MAT 331-Fall 20: Project 3

In this project, you can use any code from the previous homeworks. The libraries you are allowed to use are:

1. The PIL library (to draw).
2. The math library.
3. The string library.
4. The cmath library

We fix a complex number $c \in \mathbb{C}$ and denote by $f_c(z) = z^2 + c$. We recall that the filled Julia set $K(f_c)$ is the set

$$K(f_c) = \{z_0 | (z_n)_n \text{ is bounded, where } z_{n+1} = f_c(z_n) \forall n\}.$$

For simplicity, we shall denote by $f_c^n(z_0) := z_n$. We define the function $G_c : \mathbb{C} \to \mathbb{R}_+$ as follows:

$$G_c(z) = \lim_{n \to +\infty} \frac{1}{2^n} \max(0, \log |f_c^n(z)|) \tag{1}$$

$$G_{c,n}(z) = \frac{1}{2^n} \max(0, \log |f_c^n(z)|).$$

The function G_c is called the Green function associated to f_c.

One wants to color the exterior of the Julia set to distinguish the level set of the function G_c. To that end, we will use two coloring function, $u, v : \mathbb{R}_+ \mapsto (\mathbb{Z}/(256\,\mathbb{Z}))^3$, given by the formulas:

$$u(x) = \begin{cases} (0,0,0) \text{ if } x = 0, \\ \left(128 + \lfloor \frac{10000x}{128} \rfloor, \lfloor \log\log |x| \rfloor, \lfloor 100 \cdot \log |x| \rfloor \right) \text{ otherwise} \end{cases}$$

and

$$v(x) = \left(\left\lfloor 255 \cdot \frac{1 + \cos(2\sqrt{2}\pi log|x|)}{2} \right\rfloor, \left\lfloor 255 \cdot \frac{1 + \cos(2\sqrt{3}\pi log|x|)}{2} \right\rfloor, \right.$$
$$\left. \left\lfloor 255 \cdot \frac{1 + \cos(2\sqrt{2}\pi \log log|x|)}{2} \right\rfloor \right),$$

if $x > 0$ and $v(0) = (0,0,0)$, Note that the three values between 0 and 255 correspond to the triple associated to a color in the RGB convention. We will set the following values of c:

$$c_0 = 0 c_1 = -0.123 + i0.745$$
$$c_2 = -1$$
$$c_3 = 0.3 + i0.5$$
$$c_4 = -0.8 + 0.156i$$

Exercice 1. *(Coloring the Julia set) Ideally, one wishes to color the exterior of the Julia by different colors depending on the values of $G_c(z)$. Namely associate to z the color $u(G_c(z))$ or $v(G_c(z))$.*

(1) (2 points) If $z \in K(f)$, show that $G_c(z) = 0$.

(2) (4 points) Assume that $|z_n|$ diverges to ∞.

(a) Express $log|z_{n+1}|$ in terms of z_n and c and deduce an expression of $G_{n+1}(z)$ in terms of $G_n(z)$ and $log|1 + c/z_n^2|$.

(b) Show that if $z \notin K(f)$, then the function $G_c(z) > 0$.

(3) (5 points) Write a program that draws the colors the exterior of $K(f_{c_0}), K(f_{c_1}), K(f_{c_2}), K(f_{c_3})$ and $K(f_{c_4})$ using the color function u.

(4) (5 points) Write a program that draws the colors the exterior of $K(f_{c_0}), K(f_{c_1}), K(f_{c_2}), K(f_{c_3})$ and $K(f_{c_4})$ using the color function v.

(4) (5 points) Does your previous program really colors any point $z \in \mathbb{C}$ as $u(G_c(z))$? If not explain the approximation you did and discuss the complexity of your code.

We recall that the Mandelbrot set is the set:

$$\mathcal{M} = \{c \in \mathbb{C} \mid (f_c^n(0))_n \text{ is bounded}\} \tag{2}$$

Similarly, one wishes to color the exterior of $\mathcal{M}$ as follows. For any point $c \in \mathbb{C}$, we color it by $u(G_c(0))$ or $v(G_c(0))$

Exercice 2. *(Coloring of the Mandelbrot set)*

(1) (2 points) Assume that $c \in \mathcal{M}$, show that $G_c(0) = 0$.

(2) (2 points) If $c \notin \mathcal{M}$, show that $G_c(0) > 0$.

(3) (5 points) Write a program that colors the exterior of $\mathcal{M}$ using the color function u.

(4) (5 points) Write a program that colors the exterior of $\mathcal{M}$ using the color function v.

(5) (10 points)This last question is more experimental. Choose a point $c' \in \mathbb{C}$ near the boundary of the Mandelbrot set. One wishes to produce a picture that shows more detail on the boundary on a square $S = \{c \in \mathbb{C} \mid \max(|Re(c - c')|, |Im(c - c')|) \leqslant 1/2\}$. Define an appropriate color function w which would capture the properties of the boundary of $\mathcal{M}$ on S.

(6) (10 points) On the previous question, explain carefully what you did to choose the function w and explain the advantages and defect of the picture of S you produced, what detail does your color function capture ?