MAT 331-Fall 20: Project 2

In this project, you can use any code from the previous homeworks. The libraries you are allowed to use are:

1. The PIL library (to draw).
2. The math library.
3. The string library.

Exercice 1. *(Apollonian packing) In this project, we will draw an Apollonian packing. We first prove a mathematical statement on circle packings.*

(1) Consider three circles C_0, C_1, C_2 in the complex plane which are mutually tangent. We show that there are exactly two other circles C_3, C_4 which are tangent to the 3 circles C_0, C_1, C_2. We consider the map $f_0 : z \mapsto 1/z$, it is called the inversion with respect to the point $0 \in \mathbb{C}$. In general the inversion with respect to a point $a \in \mathbb{C}$ is $f_a : z \mapsto a + 1/(z - a)$. One important fact about inversion we will admit is that if two curves are tangent, then their images by the inversion will also be tangent.

(1.a) (5 points) If C is a circle containing 0, show that $f_0(C)$ is a line (Hint: first treat the case where the center lies on the real axis).

(1.b) (5 points) If C is a circle such that 0 belongs to the interior of the disk bounded by C, then $f_0(C)$ is another circle satisfying the same property (Hint: first treat the case where the center lies on the real axis, write the equation of C and divide by $z\bar{z}$).

(1.c) (5 points) If C is a circle such that 0 belongs to the exterior of the disk bounded by C, then $f_0(C)$ is another circle satisfying the same property (Hint: first treat the case where the center lies on the real axis).

(1.d) (5 points) Given three mutually tangent circles C_0, C_1, C_2, show that there are exactly two circles C such that C is tangent to all three circles (Hint: Consider the image of these circles by the inversion with respect to a point in the intersection $C_0 \cap C_1$).

(2) (10 points) Consider the circle C_0 in the complex plane given by:

$$C_0 = \{z \in \mathbb{C} \mid \left| z - \frac{1}{2} \right| <= \frac{1}{2}\}. \tag{1}$$

Denote by D_0 the disk of radius $1/2$ whose boundary is determined by C_0. We consider three other circles $C_1, C_2, C_3 \subset D_0$ which are mutually tangent and such which are all tangent to C_0. These circle determine four regions R_0, R_1, R_2, R_3 where each R_i is delimited the arcs joining the points $C_j \cap C_k$ for $j, k \neq i$. For each region R_i, we construct a unique circle tangent to each circle C_j for all $j \neq i$, distinct from C_i and we obtain the Apollonian packing of order $n = 1$. We obtain 4 new circle, then repeat the process inductively on each of the 12 new regions. We obtain the following figure:

Write a program that takes in input two circles C_1, C_2 tangent to C_0 in the disk D_0, an integer n, and draws the Apollonian packing of order n.

(3) *(10 points) Explain carefully the intermediate calculations you needed to write the above program. Estimate the complexity of your algorithm as a function of the order n of the Apollonian gasket.*