MAT 331-Fall 20: Project 4

In this project, you can use any code from the previous homeworks. The libraries you are allowed to use are:

1. The math library.
2. The string library.
3. The cmath library
4. The linear algebra package linalg inside numpy.

Any external libraries other than the above mentioned are not allowed. You will give turn back,

1. Your python code
2. Your math project containing the proofs and details about your methods.

Definition 1. *Given two vector $u, v \in \mathbb{R}^2$, we say that w is a positive linear combination of u, v if $w = au + bv$ where $a, b \geq 0$.*

Definition 2. *Given n points $p_1, \ldots, p_n$ in the plane, the convex enveloppe of these points is the unique convex polytope containing these n points, it is denoted $Conv(p_1, \ldots, p_n)$.*

For the first exercice, we will also need the following notion.

Definition 3. *Given 3 points $M_1, M_2, M_3 \in \mathbb{R}^2$ which are not on the same line, we say that M is in the convex hull of the three points M_i if $M \in Conv(M_1, M_2, M_3)$.*

Given three points $M_1, M_2, M_3 \in \mathbb{R}^2$ which are not on the same line, the barycentric coordinates $(a, b, c) \in \mathbb{R}^3$ of M associated to (M_1, M_2, M_3) is defined by

$$a\overrightarrow{MM_1} + b\overrightarrow{MM_2} + c\overrightarrow{MM_3} = 0.$$

Definition 4. *Given P, Q two polytopes in $\mathbb{R}^2$, the Minkowski sum, denoted $P + Q$ is the set:*

$$P + Q = \{p + q | x \in P, y \in Q\}. \tag{1}$$

The main object of this project is the mixed volumes between two polytopes P, Q, denoted by $V(P, Q)$. They are given as the coefficients of the polynomial in $\lambda, \lambda' >> 1$:

$$\text{Area}(\lambda P + \lambda' Q) = \lambda^2 \text{Area}(P) + \lambda\lambda' V(P, Q) + \lambda'^2 \text{Area}(Q). \tag{2}$$

Exercice 1. *(Convex polytopes)]We start with any n points $p_1, \ldots, p_n \in \mathbb{R}^n$ and denote by P the polytope $Conv(p_1, \ldots, p_n)$.*

(1) The purpose of this question is to determine the boundary vertices of P.

(a) (5 points) Show that for all M_1, M_2, M_3 which are not on the same line, $M \in Conv(M_1, M_2, M_3) \Leftrightarrow$ the barycentric coordinates (a, b, c) of M associated to (M_1, M_2, M_3) are all non-negative.

*(b) (5 points) Write a function called **extremal** that takes a list of points $p_1, \ldots, p_n$ and returns a list of the vertices of the polytope $P = Conv(p_1, \ldots, p_n)$.*

(c) (10 points) Explain the argument needed in the previous question and how you proceeded.

(2) Given a vertex p of P which is a point among $p_1, \ldots, p_n$. The purpose of this question is to find the two edges e_1, e_2 of P containing p.

*(a) (5 points) We take two vectors $u, v \in \mathbb{R}^2$ that are not parallel. Write a function called **positive_linear** that tells whether a vector w is a positive linear combination of u, v.*

*(b) (5 points) Write a function **extremal_edges** that takes a vertex p of P and returns the two other points p_i, p_j such that $[p, p_i]$ and $[p, p_j]$ form the two edges of P containing p.*

(c) (10 points) Explain the argument needed in the previous question and how you proceeded.

(d) (1 point) What can we say about the angle between the two edges e_1, e_2 ?

*(3) (5 points + 5 points) Write a function **cyclic_vertices** that takes a list of points $p_1, \ldots, p_n$ and returns a cyclic list of vertices of $P = Conv(p_1, \ldots, p_n)$ (in the math part, explain carefully your reasoning for 5 points).*

(4) In this question, we want to compute the area of P.

(a) (10 points) Given three points $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_1 = (x_3, y_3)$, give a formula for the area of the triangle $Conv(p_1, p_2, p_3)$ and explain how you obtained that formula ?

(b) (5 points) Given two convex subset $A, B \subset P$ such that $A \cup B = P$, determine an expression of $\mathrm{Area}(P)$ *in terms of the area of A, B and $A \cap B$.*

*(c) (5 points) Write a function **area_convex** that takes a list of points $p_1, \ldots, p_n$ and returns the area of the polytope $P = conv(p_1, \ldots, p_n)$.*

(5) (20 points) Write a function that takes two list of points $p_1, \ldots, p_n$ and $q_1, \ldots q_r$, and computes the mixed volume $V(P, Q)$ where $P = conv(p_1, \ldots, p_n)$ and $Q = conv(q_1, \ldots, q_r)$.

(6) (10 points) Explain the method you used to write the previous functions, as well as the intermediate calculations needed.