alpha_2$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **monic = injective**.

A morphism  $f : A \rightarrow B$  in a category  $C$  is called **epi** if for any morphisms  $\alpha_1, \alpha_2 : B \rightarrow X$  the equality  $\alpha_1 \circ f = \alpha_2 \circ f$  implies  $\alpha_1 = \alpha_2$ .

In the category of sets, **epi = surjective**.

One might expect that **monic + epi = isomorphism**,

but in some categories this is not true.

Categories of Linear Algebra

A binary relation  $\prec$  is called a **partial order** if it is **reflexive, anti-symmetric and transitive.**

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is **reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is **reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set**.

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable,  
 then  $(X, \prec)$  is said to be **linearly ordered set** or **loset.**

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable,  
 then  $(X, \prec)$  is said to be **linearly ordered set** or **loset.**

Notice that any loset is a poset.

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set**.

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable,  
 then  $(X, \prec)$  is said to be **linearly ordered set** or **loset**.

Notice that any loset is a poset.

**Examples** of losets:

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable,  
then  $(X, \prec)$  is said to be **linearly ordered set** or **loset.**

Notice that any loset is a poset.

**Examples** of losets:  $(\mathbb{R}, \leq)$ ,

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is  
**reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable,  
then  $(X, \prec)$  is said to be **linearly ordered set** or **loset.**

Notice that any loset is a poset.

**Examples** of losets:  $(\mathbb{R}, \leq)$ , (a dictionary, lexicographic order on the set of words).

**Direct product** of two posets:  $(X, \prec_X) \times (Y, \prec_Y) = (X \times Y, \prec_{X \times Y})$   
with  $(a, b) \prec_{X \times Y} (c, d) \iff a \prec_X c \wedge b \prec_Y d$

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is **reflexive, anti-symmetric and transitive.**

A **set** equipped with a partial order is called a **poset** or **partially ordered set.**

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable, then  $(X, \prec)$  is said to be **linearly ordered set** or **loset.**

Notice that any loset is a poset.

**Examples** of losets:  $(\mathbb{R}, \leq)$ , (a dictionary, lexicographic order on the set of words).

**Direct product** of two posets:  $(X, \prec_X) \times (Y, \prec_Y) = (X \times Y, \prec_{X \times Y})$   
with  $(a, b) \prec_{X \times Y} (c, d) \iff a \prec_X c \wedge b \prec_Y d$

**Lexicographic product** of two posets:  $(X, \prec_X) LG \times (Y, \prec_Y) = (X \times Y, \prec_{X \times Y})$   
with  $(a, b) \prec_{X \times Y} (c, d) \iff (a \prec_X c \wedge a \neq c) \vee (a = c \wedge b \prec_Y d)$

## Digression: posets and losets

A binary relation  $\prec$  is called a **partial order** if it is **reflexive, anti-symmetric and transitive**.

A **set** equipped with a partial order is called a **poset** or **partially ordered set**.

**Examples** of posets:  $(\mathcal{P}(X), \subset)$ ,  $(\mathbb{N}, |)$ ,

Elements  $a, b$  of poset  $(X, \prec)$  are said to be **comparable**, if  $a \prec b \vee b \prec a$ .

If any two elements of a poset  $(X, \prec)$  are comparable, then  $(X, \prec)$  is said to be **linearly ordered set** or **loset**.

Notice that any loset is a poset.

**Examples** of losets:  $(\mathbb{R}, \leq)$ , (a dictionary, lexicographic order on the set of words).

**Direct product** of two posets:  $(X, \prec_X) \times (Y, \prec_Y) = (X \times Y, \prec_{X \times Y})$   
with  $(a, b) \prec_{X \times Y} (c, d) \iff a \prec_X c \wedge b \prec_Y d$

**Lexicographic product** of two posets:  $(X, \prec_X) LG \times (Y, \prec_Y) = (X \times Y, \prec_{X \times Y})$   
with  $(a, b) \prec_{X \times Y} (c, d) \iff (a \prec_X c \wedge a \neq c) \vee (a = c \wedge b \prec_Y d)$

Let  $X, Y$  be posets, a map  $X \rightarrow Y$  is called **monotonic**, or **monotone**, or **increasing** if  $a \prec b \implies f(a) \prec f(b)$ .

Posets and their monotone maps form a category.

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

In both categories, isomorphism classes of objects are called **order types**.

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

In both categories, isomorphism classes of objects are called **order types**.

Each poset  $(X, \prec)$  is a category

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

In both categories, isomorphism classes of objects are called **order types**.

Each poset  $(X, \prec)$  is a category, in which  $X$  is the set of objects

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

In both categories, isomorphism classes of objects are called **order types**.

Each poset  $(X, \prec)$  is a category, in which  $X$  is the set of objects and for any  $a, b \in X$  with  $a \prec b$  there is a unique morphism  $a \rightarrow b$ .

Posets and their monotone maps form a category.

Losets and their monotone maps also form a category.

In both categories, isomorphism classes of objects are called **order types**.

Each poset  $(X, \prec)$  is a category, in which  $X$  is the set of objects and for any  $a, b \in X$  with  $a \prec b$  there is a unique morphism  $a \rightarrow b$ .

If  $a \prec b$  is false, then there is no morphism  $a \rightarrow b$ .

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$   
are said to form a **(covariant) functor**, if

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$   
are said to form a **(covariant) functor**, if  
 $F(f \circ g) = F(f) \circ F(g)$  for any  $f \in Mor_{\mathcal{C}}(A, B)$  and any  $g \in Mor_{\mathcal{C}}(B, C)$ ,

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$

are said to form a **(covariant) functor**, if

$F(f \circ g) = F(f) \circ F(g)$  for any  $f \in Mor_{\mathcal{C}}(A, B)$  and any  $g \in Mor_{\mathcal{C}}(B, C)$ , and  
 $F(id_A) = id_{F(A)}$  for any  $A \in Ob(\mathcal{C})$ .

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$

are said to form a **(covariant) functor**, if

$F(f \circ g) = F(f) \circ F(g)$  for any  $f \in Mor_{\mathcal{C}}(A, B)$  and any  $g \in Mor_{\mathcal{C}}(B, C)$ , and  
 $F(id_A) = id_{F(A)}$  for any  $A \in Ob(\mathcal{C})$ .

**Example.** Forgetting functor from the category of rings to the category of sets.

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$

are said to form a **(covariant) functor**, if

$F(f \circ g) = F(f) \circ F(g)$  for any  $f \in Mor_{\mathcal{C}}(A, B)$  and any  $g \in Mor_{\mathcal{C}}(B, C)$ , and  
 $F(id_A) = id_{F(A)}$  for any  $A \in Ob(\mathcal{C})$ .

**Example.** Forgetting functor from the category of rings to the category of sets.

Similar **more general example:**

forgetting functor from a category of set with an **additional structure** to the category of sets.

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

Maps  $F : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$  and  $F : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathcal{D}}(F(A), F(B))$

are said to form a **(covariant) functor**, if

$F(f \circ g) = F(f) \circ F(g)$  for any  $f \in Mor_{\mathcal{C}}(A, B)$  and any  $g \in Mor_{\mathcal{C}}(B, C)$ , and  
 $F(id_A) = id_{F(A)}$  for any  $A \in Ob(\mathcal{C})$ .

**Example.** Forgetting functor from the category of rings to the category of sets.

Similar **more general example:**

forgetting functor from a category of set with an **additional structure** to the category of sets.

Any reasonable general construction converting a set to other set can be upgraded to a functor.

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y),$$

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X$$

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X),$$

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ ,

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.**

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps,

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

## Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”.

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$  ?

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction

## Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above.

## Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above. This is a **contravariant** functor.

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above. This is a **contravariant** functor. Above we defined **covariant** functors.

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above. This is a **contravariant** functor. Above we defined **covariant** functors.

$f_*$  is a covariant functor

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above. This is a **contravariant** functor. Above we defined **covariant** functors.

$f_*$  is a covariant functor, while  $f^*$  is a contravariant functor.

# Two functors of power set

Any map  $f : X \rightarrow Y$  induces maps

$$f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

$$f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \text{ for any } B \subset Y.$$

(We remember that  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$  is the preimage of  $B$ , and **not** the result of applying the inverse map  $f^{-1}$ .)

The maps  $f_*$  and  $f^*$  are well defined since

for any  $A \subset X$ ,  $f(A) \subset Y$  and, therefore,  $f(A) \in \mathcal{P}(Y)$  and

for any  $B \subset Y$ ,  $f^{-1}(B) \subset X$  and, therefore,  $f^{-1}(B) \in \mathcal{P}(X)$ .

**Exercise 1.** Prove that if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps, then

$$(g \circ f)_* = g_* \circ f_* .$$

**Exercise 2.** Formulate and prove a similar identity for  $(g \circ f)^*$ .

$f_*$  is a part of functor “Power set”. It could be denoted as  $\mathcal{P}(f)$ .

But what is  $f^*$ ? It has a “wrong” direction, not like in the definition of functor above. This is a **contravariant** functor. Above we defined **covariant** functors.

$f_*$  is a covariant functor, while  $f^*$  is a contravariant functor.

A **contravariant functor**  $C \rightarrow D$  is

a (covariant) functor from  $C$  to the category dual to  $D$ .