Teaching mathematics to non-mathematicians

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From Vershik's foreword: *Vladimir Abramovich Rokhlin (1919-1984), a remarkable mathematician, widely known for his ground-breaking work in the theory of dynamical systems and topology, mentored several top-rate mathematicians, and was a brilliant lecturer. He thought a lot about problems concerning the teaching of mathematics. His own pedagogical experience was wide and diverse, as he taught at one technical university, several pedagogical universities, and (during the last 20 years of his life) at Leningrad State University.*

Unfortunately, the beginning of the lecture (about 15 minutes) has been lost. In these 15 minutes V.A. talked, in part, about his experience teaching

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http://www.math.stonybrook.edu/~oleg/Rokhlin/LectLMO-eng.html or:
http://www.math.stonybrook.edu/~oleg/Rokhlin/LectLMO-eng.pdf . The Russian original can be found at: http://www.math.stonybrook.edu/~oleg/Rokhlin/matProsv.pdf

mathematics to non-mathematics majors at the universities of Arkhangelsk, Ivanovo and Kolomna in the 1950's.

...In elementary school, of course, no serious proofs are given to children, but some rules are formulated. For example, children are taught how to divide a fraction by a fraction. There is a rule which is formulated, and children have to know the formulation. Once I attended a talk by a specialist in the methodology of education. He explained how to teach children to divide a fraction by a fraction. He said that he formulated the rule and applied it to several examples. Then children did some examples too, and then they had a written test. And in this written test almost all of them made the same mistake. (All who made the mistake made it in the same instances.) They for whatever reason sometimes didn't invert the dividing fraction, but just multiplied. In some cases they first inverted it, and in some cases they didn't.

What happened? The specialist in the methodology of education explained that the analysis of the examples had revealed the cause of the mistake. In all the examples given to the children prior to the test, the dividing fractions were proper.([1]) The children figured out from the examples how to act, and acted accordingly.

So the rule that was formulated in the beginning by the teacher was not at all the rule that the children understood and followed. The rule had been dictated to them as a matter of routine. It had to be dictated, and indeed it was dictated.

The idea that children at this age must learn to *understand* the rule and use *its statement* rather than just the examples that they have been shown had been totally missed by the specialist in the methodology of education, and also by everybody attending his lecture.

I think that giving examples immediately after the formulation of a rule causes a harmful confusion. I have no doubts that children should themselves, guided by the rule, calculate the first few examples. Of course, later they have to acquire skills, become fluent and apply the rule automatically, but the most valuable opportunity, the opportunity that children on such an occasion can learn and should learn to understand the rule itself, was totally missed.

This is only one example. The story that I just have told you reveals one of the basic defects of educational methods. The defects are wide-spread. ([2])

Nowadays one can graduate from high school without solving a single mathematics problem. Templates, patterns and examples are provided to students and students can fulfill their school duties just by imitating them.

When during a recitation session first year students - and often not only first year - are given a problem to work on, observe what they do.

Of course, I am not talking about the math majors. ([3]) My subject is different – teaching mathematics to non-mathematicians. But, according to my experience, in a class of students who are not math majors, only 2 or 3 persons actually solve assigned problems; the rest of them sit and wait. They actually don't understand what is expected of them. They wait for the problem to be solved on the blackboard, or for someone to tell them how to solve such a problem. They show no initiative; they are completely unaccustomed to being questioned.

It sounds strange, because, if you ask any of these young people to buy some groceries and some medicine and also tell them when the grocery store and when the pharmacy are closed for a lunch break, they will solve this problem beautifully. They will pick the right times to go to the stores and will not go to the pharmacy when it's closed. Well, there are some strange cases even here (laughter).([4])

The problems which the students are given on recitations are far simpler than the problem I am talking about, with the grocery and pharmacy. Nevertheless, the idea of beginning to solve these problems even doesn't cross the minds of these youngsters, who are about 17 or 18 years old. This is an astonishing phenomenon, and it is, of course, the result of how they were taught mathematics.

Similarly, the level of knowledge in the area of the exact sciences (not only mathematics) is surprisingly low among the adult population, i.e., among people who are no longer students, but who have graduated from high school or university some time ago and are considered educated.

If you take writers, musicians, actors, film directors, many medical doctors, naturally, let alone all the other people educated in the humanities, you will discover absolutely amazing things.

They say and write with pride that they are not good in math or physics. They tell it with a sneer, and, in general, don't express any difference between the two. Perhaps I exaggerate but it seems to me that math for them is some area of technology or physics..., something not very refined, not worthy of respect, but, in any case, something that should serve their needs. With medical doctors, of course, this situation is changing.

These days mathematics is taught even to philosophy students and students in other humanities. It is taught to everybody using the same template, without achieving any understanding of the subject.

If you read texts carefully - some of them are quite interesting - written by people educated in humanities, you will notice that they love to use expressions borrowed from mathematics and physics. It is fashionable, it is modern... But my goodness, what they write! It's amazing but they have less than a vague idea of what a factor is and what a divisor is, what a degree is and what the words positive and negative mean. From the texts that one can come across in newspapers and magazines one can see clearly that they studied all of this. For example, the idea that the negative is somehow related to division, to degree... Well, take, for example, such a phrase: "something is negative and therefore it is not zero, but infinity!" (laughter).

I am not exaggerating. I can give you references where you can read such things. All of these are written by well-known, even famous people, published in newspapers. These people remembered something from an elementary school course and somehow they think that they remembered it correctly.

What has to be done about all this? Well, that's a difficult question.

I don't believe that this problem, that is the problem of getting a higher level of universal, general education in exact sciences (and mathematics in particular) can be solved quickly. It is a difficult problem, and solving it will take a long time and a huge effort.

One more question arises: is it possible or desirable to solve this problem? Up till now, till very recent times, people in all the civilized societies were educated in humanities. In the whole history of mankind, there was no human society for which education in the exact sciences was universal.

In some instances classical education was very wide-spread among the educated strata of the society, but the exact sciences never belonged to the background even of the educated part of society to any serious degree.

Many modern developing countries have a long history, going back centuries and millennia. Many have their own intellectual elite but again, this elite is educated mostly in humanities. They are eager to study the humanities, but they study the exact sciences with reluctance, and not very successfully.

Concluding these preliminary remarks, I want to say that nobody really knows, and, of course, I don't know, what would be the result of a serious universal education in mathematics and exact sciences. Even if it were possible, whether it would make things better or worse, what would improve and what would get worse. I don't know and nobody does. It has never been tried. Nevertheless, for some reason we would like to achieve this goal. We are trying. Somehow intuitively we feel that it would be good if our children and grandchildren would become attached to a logical culture, to a mathematical culture, if they would understand the exact sciences better.

It may well be that this will lead to an incredible revolution, to unprecedented results. Who knows? I don't know. But in any case, such a thing is far from us today.

Turning to the more narrow subject of my lecture, I have to say that teaching mathematics to future mathematicians is infinitely easier than teaching mathematics to non-mathematicians.

After all, we speak to future mathematicians honestly: we have a subject that we know, and we do our best to teach this subject to the future mathematicians. No matter how masterful or how mediocre we are in lecturing or conducting recitation sections, we know the subject and can transmit our knowledge to interested people.

But how should we deal with those who are not interested and think they have no ability, or just say so?

Very often people, who say so, are simply stuck in what one might call intellectual laziness. This intellectual laziness is a very common phenomenon, and sometimes you can easily detect it. After talking briefly to someone, who claims to have no mathematical abilities and to being infinitely far from all this, by talking to him just a little, you discover that he understands perfectly all that you tell him.

So, the question of ability in this field is a complicated issue. You don't have just to take someone's word for it when he claims to have no inclination for these subjects, claiming to be interested only in humanities.

Before I really turn my attention to teaching mathematics to non-mathematicians, I should first explain whom I call a non-mathematician. It's rather useless to discuss this question in general. I will just say what I have in mind right now, when I am talking about this. I have in mind teaching mathematics to people who have no intention to work in mathematics, who study it either for applications or because they have a non-professional interest in it.

Of course, the presence of interest makes the task of the teacher easier, but very often there is no interest. Instead, there is disgust. There are plenty of students now who have to learn some exact sciences, but have no interest in them. Nevertheless, they have to pass the exams, etc. How should we treat such people?

What should we do with people who are interested in mathematics, but are enrolled in programs which make it impossible to learn mathematics?

There are many mathematical curricula for the students of technical universities, they vary in their length and their content. However very often these curricula and their corresponding textbooks are not independent, but are simply deteriorated courses for math majors, with the same order of presentation, the same limits, the same derivatives, the same integrals, the same second degree curves, and so on and so forth.

The material is presented in the same order, but less intelligibly. There are no proofs that would help to understand the matter. The authors of the textbooks have no talent for writing. Everything is boring, is not understandable. Students are lucky if the lecturer gets them interested by explaining something beyond the scope of their textbook.

This situation is very common, and not only in our country. This phenomenon is quite international, and, as I think, the reason is the following.

Apparently, institutes of technology, teachers' colleges and high schools require their own particular courses in mathematics. Each category of students (if it is big enough, of course) seems to require its own course, and it should clearly differ from the one for math majors.

I think the main defect of the existing courses, the main reason for their failures is the following. No first rate mathematician ever worked on putting together, a course of mathematics for non-mathematicians. I am talking now not about a high school course, but about a university course, and I have in mind the following.

Usually, before doing differential and integral calculus, students are taught the theory of limits. The same happens in high school now. Limits are taught there, too. However, and it is a striking example of the current state of affairs, that limits form the part of the course that is most difficult to understand, and, most interestingly, it is absolutely unnecessary. All the differential calculus, all the integral calculus, and, in general, all the classical mathematics, to say nothing of the finite mathematics, can be presented perfectly without limits. They are not needed there at all. They are an absolutely extraneous phenomenon, an extraneous subject that was introduced into this area by the people who strived to build a proper foundation for analysis.

Of course, the task of building the foundations is not achieved in a technical college course, it is not even formulated there. Already from this example, one can see that these courses were not thought out. They are just deteriorated university analysis courses for math majors.

I will explain my thoughts about limits in a little more detail. When

I went to high school (perhaps, it's still the same now), I was told what the area of a circle is. I was told that this is some sort of limit, and then something was written or was stated, and we got a formula for the area of the circle. What was said was difficult to understand then, but when I became a mathematician, it became totally clear to me, why it was so difficult to understand. It was all sheer nonsense.

None of my fellow-students had any doubts that he knew what the area of the circle is. Rather it looked strange to us that for the circle the area was defined, but somehow it was not defined for other figures. It was strange to us that the area of a circle that was absolutely clear to us, is defined by using some limits that were completely incomprehensible. It was strange to us, of course, (and it is strange to all the children who thought about it a little bit) that some theorems about limits are needed to establish some very clear and simple things that we never had doubts about.

But really, why should we define the area of a circle and prove that it is πR^2? Why not just announce that the area of a circle is πR^2 by definition, what is the difference? Apparently, the difference, and a really serious one, is that not only circles have area, that the area is a general notion, that the area is defined for a wide class of figures, that it has properties, known and used by everyone, and these properties make the area a useful notion.

So, the attitude towards the area adopted in the high school course was then (and maybe in many cases still remains now) absolutely bizarre.

But in mathematics taught in technical colleges, the same attitude is adopted towards integrals, derivatives, volumes and masses, density, charges, moments of inertia and, in general, towards all mathematical and physical magnitudes of this integral or differential nature.

From the point of view of a person who is not a professional mathematician, all these things exist, they don't require a definition, they require a computation, and they have to be ready for applications. That's what is needed.

The point of view that these properties should be defined is not appropriate in teaching here, at least in teaching a person who has no doubt about existence of area of all the figures, or properties of the area. You don't have to define the area to such a person. Such a person needs to learn its properties and needs to learn how to calculate the area.

The same is true about the other mathematical notions. Let's take the notion of integral. On this occasion I would like to ask a question of a historical nature. Tell me, please, did Archimedes have a notion of the integral or not? There are different points of view on it. Some say he did, some say he didn't. I will express my own opinion. I think that Archimedes, maybe the greatest mathematician of all times, did not have the notion of the integral. Here is why I think so.

Archimedes many times, by many different methods, calculated the integral $\int_0^1 x^2dx$. He calculated it when he studied the areas bounded by segments of straight lines and a segment of parabola. He calculated this integral when he calculated the volume of a ball, and in many other situations. Each time he used a special approach, very ingenious, brilliant. But apparently he did not know that it was all the same. He probably felt that it was.

The reason is completely clear. The Greeks did not possess the notion of real number. The volume and area were totally different entities for them. They were geometrical entities which could not be compared to one another. For example, Archimedes would have protested against such an expression as $x + x^2$. He would have said that x and x^2 cannot be added, that it is the same as adding a line segment to a planar figure. But we do this. We have numbers. And it looks like Archimedes didn't possess the numerical notion of integral. If he did, he undoubtedly would not have calculated the same integral many times.

On the other hand, Archimedes left us the method of exhaustion which may be singularly appropriate for teaching mathematics to non-mathematicians. Because of this method, instead of all the theory of limits only one single fact is needed. I will formulate it now. This fact is very simple. The fact is that if a non-negative number is less than any positive number, then it is zero. I repeat, *if a non-negative number is less that any positive number, then it is zero.*

Of course, this fact is not difficult to prove, but perhaps here a proof is not even needed. After you are familiar with this fact, you can use it to prove all the equalities that are encountered in the differential and integral calculus and in its applications, and, more generally, in all of analysis, provided that you do not deal with existence theorems.

The theory of limits exists to prove existence theorems.

If you do not need to prove the existence of an area, then the theory of limits is not needed for areas. If you do not need to prove the existence of the integral, the limit theory is not needed for integrals, and so on. If you only need to calculate, you can get by without a limit theory. This immediately makes differential and integral calculus infinitely easier.

[INTERMISSION]

I was talking about the theory of limits. Why does this theory give me a strong desire to crack down on it so crudely, to expel it from the course of mathematics for non-mathematicians?

I don't mean to say that it should actually be expelled from everywhere, no. I want to say only the following: nowadays, the theory of limits works not as a tool for introducing the basic notions of calculus, but as a very high and difficult barrier that one has to climb over in order to understand anything.

And this barrier is absolutely unnecessary! For a student-non-mathematician, as a rule, it is, impossible to cross this barrier, and furthermore this barrier is completely extraneous.

Let me give an example that will demonstrate this. I could have taken as an example either differential or integral calculus. I take the simpler of the two cases, or, better said, subjects – the one that can be explained more quickly, which is integral calculus.

You have to explain to beginners what an integral is. Of course, you can start simply with an area (and this is probably the right way) as is usually done.

But why not declare that an integral is simply an area? Indeed, no one in your audience will doubt that area exists. You would have to spend a lot of time if you wanted to raise any doubt about this with your audience.

Of course, having the notion of area at hand, you can easily construct integral sums. You will say to your listeners that area has well-known properties. I recall them.

- If one figure is contained in the other, then the area of the first figure is no more than the area of the second. It is difficult not to agree with that, and everyone will agree with you.
- You can say that if you add two figures; that is, put together two figures without any common interior points, then their areas will sum up, and everyone will agree with this, too.
- Then you will say that the area does not change if the figure moves around on the plane as a rigid body. They will agree.
- And finally, you will say that the area of a unit square is 1.

These four properties, as is well known, uniquely define the area on a wide class of figures, namely on the class of all squarable figures.

The same properties, slightly rephrased, determine the integral. They uniquely determine the integral on a wide class of functions. This way, starting with the area, you can define the integral by its properties that nobody doubts, because we are talking about area.

Later on, you point out that, if you construct the so-called curvilinear trapezoid upper and the lower Riemann (or Lebesgue, no difference!) sums, then the area you are interested in will be between these two auxiliary areas. It follows directly from what I have just said. You write the very same inequality that is usually written in the integral calculus texts: the lower sum is no greater than the integral which is no greater than the upper sum. To put it briefly, the integral is the only number which lies between all lower and all upper sums.

All this is formulated in terms of areas and is quite obvious; it doesn't cause any doubt and is easy to digest. On the other hand, this gives you a method to calculate areas. No limits are mentioned. If you want to demonstrate some identity, say, between two integrals or between an integral and a number, you simply notice that both numbers that you want to be equal are enclosed between the upper and the lower sums. Therefore they are equal, because the difference between the upper and the lower sum can be made less than any positive number by taking an appropriate partition. This difference is non-negative, and therefore it is zero.

The question about the technique used in any particular case does not arise. All these techniques can be used. Techniques are described in detail in all the usual courses, but in these course everything is turned upside-down.([5])

Very similarly, you can define the derivative in many ways.([6]) It will be better, of course, if you start with the intuitive meaning of the derivative, for example with the tangent line or the velocity, which is how it is usually done. But there is no need for limit theory here.

It doesn't mean that later, when you want, or when the curriculum requires it in a really reasonable way, and when your students really have to get familiar with limits, you cannot explain that in fact the derivative is such-and such a limit. But at the beginning it is absolutely unnecessary, and many students don't ever need it.

In short, I would suggest the following approach to the understanding of analysis that for convenience I will call *naively-axiomatic.* The idea of this approach is that, you define the notions, which you are interested in, essentially by axioms.

For example, for the integral the axioms are the following.

The integral of a constant is the product of this constant by the length of the interval of integration. Of course, you don't pull this axiom out of the thin air. At the beginning you will talk about the area. Everything will be prepared. And only then you formulate this axiom number one.

Axiom number two: if one function is not greater than the other at every point, then the integral of the first function is not greater than the integral of the second.

And axiom number three: if you integrate a function over the interval that is the union of two smaller non-overlapping intervals, then the corresponding integral will be the sum of the integrals over these smaller parts.

That's it! It is difficult to think of a less complicated approach. Of course, about the area we can say these things right away, as everyone is used to the area, about integrals we can say it a little later, but, based on these properties, all the integrals can be calculated beautifully. In fact, that's how they are calculated in all the textbooks.

Moreover, this approach immensely simplifies all the applications of the notion of integral in natural sciences and in mathematics itself. If you want, for example, to show that some volume is expressed as some integral, you simply check that all these three properties are satisfied. You don't have to prove anything. Since you have uniqueness, the volume turns out to be an integral, and you get the formula at once, whether you deal with the volume of a solid of revolution or of volumes in other situations.

The same applies to torques, calculations of centers of gravity, moments of inertia and all the other mechanical and physical quantities. There is no need for any "passing to the limit" and in all the long and boring presentation that textbooks for technical colleges are full of. There is no need for this at all.

I am just giving some examples, as you see. It is possible to give many more. If we want to talk seriously, we have to admit that a course of mathematics for technical colleges simply has not been created, not put together yet. Not in the sense that there are no curricula or no textbooks. There are curricula, there are textbooks. By a course I mean something else.

To avoid any misunderstanding, I will tell briefly what I mean by the expression "to create a course." Imagine a course guide. Of course, it would be longer, than a list of the contents of the course, but shorter than an actual textbook. It would be a guide from which a competent person could learn precisely what to present and how to present it, including all details.

It is not necessary that students should understand this guide. It has to be understandable to their teachers. Unfortunately, if such a guide were written for high schools, it would not be understood by the teachers. But such a guide, written down or kept in one's head is the substance of the course.

What I want to say is that a course in mathematics that is composed in this sense does not exist yet. Of course, there are many possible approaches to achieving it; I hope that such courses will be composed and that the corresponding textbooks will be written.

The few remarks that I have made about integral calculus, apply not only to it. I think that even the university math majors would benefit from taking a one semester preliminary course in introductory analysis in which the basic notions were introduced not from the point of view of mathematical hairsplitting, but were described meaningfully, with the view towards applications, with a discussion of the geometric and physical meaning and with plenty of material for exercises. After this, a math major can begin a more systematic and definitive study of the subject.

To some extent, such experiments were done. I don't know what the situation now here is, at the faculty of mathematics and mechanics at Leningrad State University. I don't know if such things were done here.

In any case, it seems to me that this naively-axiomatic approach could be useful at the beginning even in teaching professional mathematicians. However, I think it is absolutely necessary for teaching mathematics to non-mathematicians.

Now I will say a few words about the more advanced parts of the course. Indeed, some non-mathematicians are taught not only calculus and infinite series. They are taught, for example, integrals over curves and surfaces, change of variables in multiple integrals, etc.

These things are already not so straightforward and pleasant, they may present some technical difficulties. What should one do about them? Here one cannot rely solely on the naively-axiomatic method. Instead we have to look for compromises.

Very recently I have discussed similar matters with a professor from Leningrad who had to explain to his students the change of variables in double integrals. How should he do this? Being a mathematician, this teacher doesn't want to swindle anyone. He is ashamed to swindle anyone, including his students. He just wants to prove something.

However, change of variables in double integrals is a rather complicated subject. Several different methods are possible. We can present our integral as an iterated integral and use the formula for change of variables in a one-dimensional integral. This is one way; there are many other ways. Here, as it looks to me, one habit dominates, the habit of a professional mathematician to prove everything.

But really, what should we teach a future engineer, or a future physicist, to say nothing about a future philosopher? First and foremost we have to teach understanding. The students have to understand the subject. We have to, (and this is probably the most important thing), stop teaching things that our students cannot understand or stop teaching in ways that the students do not understand. The same is true when teaching high school students.

It is very common to teach students things that they do not understand. Well, what is the good if a student at a technical college will be exposed to a proof (a proof in a rather restricted sense, of course) of the formula for change of variables in double integrals? Wouldn't it be better if the student would understand this formula, at least intuitively?

For example, could you explain, first, to the students how area behaves under a linear transformation? Here is the plane, and a linear transformation is applied to it. How does the area of a triangle behave, or the area of a polygon? You probably can explain that. You can also explain that the area of a small region behaves approximately in the same manner also under a smooth nonlinear transformation. So the appearance of the Jacobian will not be surprising to anybody. Of course, one has to get used to the Jacobian first.

[Here the recording was interrupted, probably for a tape change. In the lost piece of the lecture there was a discussion of the implicit function theorem.]

... axioms that establish the equivalence of different approaches to, say, defining a surface in the space. A surface in the space can be defined parametrically, by three equations, it can be defined "implicitly" by one equation, and finally, it can be defined as a graph of a function. All these three approaches are equivalent, and this equivalence is established by the theory of implicit functions.

To my surprise, few people know about it, (even among the student math majors) before they actually meet it, in some other subject where it is used. In analysis courses it is very rarely discussed. I haven't seen it in textbooks, either. The connection of this stuff to mappings is discussed in very few places. But maybe to the students of a technical college all this can be explained without hairsplitting, so to speak, in such a way that they could understand (through some understandable examples, by some general formulations). To summarize, it looks like, in addition to our naively-axiomatic approach that I talked about earlier, we need also greater freedom of dealing with the subject matter, when we teach mathematics to non-mathematicians.

Maybe a teacher should try to recall how he experienced all this material when he was learning. We usually forget such things. For example, I can say about myself that I absolutely don't remember what and how I learned in high school. Maybe I remembered when I was a university student, but then I forgot it, little by little. Maybe it is necessary to study this problem also by involving the students in a discussion and by listening to what they have to say. We don't do enough of that.

As an example, I mention the following observation. Without any doubts, students that take this or that class give grades to their professors. They don't put these grades into professors' transcripts, but they do grade their teachers. More than that, each professor or teacher has a rather stable average grade. It's like the rating in chess, so to speak. Unlike the chess rating, this rating is not published and even usually kept secret. I will not discuss whether having ratings is good or bad. I think everybody knows whether it is good or bad. But undoubtedly, the help of the students is invaluable here. ([7])

I think student's help is also invaluable in composing a course that they take. Of course, the generations change, some students will help us to compose a course, and we will teach their successors. Nevertheless, I think that this activity, undertaken with such arrogance by the professors, cannot be successful without help of their students. The students should be consulted first and foremost.

I nave recently met a girl who attends a high school and lives next door, my next door neighbor. I had invited her to visit me a while ago. She came one evening an said: "Here I am, you called me, and I came. There is a problem that I cannot solve" (laughter)

Well, I started to look at the problem and was horrified. First of all, it is impossible to solve this problem, because it is not clear what is being asked. Professionally speaking, the problem was written ungrammatically. One could guess, of course, what the author of the problem wanted. We started guessing. But then I discovered, to my surprise, that I was guessing, but she understood everything from the very beginning. (laughter) More than that, when I started to solve the problem, it turned out that she knew very well everything that I was talking about. "She knows everything" I thought at first. But after we talked a bit more, it turned out that she did not understand anything. She knew all the words. And it turned out that these words sufficed for solving the problem. And she knew all of them.

This kind of experience is invaluable for a teacher and for people who compose curricula. Keep in mind that we are talking about education for the masses, not about teaching children who attend special schools, who have heard mathematical language and speak it themselves. No, these children are not taught mathematics professionally and have no intentions to work in it. They hear all these words, they learn to pronounce them, but their understanding of these words is somewhat strange; they either have no understanding at all, or it is strange. In many cases you discover some really unexpected understanding.

It turned out that this girl and I understood some words very differently. Well, clearly we should wonder how her teacher understood all these words. And that is the whole problem, of course.

Let me draw your attention to yet another peculiarity of the teaching of mathematics and of how this teaching is perceived by the students. This peculiarity is the following. The teaching is performed largely as a form of magic, one could almost call it an occult process.

Many years ago I saw a program of an entrance exam to universities. That program had been approved and signed by top officials, and in that program I saw something very strange. The topic was solving first degree equations with one unknown and there were two types of equations:

One was $ax = b$, and the other was $ax + b = 0$. ([8])

(Laughter) These are different types! What was the issue?

Some other girl helped me to figure it out. From a conversation with her I understood that all the numbers are positive. (Laughter) The negative numbers don't exist, and, in fact, a negative number is the following. It is a positive number in front of which there, for everyone to see, stands the minus sign. (Laughter) But if that is so, then, of course, these equations are of different types. Indeed, if you move b to the other side of the equation, you have to change the sign, but all the numbers must be positive! So, a must be positive, b must be positive, clearly these are two types of equations.

But how had this stuff found a way to the program? Well, it got there from the high school program. There was a requirement that the program of the entrance exams should not differ from the high school program. If not, then what would happen? How could one take such exam? Now, from the school program it had migrated into the entrance exam program.

And how did it get into the school program? Well, it's clear how. The school programs are developed by specialists in mathematics education who know how to teach mathematics, and tell us how to do it. Well, they know how to teach mathematics, but they really think that all the numbers are positive. (Laughter in the hall.)

Of course, no teacher - well, maybe there are very few, but no ordinary teacher does tell in the class that all numbers are positive. It is not written in a textbook, so how could a teacher say this? But he thinks so! And if he does, so will do his students. How is it transmitted to them? Of course, this is an interesting question, but it is true beyond any doubt.

The understanding of the subject by a teacher is passed to his students. The understanding of the subject by a lecturer is passed to his listeners. It is transmitted in a mysterious way, but very reliably. We have to keep this in mind. No extra education of the teacher, no correct presentation in the textbook or curriculum can help if the teacher thinks differently.

The teacher, the instructor is, in this sense, the central figure, the decisive figure. I want to repeat again that I absolutely do not believe that one can somehow improve, or change teaching by improving programs, textbooks, but without changing, very seriously, the training of the teachers.

True, there are some methods of retraining, various continuous education programs and so on. How effective are these? I don't have any factual data. I have only my personal experience and the personal experience of my friends. And here I have to express a rather grim prognosis. According to my observations, any retraining or additional training of individuals who had learned mathematics at some institution of higher education, such as a pedagogical university, gives nothing.

If during all their student years they learned nothing, if during the following long period of teaching they managed to ... I am afraid to say forget what they learned, it would have been better. If they have managed to ... (well, o.k.) forget what they had learned, then, of course, additional training would lead only to superficial changes. They would get used to new words, to new teaching methods, but it would not change anything that matters.

It seems to me that the universal teaching of mathematics can be improved only in one way. It is a slow, long and difficult way, but it may be possible. The way is to increase gradually the preparation of qualified teachers.

There are certainly enough capable people available. Unfortunately, we don't teach them properly. The reason is understandable: there are not enough teachers to teach them well.

In the good old days, when all the mathematics teachers in high schools were graduates of the top notch universities (it was long ago, before there was universal education), the situation was better. We often hear and read that long ago mathematics teachers were better. They knew their subject better and taught better. I don't know if it is true, but if it is, the reason is that these teachers studied not in pedagogical universities, but in a *few* top quality universities. Now they study in pedagogical universities and *many* other mediocre universities.

Well, I would not want this lecture to leave a grim memory. I want to say something optimistic at the end. I think that if we need a hundred years to prepare gradually a sufficient number of qualified mathematics teachers and, to the delight of high schools, colleges and universities, to build a good mathematics educational system, if a hundred years were enough for that, then it would be good. (Liveliness in the hall).

Notes by Oleg Viro

([1]) Recall that a fraction is called proper if its numerator is less than its denominator.

([2]) Perhaps, at the beginning of his lecture (which was lost), Rokhlin talked about the harm made by universally accepted methods of teaching mathematics. Unfortunately, only his criticism of the principle that any rule should be immediately illustrated by examples of its application survived.

([3]) In fact, Rokhlin said "students of Mat-Mekh" instead of math majors. Mat-Mekh is an abbreviation for the faculty of mathematics and mechanics of Leningrad State University. Students of Mat-Mekh were of various levels, but in average they were more used to solve problems than our math majors are. However, according to my experience, the picture presented by Rokhlin could be seen in a class of Mat-Mekh students and in a class of our math majors.

([4]) Here Rokhlin referred to the Soviet realities of that time which were commonly known then, but are completely forgotten now. Almost all the stores had a one hour lunch break, all grocery stores were closed for a lunch between 1 and 2pm, all pharmacies were closed between 2 and 3pm.

([5]) This "upside-down" approach is accepted in many modern calculus textbooks, for example, in the textbooks by James Stewart used in Stony Brook University for the main calculus courses MAT 125 - 132. Here is the definition 5.1.2 from Stewart's *Single Variable Calculus*:
"The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$" Notice that the partition of S is assumed to be into strips of equal width. This allows to speak legitimately about the limit of a sequence, but deprives of a flexibility that could simplify calculations (say, an easy calculation of $\int x^n dx$ with any n, which becomes possible if the widths of strips form a geometric sequence). There are calculus textbooks which are more reasonable. In some of them, the area is not defined as a limit, but the integral is. This is what happens in *Calculus* by Jon Rogawski used in MAT 171 at Stony Brook. The limit used there in the definition of the integral is not a limit of sequence, but rather a limit over all partitions. The flexibility was saved (although not used), but at the cost of a misleading reference: this kind of limit (i.e., limits of nets) is used in Chapter 5 and defined much later, in Chapter 16, which is devoted to multiple integrals.

([6]) For example, the derivative can be defined by two simple axioms (or by two natural properties):
(1) The derivative of a linear function $ax + b$ equals a at every point.
(2) If a function f at point x_0 grows faster than any linear function $ax+f(x_0)$ if $a < d$ and slower if $a > d$, then $f'(x_0) = d$.
We say that f grows faster than g at x_0, if $f(x) - g(x)$ changes the sign at x_0 from negative to positive (i.e. $f(x) - g(x) < 0$ for some $L < x_0$ and all $x \in (L, x_0)$ and $f(x) - g(x) > 0$ for some $M > x_0$ and all $x \in (x_0, M)$.
See *Calculus Unlimited* by J. Marsden and A. Weinstein.

([7]) Student evaluation did not exist in the Soviet Union.

([8]) I do remember the questions. They were in the programs for the oral final exam of middle school in 1963, and for the oral exam for entrance to the university in 1966. Then I was surprised, but did not realize the reason. Later I was absolutely convinced by Rokhlin's explanation.