MAT 331-Fall 20:Solution for the math part of Homework 3

The last exercice will be in two parts. A programming part and a math part.

Exercice 1. *(Trapezoid integral with precision) We want to estimate the integral*

$$\int_a^b e^{-t^2} dt, \tag{1}$$

for $a, b \in \mathbb{R}$ using the trapezoidal method. We denote by $f(t)$ the function e^{-t^2}. The trapezoid integral at step n, denoted by $T_n(f)$, is obtained from f by subdividing the interval $[a, b]$ into n subintervals $[x_i = a + i(b-a)/n, x_{i+1} = a + (i+1)(b-a)/n]$ and taking the sum of the areas of the trapezoid whose corners are given by the points $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, f(x_{i+1})), (x_i, f(x_i))$.

(1) (2 points) Write an expression of $T_n(f)$.

(2) We want to estimate the difference

$$\left| \int_a^b f(t) dt - T_n(f) \right| \tag{2}$$

(2.a) (2 points) Using the mean value theorem, show that for any $a, b \in \mathbb{R}$, there exists $t \in [a, b]$ such that:

$$f'(t) = \frac{f(b) - f(a)}{b - a}. \tag{3}$$

(2.b) (2 points) The Taylor-Lagrange formula states at a that for all $t \in \mathbb{R}$, there exists $\theta \in [a, t]$ such that:

$$f(t) = f(a) + (t-a) f'(a) + \frac{(t-a)^2}{2} f''(\theta). \tag{4}$$

Using the Taylor Lagrange formula at a and b and question (2.a) show that:

$$\left| \int_a^b f(t) - \frac{f(a) + f(b)}{2} dt \right| \leqslant \frac{5(b-a)^3}{12} \max_{[a,b]} |f''(x)|. \tag{5}$$

(2.c) (2 points) By decomposing the interval into n subintervals and applying the previous inequality on each of these subintervals, prove that:

$$\left| \int_a^b f(t) dt - T_n(f) \right| \leqslant \frac{5(b-a)^3}{12n^2} \max_{[a,b]} |f''(x)| \tag{6}$$

(3) (2 points) Find the maximum value of $|f''(x)|$.

(4) (2 points) Write a function ***trapezoid(a,b,n)*** *that takes two real numbers a, b and an integer n and returns $T_n(f)$ the trapezoidal integral between a and b of step n.*

(5) *(2 points) Write a function* **n_ epsilon(a,b , ϵ)** *that take two real numbers a, b and $\epsilon > 0$ and returns an integer N_ϵ satisfying the condition*

$$\left|T_{N_\epsilon}(f) - \int_a^b f(t)dt\right| \leqslant \epsilon$$

(6) *(1 point) Write a function* **trapezoid_ integral(a,b,ϵ)** *that takes two real numbers a, b and $\epsilon > 0$ and returns the value of a trapezoidal integral $T_n(f)$ satisfying:*

$$\left|T_n(f) - \int_a^b f(t)dt\right| \leqslant \epsilon.$$

Solution to question (1). An expression of $T_n(f)$ is:

$$T_n(f) = \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{1}{2}\left(f\left(a+i\frac{b-a}{n}\right)+f\left(a+(i+1)\frac{b-a}{n}\right)\right).$$

□

Solution to question 2.a. This is exactly the statement of the mean value theorem. □

Solution to question 2.b . We fix $t \in [a, b]$. We write the Taylor Lagrange formula at a:

$$f(t) - f(a) = (t-a)f'(a) + \frac{(t-a)^2}{2}f''(\theta_t), \tag{7}$$

where $\theta_t \in [a, t]$. We write the Taylor-Lagrange at b:

$$f(t) - f(b) = (t-b)f'(b) + \frac{(t-b)^2}{2}f''(\gamma_t), \tag{8}$$

where $\gamma_t \in [t, b]$.

Set Δ to be the difference:

$$\Delta = \int_a^b \left(f(t) - \frac{f(a)+f(b)}{2}\right)dt. \tag{9}$$

Take $c \in [a, b]$ a point which satisfies the mean value theorem for f' on $[a, b]$:

$$f''(c) = \frac{f'(b) - f'(a)}{b-a}.$$

We have using the above two equations and setting $M = \max_{[a,b]} |f''(x)|$.

$$
\begin{aligned}
\Delta &= \int_a^b \frac{2f(t)-f(a)-f(b)}{2}dt \\
&= \int_a^b \frac{(f(t)-f(a))+(f(t)-f(b))}{2} \\
&= \frac{1}{2}\left(\int_a^b (t-a)f'(a)dt + \int_a^b \frac{(t-a)^2}{2}f''(\theta_t)dt + \int_a^b (t-b)f'(b)dt + \int_a^b \frac{(t-b)^2}{2}f''(\gamma_t)\right) \\
&= \frac{1}{2}\left(\frac{(b-a)^2}{2}f'(a) - \frac{(b-a)^2}{2}f'(b)\right) + \frac{1}{2}\left(\int_a^b \frac{(t-a)^2}{2}f''(\theta_t)dt + \int_a^b \frac{(t-b)^2}{2}f''(\gamma_t)\right) \\
|\Delta| &\leqslant \frac{1}{2}\left(\frac{(b-a)^2}{2}f'(a) - \frac{(b-a)^2}{2}f'(b)\right) + \frac{M}{4}\left(\int_a^b (t-a)^2dt + \int_a^b (t-b)^2dt\right) \\
|\Delta| &\leqslant \frac{(b-a)^3}{4}\frac{|f'(a)-f'(b)|}{b-a} + \frac{M}{4}\left(\int_a^b (t-a)^2dt + \int_a^b (t-b)^2dt\right) \\
|\Delta| &\leqslant \frac{(b-a)^3|f''(c)|}{4} + \frac{M}{4}\frac{2(b-a)^3}{3} \\
|\Delta| &\leqslant M(b-a)^3\left(\frac{1}{4}+\frac{1}{6}\right) \\
|\Delta| &\leqslant \frac{5M(b-a)^3}{12}.
\end{aligned}
$$

□

Solution of (3). We have:

$$f' = -2xe^{-x^2} = -2xf,$$

$$f''(x) = -2f - 2xf' = (4x^2-2)e^{-x^2}.$$

Observe that the function f'' is an even function. And the third derivative satisfies:

$$f'''(x) = 8xf + (4x^2-2)(-2xf) = (12x - 8x^3)e^{-x^2} = 4xe^{-x^2}(3-2x^2)$$

So the function $|f''(x)|$ is decreasing on $[0, 1/\sqrt{2}]$, then increasing on $[1/\sqrt{2}, \sqrt{3/2}]$ then decreasing on $[\sqrt{3/2}, +\infty]$. We deduce that

$$\max|f''(x)| = \max(f''(0), f''(\sqrt{3/2}))$$

□