Lecture 6. Dynamical Systems

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The lecture on dynamical systems was given by Matthieu Arfeux. Below you can find a concise list of definitions (borrowed from [1]) and statements on this topic.

6.1 Dynamical System

A *discrete-time dynamical system* consists of a non-empty set X and a map $f : X \to X$. For $n \in \mathbb{N}$, the *nth iterate of f* is the n-fold composition $f^n = f \circ \cdots \circ f$.

For $x \in X$,we define the *orbit* $O_f(x) = \cup_{n \geq 0} f^n(x)$. A point $x \in X$ is a *periodic point* of period $T > 0$ if $f^T(x) = x$. If $f(x) = x$ then x is called a *fixed point*.

In order to classify dynamical systems, we need a notion of equivalence. Let $f : X \to X$ and $g : Y \to Y$ be dynamical systems. A *semiconjugacy from* (Y, g) *to* (X, f) (or, briefly, from g to f) is a surjective map $\pi : Y \to X$ such that $f^n \circ \pi = \pi \circ g^n$, for all $n \in \mathbb{N}$. We express this formula schematically by saying that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[f]{} & X \end{array}$$

An invertible semiconjugacy is called a *conjugacy*. If there is a conjugacy from one dynamical system to another, the two systems are said to be conjugate; conjugacy is an equivalence relation. To study a particular dynamical system, we often look for a conjugacy or semiconjugacy with a better-understood model. To classify dynamical systems, we study equivalence classes determined by conjugacies preserving some specified structure. Note that for some classes of dynamical systems (e.g., measure-preserving transformations) the word isomorphism is used instead of "conjugacy."

6.2 Circle

Consider the unit circle $S^1 = [0,1]/\sim$, where $\sim$ indicates that 0 and 1 are identified. Addition mod 1 makes S^1 an abelian group. The natural distance on $[0,1]$ induces a distance on S^1; specifically, $d(x,y) = \min(|x-y|\, l - |x-y|)$.

We can also describe the circle as the set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, with complex multiplication as the group operation. The two notations are related by $z = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$, which is an isometry if we divide arc length on the multiplicative circle by 2π.

6.3 Binary expansion

We will mostly use the additive notation for the circle: a point on the circle S^1 is presented by real number x modulo 1. In other words, two numbers x and x' represent the same point on the circle, if and only if $x - x'$ is an integer. As above each point can be represented by $x\imath[0,1]$. Then te only non-uniqueness of the representation appears for the end points 0 and 1, which represent the same point. To avoid even this non-uniqueness, we may choose x in $[0,1)$.

Any real number $x \in [0,1)$ can be presented as $\sum_{j=1}^{\infty} \frac{x_j}{2^j}$ where each x_j is either 0 or 1. This is similar to the decimal expansion $x = \sum_{j=1}^{\infty} \frac{d_j}{10^j}$ where d_j is a digit (i.e., one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and x is presented by an infinite decimal fraction $0.d_1 d_2 d_3 \ldots d_n \ldots$. The presentation

$$x = \sum_{j=1}^{\infty} \frac{x_j}{2^j}$$

is called a *binary expansion* of x. The sequence of zeros and ones x_1, x_2, x_3, ... is called a *binary representation* of x.

A binary representation is not unique. For example, the sequences

$$1, 0, 0, 0, 0, \ldots, 0, \ldots \text{ and } 0, 1, 1, 1, 1, \ldots, 1, \ldots$$

are representations of the same number $\frac{1}{2}$.

Exercise 1. Describe all the pairs of different sequences of zeros and ones which represent the same number.

Denote by Σ the set of infinite sequences formed of zeros and ones. Define a map

$$\varphi : \Sigma \to [0,1] : \varphi((x_j)_{j\in\mathbb{N}}) = \sum_{j=1}^{\infty} \frac{x_j}{2^j}$$

Exercise 2. Find a subset $\Gamma \subset \Sigma$ such that the restriction of φ to Γ is bijective.

6.4 Square and shift

Since $|z^2| = |z|^2$ for any complex number z, taking square maps the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ to itself. The binary representation is quite useful for study of the dynamical system $S^1 \to \mathrm{Re}^1 : z \mapsto z^2$.

Theorem 1. *The map $\varphi : \Sigma \to [0, 1]$ defines a semiconjugacy*

$$\begin{array}{ccc} \Sigma & \xrightarrow{g} & \Sigma \\ \Big\downarrow & & \Big\downarrow \\ S^1 & \xrightarrow[z \mapsto z^2]{} & S^1 \end{array}$$

where $g : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$.

It is easy to describe periodic points for g of period $T \in \mathbb{N}$.

Exercise 3. Prove that $(x_1, x_2, x_3, \dots)$ is a periodic point for g of period T if and only if it is periodic sequence of the same period, that is $x_j = x_{j+T}$ for any $j \in \mathbb{N}$.

By Theorem 1, this gives also a description of periodic points for $z \mapsto z^2$.

Exercise 4. Fof an integer T, find all the complex numbers z with $|z| = 1$, which are T-periodic for the map $z \mapsto z^2$.

Exercise 5. Check if the answer to Exercise 4 can be obtained via Theorem 1 and Exercise 3.

References

[1] Michael Brin, Garrett Stuck *Introduction to Dynamical Systems* 2002, Cambridge University Press.