MAT 331-Fall 20: Homework 3

The first exercices are purely programming exercices.

Exercice 1. *(a) (2 points) Using a for loop, write a function* ***for_ decreasing_ by_ three(n)*** *that takes an integer $n \geq 0$ and prints some numbers three by three, separated by a comma, in the decreasing order from n to $2, 1$ or 0.*
For example, the function $for\_decreasing\_by\_three(10)$ will print:
$10, 7, 4, 1, Finish.$

(b) (2 points) Using a while loop, write a function ***while_ decreasing_ by_ three(n)*** *which takes an integer $n \geq 0$ and does the same as the function* ***for_ decreasing_ by_ three(n)****.*

(c) (2 points) Using a recursive function, write a function ***rec_ decreasing_ by_ three(n)*** *that does the same as the above two functions.*

Exercice 2. *(a) (2 points) Using a for loop, write a function* ***for_ factorial(n)*** *that takes an integer n, prints* ***"Error"*** *if $n < 0$ and returns -1 or returns the factorial $n!$ if $n \geq 0$. (Note that by convention $0! = 1$).*

Exercice 3. *(A multiplication table) (2 points) Write a function* ***multiplication_ table(n)*** *which takes an integer $n \geq 1$ which prints the multiplication table for all integers between 1 and n.*
For example, the output of ***multiplication_ table(3)*** *will be:*

$$\begin{array}{ccc} 1, & 2, & 3 \\ 2, & 4, & 6 \\ 3, & 6, & 9. \end{array}$$

The last exercice will be in two parts. A programming part and a math part.

Exercice 4. *(Trapezoid integral with precision) We want to estimate the integral*

$$\int_a^b e^{-t^2} dt, \tag{1}$$

for $a, b \in \mathbb{R}$ using the trapezoidal method. We denote by $f(t)$ the function e^{-t^2}. The trapezoid integral at step n, denoted by $T_n(f)$, is obtained from f by subdividing the interval $[a, b]$ into n subintervals $[x_i = a + i(b-a)/n, x_{i+1} = a + (i+1)(b-a)/n]$ and taking the sum of the areas of the trapezoid whose corners are given by the points $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, f(x_{i+1})), (x_i, f(x_i))$.

(1) (2 points) Write an expression of $T_n(f)$.

(2) We want to estimate the difference

$$\left| \int_a^b f(t)dt - T_n(f) \right| \tag{2}$$

(2.a) (2 points) Using the mean value theorem, show that for any $a, b \in \mathbb{R}$, there exists $t \in [a, b]$ such that:

$$f'(t) = \frac{f(b) - f(a)}{b - a}. \tag{3}$$

(2.b) (2 points) The Taylor-Lagrange formula states at a that for all $t \in \mathbb{R}$, there exists $\theta \in [a, t]$ such that:

$$f(t) = f(a) + (t - a)f'(a) + \frac{(t - a)^2}{2} f''(\theta). \tag{4}$$

Using the Taylor Lagrange formula at a and b and question (2.a) show that:

$$\left| \int_a^b f(t) - \frac{f(a) + f(b)}{2} dt \right| \leqslant \frac{5(b - a)^3}{12} \max_{[a,b]} |f''(x)|. \tag{5}$$

(2.c) (2 points) By decomposing the interval into n subintervals and applying the previous inequality on each of these subintervals, prove that:

$$\left| \int_a^b f(t)dt - T_n(f) \right| \leqslant \frac{5(b - a)^3}{12n^2} \max_{[a,b]} |f''(x)| \tag{6}$$

(3) (2 points) Find the maximum value of $|f''(x)|$.

*(4) (2 points) Write a function **trapezoid(a,b,n)** that takes two real numbers a, b and an integer n and returns $T_n(f)$ the trapezoidal integral between a and b of step n.*

*(5) (2 points) Write a function **n_ epsilon(a,b , ϵ)** that take two real numbers a, b and $\epsilon > 0$ and returns an integer N_ϵ satisfying the condition*

$$\left| T_{N_\epsilon}(f) - \int_a^b f(t)dt \right| \leqslant \epsilon$$

*(6) (1 point) Write a function **trapezoid_ integral(a,b,ϵ)** that takes two real numbers a, b and $\epsilon > 0$ and returns the value of a trapezoidal integral $T_n(f)$ satisfying:*

$$\left| T_n(f) - \int_a^b f(t)dt \right| \leqslant \epsilon.$$