3 | TECHNIQUES OF INTEGRATION

Figure 3.1 Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

Chapter Outline

Introduction

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer (see Example 3.49).

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

3.1 | Integration by Parts

Learning Objectives

- **3.1.1** Recognize when to use integration by parts.
- **3.1.2** Use the integration-by-parts formula to solve integration problems.
- **3.1.3** Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x \sin(x^2) dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x \sin x \, dx$ defies us. Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts**.

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation: $\int h'(x)dx = \int (g(x)f'(x) + f(x)g'(x))dx$.

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx.$$

Now we solve for $\int f(x)g'(x)dx$:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form

$$\int u \, dv = uv - \int v \, du.$$

Theorem 3.1: Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u \, dv = uv - \int v \, du. \tag{3.1}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 3.1

Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x \, dx$ to evaluate $\int x \sin x \, dx$.

Solution

By choosing $u = x,$ we have $du = 1dx$. Since $dv = \sin x\,dx,$ we get $v = \int \sin x\,dx = -\cos x$. It is handy to keep track of these values as follows:

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= 1dx & v &= \int \sin x\,dx = -\cos x. \end{aligned}$$

Applying the integration-by-parts formula results in

$$\begin{aligned} \int x \sin x\,dx &= (x)(-\cos x) - \int(-\cos x)(1dx) && \text{Substitute.} \\ &= -x\cos x + \int \cos x\,dx && \text{Simplify.} \\ &= -x\cos x + \sin x + C. && \text{Use } \int \cos x\,dx = \sin x + C. \end{aligned}$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts, we have $\int x \sin x\,dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x\,dx$. Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x\,dx = -\cos x,$ we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x\,dx &= (x)(-\cos x + K) - \int(-\cos x + K)(1dx) \\ &= -x\cos x + Kx + \int \cos x\,dx - \int Kdx \\ &= -x\cos x + Kx + \sin x - Kx + C \\ &= -x\cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x\cos x + \sin x + C$:

$$\begin{aligned} \frac{d}{dx}(-x\cos x + \sin x + C) &= (-1)\cos x + (-x)(-\sin x) + \cos x \\ &= x \sin x. \end{aligned}$$

Therefore, the antiderivative checks out.

Watch this **video (http://www.openstaxcollege.org/l/20_intbyparts1)** and visit this **website (http://www.openstaxcollege.org/l/20_intbyparts2)** for examples of integration by parts.

3.1 Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula with $u = x$ and $dv = e^{2x}\,dx$.

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym LIATE can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u.

The type of function in the integral that appears first in the list should be our first choice of u. For example, if an integral contains a logarithmic function and an algebraic function, we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in **Example 3.1** has a trigonometric function $(\sin x)$ and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

Example 3.2

Using Integration by Parts

Evaluate $\int \frac{\ln x}{x^3} dx$.

Solution

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3} dx = \int x^{-3} \ln x \, dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3} dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x} dx$. Also, $v = \int x^{-3} dx = -\frac{1}{2} x^{-2}$. Summarizing,

$$\begin{aligned} u &= \ln x & dv &= x^{-3} dx \\ du &= \frac{1}{x} dx & v &= \int x^{-3} dx = -\frac{1}{2} x^{-2}. \end{aligned}$$

Substituting into the integration-by-parts formula (**Equation 3.1**) gives

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= \int x^{-3} \ln x \, dx = (\ln x)\left(-\tfrac{1}{2}x^{-2}\right) - \int \left(-\tfrac{1}{2}x^{-2}\right)\left(\tfrac{1}{x}dx\right) \\ &= -\frac{1}{2}x^{-2} \ln x + \int \frac{1}{2}x^{-3} \, dx && \text{Simplify.} \\ &= -\frac{1}{2}x^{-2} \ln x - \frac{1}{4}x^{-2} + C && \text{Integrate.} \\ &= -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C. && \text{Rewrite with positive integers.} \end{aligned}$$

3.2 Evaluate $\int x \ln x \, dx.$

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

Example 3.3

Applying Integration by Parts More Than Once

Evaluate $\int x^2 e^{3x} \, dx.$

Solution

Using LIATE, choose $u = x^2$ and $dv = e^{3x} dx.$ Thus, $du = 2x\, dx$ and $v = \int e^{3x} dx = \left(\frac{1}{3}\right)e^{3x}.$ Therefore,

$$\begin{aligned} u &= x^2 & dv &= e^{3x} dx \\ du &= 2x\, dx & v &= \int e^{3x} dx = \tfrac{1}{3}e^{3x}. \end{aligned}$$

Substituting into **Equation 3.1** produces

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x} dx.$$

We still cannot integrate $\int \frac{2}{3}xe^{3x} dx$ directly, but the integral now has a lower power on $x.$ We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{3}e^{3x} dx.$ Thus, $du = dx$ and $v = \int \left(\frac{2}{3}\right)e^{3x} dx = \left(\frac{2}{9}\right)e^{3x}.$ Now we have

$$\begin{aligned} u &= x & dv &= \tfrac{2}{3}e^{3x} dx \\ du &= dx & v &= \int \tfrac{2}{3}e^{3x} dx = \tfrac{2}{9}e^{3x}. \end{aligned}$$

Substituting back into the previous equation yields

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x} dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

Example 3.4

Applying Integration by Parts When LIATE Doesn’t Quite Work

Evaluate $\int t^3 e^{t^2} dt.$

Solution

If we use a strict interpretation of the mnemonic LIATE to make our choice of $u,$ we end up with $u = t^3$ and $dv = e^{t^2}dt.$ Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}dt.$ However, since we can evaluate $\int te^{t^2}dx,$ we can try choosing $u = t^2$ and $dv = te^{t^2}dt.$ With these choices we have

$$\begin{array}{rclrcl} u & = & t^2 & dv & = & te^{t^2}dt \\ du & = & 2t\,dt & v & = & \int te^{t^2}dt = \frac{1}{2}e^{t^2}. \end{array}$$

Thus, we obtain

$$\begin{aligned} \int t^3 e^{t^2}dt &= \frac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2}2tdt \\ &= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C. \end{aligned}$$

Example 3.5

Applying Integration by Parts More Than Once

Evaluate $\int \sin(\ln x)dx.$

Solution

This integral appears to have only one function—namely, $\sin(\ln x)$ —however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1dx.$ (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x)dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)dx$ and $v = \int 1dx = x.$ After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1dx,$ making $du = -(1/x)\sin(\ln x)dx$ and $v = \int 1dx = x.$ Substituting, we have

$$\int \sin(\ln x)dx = x\sin(\ln x) - \left(x\cos(\ln x) - \int -\sin(\ln x)dx\right).$$

After simplifying, we obtain

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)dx.$ Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)dx$ again, we have

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right)\\ &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right)\\ &= \sin(\ln x).\end{aligned}$$

3.3 Evaluate $\int x^2 \sin x\, dx.$

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Theorem 3.2: Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du. \quad (3.2)$$

Example 3.6

Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0,\ 1]$.

Solution

This region is shown in **Figure 3.2**. To find the area, we must evaluate $\int_0^1 \tan^{-1} x\,dx$.

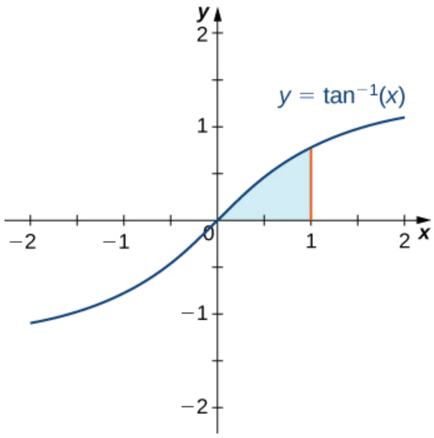


Figure 3.2 To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = \tan^{-1} x$ and $dv = dx$, thereby making $du = \frac{1}{x^2+1}dx$ and $v = x$. After applying the integration-by-parts formula (**Equation 3.2**) we obtain

$$\text{Area} = x\tan^{-1} x|_0^1 - \int_0^1 \frac{x}{x^2+1}dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\ln\left|x^2+1\right|\Big|_0^1.$$

Thus,

$$\text{Area} = x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln\left|x^2+1\right|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388$, and from **Figure 3.2** we expect our area to be slightly less than 0.5, this solution appears to be reasonable.

Example 3.7

Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (see the following graph).

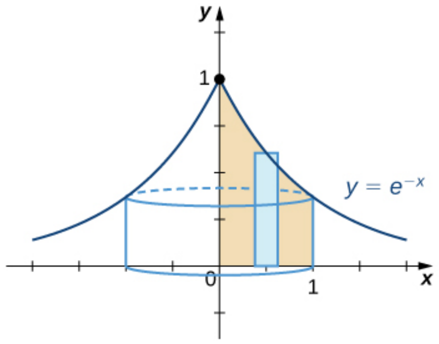


Figure 3.3 We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate $2\pi \int_0^1 xe^{-x}\,dx$. To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x} = -e^{-x}$. Substituting into **Equation 3.2**, we obtain

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^1 xe^{-x}\,dx = 2\pi\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) && \text{Use integration by parts.}\\ &= -2\pi xe^{-x}\Big|_0^1 - 2\pi e^{-x}\Big|_0^1 && \text{Evaluate } \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1.\\ &= 2\pi - \frac{4\pi}{e}. && \text{Evaluate and simplify.}\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{3}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2\frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} - \frac{\pi}{3} \approx 1.8177.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

3.4 Evaluate $\int_0^{\pi/2} x\cos x\,dx$.

3.1 EXERCISES

In using the technique of integration by parts, you must carefully choose which expression is *u*. For each of the following problems, use the guidelines in this section to choose *u*. Do **not** evaluate the integrals.

1. $\int x^3 e^{2x} dx$

2. $\int x^3 \ln(x) dx$

3. $\int y^3 \cos y dx$

4. $\int x^2 \arctan x\, dx$

5. $\int e^{3x} \sin(2x) dx$

Find the integral by using the simplest method. Not all problems require integration by parts.

6. $\int v \sin v dv$

7. $\int \ln x\, dx$ (*Hint:* $\int \ln x\, dx$ is equivalent to $\int 1 \cdot \ln(x) dx.$)

8. $\int x \cos x\, dx$

9. $\int \tan^{-1} x\, dx$

10. $\int x^2 e^x dx$

11. $\int x \sin(2x) dx$

12. $\int x e^{4x} dx$

13. $\int x e^{-x} dx$

14. $\int x \cos 3x\, dx$

15. $\int x^2 \cos x\, dx$

16. $\int x \ln x\, dx$

17. $\int \ln(2x + 1) dx$

18. $\int x^2 e^{4x} dx$

19. $\int e^x \sin x\, dx$

20. $\int e^x \cos x\, dx$

21. $\int x e^{-x^2} dx$

22. $\int x^2 e^{-x} dx$

23. $\int \sin(\ln(2x)) dx$

24. $\int cos(\ln x) dx$

25. $\int (\ln x)^2 dx$

26. $\int \ln(x^2) dx$

27. $\int x^2 \ln x\, dx$

28. $\int \sin^{-1} x\, dx$

29. $\int \cos^{-1}(2x) dx$

30. $\int x \arctan x\, dx$

31. $\int x^2 \sin x\, dx$

32. $\int x^3 \cos x\, dx$

33. $\int x^3 \sin x\, dx$

34. $\int x^3 e^x dx$

35. $\int x \sec^{-1} x\, dx$

36. $\int x \sec^2 x\, dx$

37. $\int x \cosh x\, dx$

Compute the definite integrals. Use a graphing utility to confirm your answers.

38. $\int_{1/e}^{1} \ln x \, dx$

39. $\int_{0}^{1} xe^{-2x} dx$ (Express the answer in exact form.)

40. $\int_{0}^{1} e^{\sqrt{x}} dx (\text{let } u = \sqrt{x})$

41. $\int_{1}^{e} \ln(x^2) dx$

42. $\int_{0}^{\pi} x \cos x \, dx$

43. $\int_{-\pi}^{\pi} x \sin x \, dx$ (Express the answer in exact form.)

44. $\int_{0}^{3} \ln(x^2 + 1) dx$ (Express the answer in exact form.)

45. $\int_{0}^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

46. $\int_{0}^{1} x5^x dx$ (Express the answer using five significant digits.)

47. Evaluate $\int \cos x \ln(\sin x) dx$

Derive the following formulas using the technique of integration by parts. Assume that *n* is a positive integer. These formulas are called *reduction formulas* because the exponent in the *x* term has been reduced by one in each case. The second integral is simpler than the original integral.

48. $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

49. $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

50. $\int x^n \sin x \, dx =$ ______

51. Integrate $\int 2x\sqrt{2x-3} dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3} \, dx$
b. Substitution, letting $u = 2x - 3$

State whether you would use integration by parts to evaluate the integral. If so, identify *u* and *dv*. If not, describe the technique used to perform the integration without actually doing the problem.

52. $\int x \ln x \, dx$

53. $\int \frac{\ln^2 x}{x} dx$

54. $\int xe^x dx$

55. $\int xe^{x^2-3} dx$

56. $\int x^2 \sin x \, dx$

57. $\int x^2 \sin(3x^3 + 2) dx$

Sketch the region bounded above by the curve, the *x*-axis, and $x = 1,$ and find the area of the region. Provide the exact form or round answers to the number of places indicated.

58. $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59. $y = e^{-x} \sin(\pi x)$ (Approximate answer to five decimal places.)

Find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

60. $y = \sin x,\ y = 0,\ x = 2\pi,\ x = 3\pi$ about the y-axis (Express the answer in exact form.)

61. $y = e^{-x}$ $y = 0,\ x = -1 x = 0;$ about $x = 1$ (Express the answer in exact form.)

62. A particle moving along a straight line has a velocity of $v(t) = t^2 e^{-t}$ after *t* sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63. Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1.$ (Round the answer to two significant digits.)

64. Find the area between $y = (x - 2)e^x$ and the *x*-axis from $x = 2$ to $x = 5.$ (Express the answer in exact form.)

65. Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. (Express the answer in exact form.)

66. Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

67. Find the volume of the solid generated by revolving the region bounded by the curve $y = 4\cos x$ and the x-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

68. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

3.2 | Trigonometric Integrals

Learning Objectives

3.2.1 Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
3.2.2 Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
3.2.3 Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in **Trigonometric Substitution**. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin*x* and cos*x*

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x\,dx$ or $\int \cos^j x \sin x\,dx$. After rewriting these integrals, we evaluate them using *u*-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example 3.8

Integrating $\int \cos^j x \sin x\,dx$

Evaluate $\int \cos^3 x \sin x\,dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x\,dx$. Thus,

$$\begin{aligned}\int \cos^3 x \sin x\,dx &= -\int u^3\,du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

3.5 Evaluate $\int \sin^4 x \cos x\,dx$.

Example 3.9

A Preliminary Example: Integrating $\int \cos^j x \sin^k x\,dx$ Where *k* is Odd

Evaluate $\int \cos^2 x \sin^3 x\, dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x\, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

$$\begin{aligned}\int \cos^2 x \sin^3 x\, dx &= \int \cos^2 x(1 - \cos^2 x)\sin x\, dx \quad \text{Let } u = \cos x; \text{ then } du = -\sin x\, dx.\\ &= -\int u^2\left(1 - u^2\right)du\\ &= \int \left(u^4 - u^2\right)du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.\end{aligned}$$

3.6 Evaluate $\int \cos^3 x \sin^2 x\, dx$.

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1 - \cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1 + \cos(2x)}{2}$$

are invaluable. These identities are sometimes known as *power-reducing identities* and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

Example 3.10

Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\, dx$.

Solution

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\begin{aligned}\int \sin^2 x\, dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx\\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

3.7 Evaluate $\int \cos^2 x\, dx$.

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of sin *x* and cos *x*

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (*Note*: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 3.11

Integrating $\int \cos^j x \sin^k x\, dx$ where *k* is Odd

Evaluate $\int \cos^8 x \sin^5 x\, dx$.

Solution

Since the power on $\sin x$ is odd, use strategy 1. Thus,

$$\begin{aligned}
\int \cos^8 x \sin^5 x\, dx &= \int \cos^8 x \sin^4 x \sin x\, dx && \text{Break off } \sin x. \\
&= \int \cos^8 x (\sin^2 x)^2 \sin x\, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2. \\
&= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx && \text{Substitute } \sin^2 x = 1 - \cos^2 x. \\
&= \int u^8 (1 - u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x\, dx. \\
&= \int \left(-u^8 + 2u^{10} - u^{12}\right) du && \text{Expand.} \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C && \text{Evaluate the integral.} \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C. && \text{Substitute } u = \cos x.
\end{aligned}$$

Example 3.12

Integrating $\int \cos^j x \sin^k x\,dx$ where k and j are Even

Evaluate $\int \sin^4 x\,dx$.

Solution

Since the power on $\sin x$ is even $(k=4)$ and the power on $\cos x$ is even $(j=0)$, we must use strategy 3. Thus,

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2 dx && \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2. \\
&= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x). \\
&= \int \left(\tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\cos^2(2x)\right)dx && \text{Expand} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)^2. \\
&= \int \left(\tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)dx.
\end{aligned}$$

Since $\cos^2(2x)$ has an even power, substitute $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$:

$$\begin{aligned}
&= \int \left(\tfrac{3}{8} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x)\right)dx && \text{Simplify.} \\
&= \tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

3.8 Evaluate $\int \cos^3 x\,dx$.

3.9 Evaluate $\int \cos^2(3x)dx$.

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x) \tag{3.3}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x) \tag{3.4}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x) \tag{3.5}$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 3.13

Evaluating $\int \sin(ax)\cos(bx)dx$

Evaluate $\int \sin(5x)\cos(3x)dx$.

Solution

Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)$. Thus,

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \int \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\sin(8x) + C.\end{aligned}$$

3.10 Evaluate $\int \cos(6x)\cos(5x)dx$.

Integrating Products and Powers of tan*x* and sec*x*

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\, dx = \tan x + C$
2. $\int \sec x \tan x\, dx = \sec x + C$
3. $\int \tan x\, dx = \ln|\sec x| + C$
4. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\, dx$ or $\int \sec^j x \tan x\, dx$. As we see in the following example, we can evaluate these new integrals by using *u*-substitution.

Example 3.14

Evaluating $\int \sec^j x \tan x\, dx$

Evaluate $\int \sec^5 x \tan x\, dx$.

Solution

Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

$$\begin{aligned} \int \sec^5 x \tan x\,dx &= \int \sec^4 x \sec x \tan x\,dx && \text{Let } u = \sec x;\ \text{then, } du = \sec x \tan x\,dx. \\ &= \int u^4\,du && \text{Evaluate the integral.} \\ &= \tfrac{1}{5}u^5 + C && \text{Substitute } \sec x = u. \\ &= \tfrac{1}{5}\sec^5 x + C \end{aligned}$$

You can read some interesting information at this **website (http://www.openstaxcollege.org/l/20_intseccube)** to learn about a common integral involving the secant.

3.11 Evaluate $\int \tan^5 x \sec^2 x\,dx$.

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\,dx$

To integrate $\int \tan^k x \sec^j x\,dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x\,dx$. (*Note*: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example 3.15

Integrating $\int \tan^k x \sec^j x\,dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x\,dx$.

Solution

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\,dx &= \int \tan^6 x\left(\tan^2 x + 1\right)\sec^2 x\,dx && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\
&= \int u^6\left(u^2 + 1\right)du && \text{Expand.} \\
&= \int (u^8 + u^6)du && \text{Evaluate the integral.} \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C && \text{Substitute } \tan x = u. \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.
\end{aligned}$$

Example 3.16

Integrating $\int \tan^k x \sec^j x\,dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\,dx$.

Solution

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\tan^5 x \sec^3 x &= \tan^4 x \sec^2 x \sec x \tan x. && \text{Write } \tan^4 x = (\tan^2 x)^2. \\
\int \tan^5 x \sec^3 x\,dx &= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\,dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\
&= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\,dx && \text{Let } u = \sec x \text{ and } du = \sec x \tan x\,dx. \\
&= \int (u^2 - 1)^2 u^2\,du && \text{Expand.} \\
&= \int \left(u^6 - 2u^4 + u^2\right)du && \text{Integrate.} \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C && \text{Substitute } \sec x = u. \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.
\end{aligned}$$

Example 3.17

Integrating $\int \tan^k x\,dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\,dx$.

Solution

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x - 1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}\int \tan^3 x\, dx &= \int (\tan x \sec^2 x - \tan x) dx \\ &= \int \tan x \sec^2 x\, dx - \int \tan x\, dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

Example 3.18

Integrating $\int \sec^3 x\, dx$

Integrate $\int \sec^3 x\, dx$.

Solution

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int \tan x \sec x \tan x\, dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\, dx && \text{Simplify.} \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx && \text{Rewrite.} \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx. && \text{Evaluate } \int \sec x\, dx.\end{aligned}$$

We now have

$$\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx.$$

Since the integral $\int \sec^3 x\, dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\, dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

3.12 Evaluate $\int \tan^3 x \sec^7 x\, dx$.

Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To make the process easier, we can derive and apply the following **power reduction formulas**. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \quad (3.6)$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \quad (3.7)$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example 3.19

Revisiting $\int \sec^3 x\,dx$

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution

By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example 3.20

Using a Reduction Formula

Evaluate $\int \tan^4 x\,dx$.

Solution

Applying the reduction formula for $\int \tan^4 x\,dx$ we have

$$\begin{aligned}\int \tan^4 x\,dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\ &= \frac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x\,dx\right) && \text{Apply the reduction formula to} \int \tan^2 x\,dx. \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C. && \text{Evaluate} \int 1\,dx.\end{aligned}$$

3.13 Apply the reduction formula to $\int \sec^5 x\,dx$.

3.2 EXERCISES

Fill in the blank to make a true statement.

69. $\sin^2 x + \_\_\_\_\_\_\_ = 1$

70. $\sec^2 x - 1 = \_\_\_\_\_\_\_$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

71. $\sin^2 x = \_\_\_\_\_\_\_$

72. $\cos^2 x = \_\_\_\_\_\_\_$

Evaluate each of the following integrals by *u*-substitution.

73. $\int \sin^3 x \cos x\, dx$

74. $\int \sqrt{\cos x} \sin x\, dx$

75. $\int \tan^5(2x)\sec^2(2x)dx$

76. $\int \sin^7(2x)\cos(2x)dx$

77. $\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

78. $\int \tan^2 x \sec^2 x\, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

79. $\int \sin^3 x\, dx$

80. $\int \cos^3 x\, dx$

81. $\int \sin x \cos x\, dx$

82. $\int \cos^5 x\, dx$

83. $\int \sin^5 x \cos^2 x\, dx$

84. $\int \sin^3 x \cos^3 x\, dx$

85. $\int \sqrt{\sin x} \cos x\, dx$

86. $\int \sqrt{\sin x} \cos^3 x\, dx$

87. $\int \sec x \tan x\, dx$

88. $\int \tan(5x)dx$

89. $\int \tan^2 x \sec x\, dx$

90. $\int \tan x \sec^3 x\, dx$

91. $\int \sec^4 x\, dx$

92. $\int \cot x\, dx$

93. $\int \csc x\, dx$

94. $\int \frac{\tan^3 x}{\sqrt{\sec x}}dx$

For the following exercises, find a general formula for the integrals.

95. $\int \sin^2 ax \cos ax\, dx$

96. $\int \sin ax \cos ax\, dx.$

Use the double-angle formulas to evaluate the following integrals.

97. $\int_0^{\pi} \sin^2 x\, dx$

98. $\int_0^{\pi} \sin^4 x\, dx$

99. $\int \cos^2 3x\, dx$

100. $\int \sin^2 x \cos^2 x\, dx$

101. $\int \sin^2 x\, dx + \int \cos^2 x\, dx$

102. $\int \sin^2 x \cos^2(2x)dx$

For the following exercises, evaluate the definite integrals. Express answers in exact form whenever possible.

103. $\int_0^{2\pi} \cos x \sin 2x\, dx$

104. $\int_0^{\pi} \sin 3x \sin 5x\, dx$

105. $\int_0^{\pi} \cos(99x)\sin(101x)dx$

106. $\int_{-\pi}^{\pi} \cos^2(3x)dx$

107. $\int_0^{2\pi} \sin x \sin(2x)\sin(3x)dx$

108. $\int_0^{4\pi} \cos(x/2)\sin(x/2)dx$

109. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}dx$ (Round this answer to three decimal places.)

110. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\, dx$

111. $\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\, dx$

112. Find the area of the region bounded by the graphs of the equations $y = \sin x,\ y = \sin^3 x,\ x = 0,$ and $x = \frac{\pi}{2}.$

113. Find the area of the region bounded by the graphs of the equations $y = \cos^2 x,\ y = \sin^2 x,\ x = -\frac{\pi}{4},$ and $x = \frac{\pi}{4}.$

114. A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t).$ Find its position function $x = f(t)$ if $f(0) = 0.$

115. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi].$

For the following exercises, solve the differential equations.

116. $\frac{dy}{dx} = \sin^2 x.$ The curve passes through point $(0, 0).$

117. $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

118. Find the length of the curve $y = \ln(\csc x),\ \frac{\pi}{4} \le x \le \frac{\pi}{2}.$

119. Find the length of the curve $y = \ln(\sin x),\ \frac{\pi}{3} \le x \le \frac{\pi}{2}.$

120. Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, $0 \le x \le \frac{\pi}{36}.$

For the following exercises, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g dx.$ Two distinct functions f and g are said to be orthogonal if $\langle f, g \rangle = 0.$

121. Show that $\{\sin(2x), \cos(3x)\}$ are orthogonal over the interval $[-\pi, \pi].$

122. Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx.$

123. Integrate $y' = \sqrt{\tan x}\sec^4 x.$

For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

124. $\int \sin^{456} x \cos x\, dx$ or $\int \sin^2 x \cos^2 x\, dx$

125. $\int \tan^{350} x \sec^2 x\, dx$ or $\int \tan^{350} x \sec x\, dx$

3.3 | Trigonometric Substitution

Learning Objectives

3.3.1 Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of **trigonometric substitution** comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2 - x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2 - x^2}$, consider the integral $\int \sqrt{9 - x^2}dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta d\theta$. After substituting into the integral, we have

$$\int \sqrt{9 - x^2}dx = \int \sqrt{9 - (3\sin\theta)^2}3\cos\theta d\theta.$$

After simplifying, we have

$$\int \sqrt{9 - x^2}dx = \int 9\sqrt{1 - \sin^2\theta}\cos\theta d\theta.$$

Letting $1 - \sin^2\theta = \cos^2\theta$, we now have

$$\int \sqrt{9 - x^2}dx = \int 9\sqrt{\cos^2\theta}\cos\theta d\theta.$$

Assuming that $\cos\theta \ge 0$, we have

$$\int \sqrt{9 - x^2}dx = \int 9\cos^2\theta d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2 - x^2}$, we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2 - x^2}$ is $[-a, a]$. Thus, $-a \le x \le a$. Consequently, $-1 \le \frac{x}{a} \le 1$. Since the range of $\sin x$ over $[-(\pi/2), \pi/2]$ is $[-1, 1]$, there is a unique angle θ satisfying $-(\pi/2) \le \theta \le \pi/2$ so that $\sin\theta = x/a$, or equivalently, so that $x = a\sin\theta$. If we substitute $x = a\sin\theta$ into $\sqrt{a^2 - x^2}$, we get

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}. \text{ Simplify.} \\
&= \sqrt{a^2 - a^2\sin^2\theta} && \text{Factor out } a^2. \\
&= \sqrt{a^2(1 - \sin^2\theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x. \\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.} \\
&= |a\cos\theta| \\
&= a\cos\theta.
\end{aligned}$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in **Figure 3.4** to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2 - x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

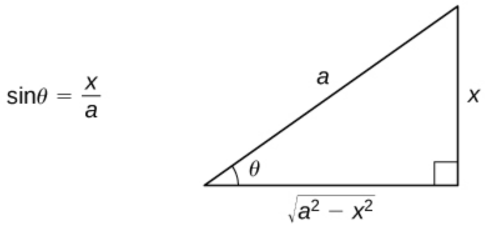


Figure 3.4 A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}}dx$, $\int \frac{x}{\sqrt{a^2 - x^2}}dx$, and $\int x\sqrt{a^2 - x^2}dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$. *Note*: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.4** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 3.21

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \sqrt{9 - x^2}dx$.

Solution

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in the following figure.

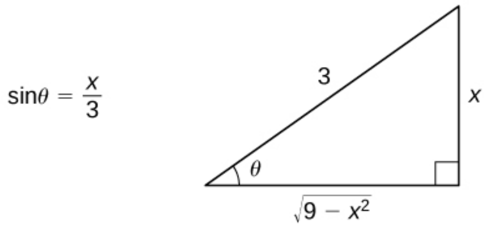


Figure 3.5 A reference triangle can be constructed for **Example 3.21**.

Thus,

$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(3\sin\theta)^2}\,3\cos\theta\,d\theta$	Substitute $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$.
$= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta\,d\theta$	Simplify.
$= \int \sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta$	Substitute $\cos^2\theta = 1-\sin^2\theta$.
$= \int 3\lvert\cos\theta\rvert 3\cos\theta\,d\theta$	Take the square root.
$= \int 9\cos^2\theta\,d\theta$	Simplify. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$ and $\lvert\cos\theta\rvert = \cos\theta$.
$= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta$	Use the strategy for integrating an even power of $\cos\theta$.
$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$	Evaluate the integral.
$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C$	Substitute $\sin(2\theta) = 2\sin\theta\cos\theta$.
$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$	Substitute $\sin^{-1}\left(\frac{x}{3}\right) = \theta$ and $\sin\theta = \frac{x}{3}$. Use the reference triangle to see that $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and make this substitution.
$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C.$	Simplify.

Example 3.22

Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate $\int \frac{\sqrt{4-x^2}}{x}dx.$

Solution

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.

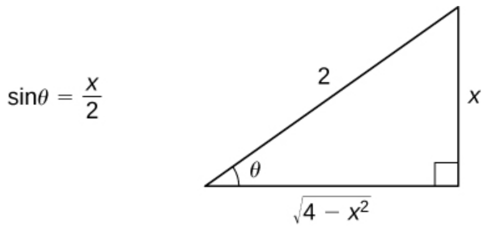


Figure 3.6 A reference triangle can be constructed for **Example 3.22**.

Thus,

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x}dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta d\theta && \text{Substitute } x = 2\sin\theta \text{ and } = 2\cos\theta d\theta.\\
&= \int \frac{2\cos^2\theta}{\sin\theta}d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.}\\
&= \int \frac{2(1-\sin^2\theta)}{\sin\theta}d\theta && \text{Substitute } \sin^2\theta = 1-\cos^2\theta.\\
&= \int (2\csc\theta - 2\sin\theta)d\theta && \text{Separate the numerator, simplify, and use } \csc\theta = \frac{1}{\sin\theta}.\\
&= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C && \text{Evaluate the integral.}\\
&= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. && \text{Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}
\end{aligned}$$

In the next example, we see that we sometimes have a choice of methods.

Example 3.23

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1 - x^2$ and then by using a trigonometric substitution.

Solution

Method 1

Let $u = 1 - x^2$ and hence $x^2 = 1 - u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) && \text{Make the substitution.}\\
&= -\frac{1}{2}\int (1-u)\sqrt{u}\,du && \text{Expand the expression.}\\
&= -\frac{1}{2}\int \left(u^{1/2}-u^{3/2}\right)du && \text{Evaluate the integral.}\\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)+C && \text{Rewrite in terms of } x.\\
&= -\frac{1}{3}\left(1-x^2\right)^{3/2}+\frac{1}{5}\left(1-x^2\right)^{5/2}+C.
\end{aligned}$$

Method 2

Let $x=\sin\theta$. In this case, $dx=\cos\theta\,d\theta$. Using this substitution, we have

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\cos^2\theta\,d\theta\\
&= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta && \text{Let } u=\cos\theta. \text{ Thus, } du=-\sin\theta\,d\theta.\\
&= \int \left(u^4-u^2\right)du\\
&= \frac{1}{5}u^5-\frac{1}{3}u^3+C && \text{Substitute } \cos\theta=u.\\
&= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C && \text{Use a reference triangle to see that } \cos\theta=\sqrt{1-x^2}.\\
&= \frac{1}{5}\left(1-x^2\right)^{5/2}-\frac{1}{3}\left(1-x^2\right)^{3/2}+C.
\end{aligned}$$

3.14 Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$, let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x=a\tan\theta$ or $x=a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x=a\tan\theta$ or, equivalently, $\tan\theta=x/a$. With this substitution, we make the assumption that $-(\pi/2)<\theta<\pi/2$, so that we also have $\theta=\tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x=a\tan\theta$ and $dx=a\sec^2\theta\,d\theta$. This substitution yields $\sqrt{a^2+x^2}=\sqrt{a^2+(a\tan\theta)^2}=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\sec^2\theta}=|a\sec\theta|=a\sec\theta$. (Since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $\sec\theta>0$ over this interval, $|a\sec\theta|=a\sec\theta$.)

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)

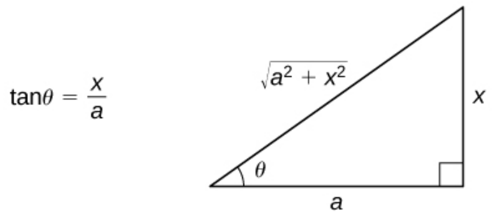


Figure 3.7 A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Example 3.24

Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

Begin with the substitution $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. Since $\tan\theta = x$, draw the reference triangle in the following figure.

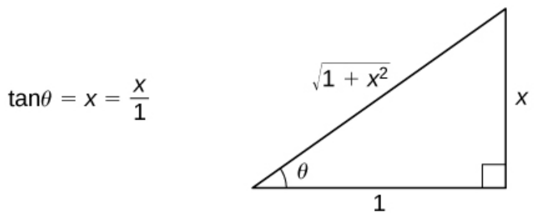


Figure 3.8 The reference triangle for **Example 3.24**.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta$$ Substitute $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. This substitution makes $\sqrt{1+x^2} = \sec\theta$. Simplify.

$$= \int \sec\theta d\theta$$ Evaluate the integral.

$$= \ln|\sec\theta + \tan\theta| + C$$ Use the reference triangle to express the result in terms of x.

$$= \ln\left|\sqrt{1+x^2} + x\right| + C.$$

To check the solution, differentiate:

$$\begin{aligned}\frac{d}{dx}\left(\ln\left|\sqrt{1+x^2}+x\right|\right) &= \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right)\\ &= \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\\ &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln\left|\sqrt{1+x^2}+x\right|+C=\ln\left(\sqrt{1+x^2}+x\right)+C$, if desired.

Example 3.25

Evaluating $\int\frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh\theta$ to evaluate $\int\frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh\theta$ has a range of all real numbers, and $1+\sinh^2\theta=\cosh^2\theta$, we may also use the substitution $x=\sinh\theta$ to evaluate this integral. In this case, $dx=\cosh\theta d\theta$. Consequently,

$$\begin{aligned}\int\frac{dx}{\sqrt{1+x^2}} &= \int\frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}d\theta && \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta d\theta. \text{ Substitute } 1+\sinh^2\theta=\cosh^2\theta.\\ &= \int\frac{\cosh\theta}{\sqrt{\cosh^2\theta}}d\theta && \sqrt{\cosh^2\theta}=|\cosh\theta|\\ &= \int\frac{\cosh\theta}{|\cosh\theta|}d\theta && |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.\\ &= \int\frac{\cosh\theta}{\cosh\theta}d\theta && \text{Simplify.}\\ &= \int 1d\theta && \text{Evaluate the integral.}\\ &= \theta+C && \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.\\ &= \sinh^{-1}x+C.\end{aligned}$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y=x$. From this equation we obtain:

$$\frac{e^y-e^{-y}}{2}=x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^{y} - 1 = 0.$$

Use the quadratic equation to solve for e^{y}:

$$e^{y} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}.$$

Simplifying, we have:

$$e^{y} = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$, it must be the case that $e^{y} = x + \sqrt{x^2 + 1}$. Thus,

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Last, we obtain

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

After we make the final observation that, since $x + \sqrt{x^2 + 1} > 0$,

$$\ln\left(x + \sqrt{x^2 + 1}\right) = \ln\left|\sqrt{1 + x^2} + x\right|,$$

we see that the two different methods produced equivalent solutions.

Example 3.26

Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Solution

Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\int_0^{1/2} \sqrt{1 + (2x)^2}\, dx = \int_0^{1/2} \sqrt{1 + 4x^2}\, dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$$\begin{aligned}\int_0^{1/2}\sqrt{1+4x^2}\,dx &= \int_0^{\pi/4}\sqrt{1+\tan^2\theta}\,\frac{1}{2}\sec^2\theta\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/4}\sec^3\theta\,d\theta \\ &= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} \\ &= \frac{1}{4}(\sqrt{2}+\ln(\sqrt{2}+1)).\end{aligned}$$

After substitution, $\sqrt{1+4x^2}=\tan\theta$. Substitute $1+\tan^2\theta=\sec^2\theta$ and simplify.

We derived this integral in the previous section.

Evaluate and simplify.

3.15 Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Integrating Expressions Involving $\sqrt{x^2-a^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty, -a]\cup[a, +\infty)$. Thus, either $x<-a$ or $x>a$. Hence, $\frac{x}{a}\leq -1$ or $\frac{x}{a}\geq 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$, it makes sense to use the substitution $\sec\theta=\frac{x}{a}$ or, equivalently, $x=a\sec\theta$, where $0\leq\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\leq\pi$. The corresponding substitution for dx is $dx=a\sec\theta\tan\theta\,d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2-a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x=a\sec\theta$ and $dx=a\sec\theta\tan\theta\,d\theta$. This substitution yields

$$\sqrt{x^2-a^2}=\sqrt{(a\sec\theta)^2-a^2}=\sqrt{a^2(\sec^2\theta+1)}=\sqrt{a^2\tan^2\theta}=|a\tan\theta|.$$

   For $x\geq a$, $|a\tan\theta|=a\tan\theta$ and for $x\leq -a$, $|a\tan\theta|=-a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta=\sec^{-1}\left(\frac{x}{a}\right)$. (*Note*: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x>a$ or $x<-a$.)

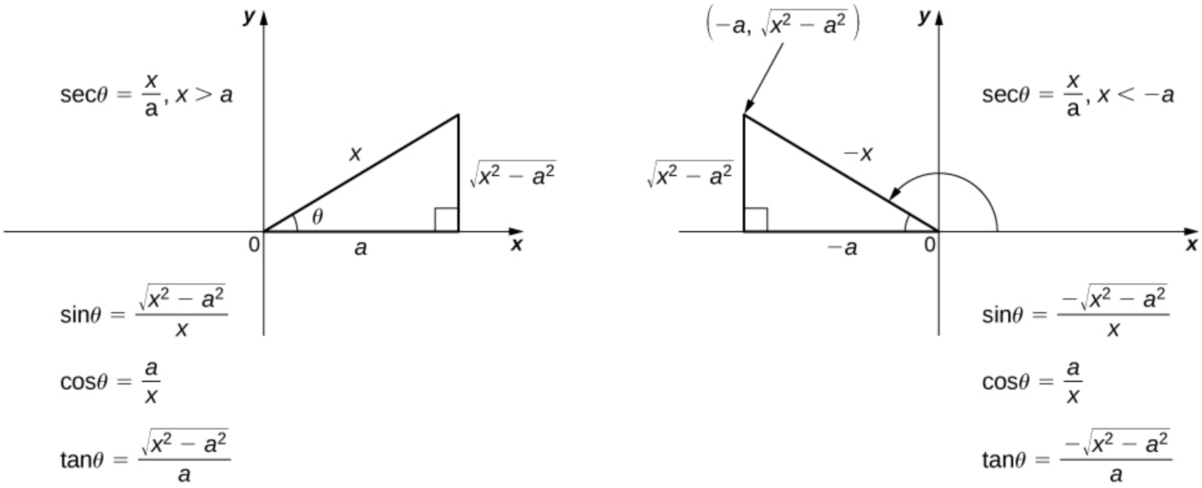


Figure 3.9 Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Example 3.27

Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

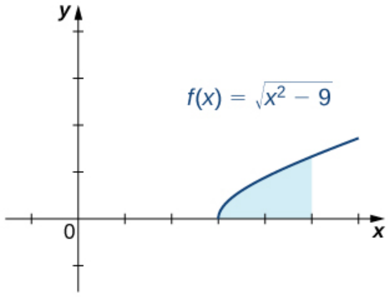


Figure 3.10 Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}\,dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$. After making these substitutions and simplifying, we have

$$\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2 - 9}\, dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta \sec\theta\, d\theta && \text{Use } \tan^2\theta = 1 - \sec^2\theta. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta\, d\theta && \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta)\, d\theta && \text{Evaluate the integral.} \\
&= \left(\frac{9}{2}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\sec\theta\tan\theta\right) - 9\ln|\sec\theta + \tan\theta| \Big|_0^{\sec^{-1}(5/3)} && \text{Simplify.} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| \Big|_0^{\sec^{-1}(5/3)} && \text{Evaluate. Use } \sec\left(\sec^{-1}\frac{5}{3}\right) = \frac{5}{3} \text{ and } \tan\left(\sec^{-1}\frac{5}{3}\right) = \frac{4}{3}. \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln\left|\frac{5}{3} + \frac{4}{3}\right| - \left(\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1 + 0|\right) \\
&= 10 - \frac{9}{2}\ln 3.
\end{aligned}$$

3.16 Evaluate $\int \frac{dx}{\sqrt{x^2 - 4}}$. Assume that $x > 2$.

3.3 EXERCISES

Simplify the following expressions by writing each one using a single trigonometric function.

126. $4 - 4\sin^2\theta$

127. $9\sec^2\theta - 9$

128. $a^2 + a^2\tan^2\theta$

129. $a^2 + a^2\sinh^2\theta$

130. $16\cosh^2\theta - 16$

Use the technique of completing the square to express each trinomial as the square of a binomial.

131. $4x^2 - 4x + 1$

132. $2x^2 - 8x + 3$

133. $-x^2 - 2x + 4$

Integrate using the method of trigonometric substitution. Express the final answer in terms of the variable.

134. $\int \frac{dx}{\sqrt{4 - x^2}}$

135. $\int \frac{dx}{\sqrt{x^2 - a^2}}$

136. $\int \sqrt{4 - x^2}dx$

137. $\int \frac{dx}{\sqrt{1 + 9x^2}}$

138. $\int \frac{x^2 dx}{\sqrt{1 - x^2}}$

139. $\int \frac{dx}{x^2\sqrt{1 - x^2}}$

140. $\int \frac{dx}{(1 + x^2)^2}$

141. $\int \sqrt{x^2 + 9}dx$

142. $\int \frac{\sqrt{x^2 - 25}}{x}dx$

143. $\int \frac{\theta^3 d\theta}{\sqrt{9 - \theta^2}}d\theta$

144. $\int \frac{dx}{\sqrt{x^6 - x^2}}$

145. $\int \sqrt{x^6 - x^8}dx$

146. $\int \frac{dx}{\left(1 + x^2\right)^{3/2}}$

147. $\int \frac{dx}{\left(x^2 - 9\right)^{3/2}}$

148. $\int \frac{\sqrt{1 + x^2}dx}{x}$

149. $\int \frac{x^2 dx}{\sqrt{x^2 - 1}}$

150. $\int \frac{x^2 dx}{x^2 + 4}$

151. $\int \frac{dx}{x^2\sqrt{x^2 + 1}}$

152. $\int \frac{x^2 dx}{\sqrt{1 + x^2}}$

153. $\int_{-1}^{1} (1 - x^2)^{3/2} dx$

In the following exercises, use the substitutions $x = \sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable x.

154. $\int \frac{dx}{\sqrt{x^2 - 1}}$

155. $\int \frac{dx}{x\sqrt{1 - x^2}}$

156. $\int \sqrt{x^2 - 1}dx$

157. $\int \frac{\sqrt{x^2 - 1}}{x^2}dx$

158. $\int \frac{dx}{1-x^2}$

159. $\int \frac{\sqrt{1+x^2}}{x^2}dx$

Use the technique of completing the square to evaluate the following integrals.

160. $\int \frac{1}{x^2-6x}dx$

161. $\int \frac{1}{x^2+2x+1}dx$

162. $\int \frac{1}{\sqrt{-x^2+2x+8}}dx$

163. $\int \frac{1}{\sqrt{-x^2+10x}}dx$

164. $\int \frac{1}{\sqrt{x^2+4x-12}}dx$

165. Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9-x^2}\,dx.$

166. Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1.$

167. Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

168. Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

169. Evaluate the integral $\int \frac{x}{x^2+1}dx$ using the form $\int \frac{1}{u}du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

170. State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

171. State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

172. Evaluate $\int_{-1}^{1} \frac{x\,dx}{x^2+1}$

173. Find the length of the arc of the curve over the specified interval: $y=\ln x$, $[1, 5]$. Round the answer to three decimal places.

174. Find the surface area of the solid generated by revolving the region bounded by the graphs of $y=x^2$, $y=0$, $x=0$, and $x=\sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

175. The region bounded by the graph of $f(x)=\frac{1}{1+x^2}$ and the x-axis between $x=0$ and $x=1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Solve the initial-value problem for y as a function of x.

176. $(x^2+36)\frac{dy}{dx}=1,\ y(6)=0$

177. $(64-x^2)\frac{dy}{dx}=1,\ y(0)=3$

178. Find the area bounded by $y=\frac{2}{\sqrt{64-4x^2}}$, $x=0$, $y=0$, and $x=2$.

179. An oil storage tank can be described as the volume generated by revolving the area bounded by $y=\frac{16}{\sqrt{64+x^2}}$, $x=0$, $y=0$, $x=2$ about the x-axis. Find the volume of the tank (in cubic meters).

180. During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v=2t-\frac{14}{4+t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s=0$ when $t=0$.

181. Find the length of the curve $y=\sqrt{16-x^2}$ between $x=0$ and $x=2$.

3.4 | Partial Fractions

Learning Objectives

3.4.1 Integrate a rational function using the method of partial fractions.
3.4.2 Recognize simple linear factors in a rational function.
3.4.3 Recognize repeated linear factors in a rational function.
3.4.4 Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C \text{ and } \int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating $\int \frac{3x}{x^2 - x - 2}dx.$ However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose rational functions into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2 - x - 2}$ as an expression such as $\frac{1}{x+1} + \frac{2}{x-2}.$

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$.

Example 3.28

Integrating $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2 + 3x + 5}{x+1}dx.$

Solution

Since $\deg\left(x^2+3x+5\right) \geq \deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1}dx &= \int \left(x+2+\frac{3}{x+1}\right)dx \\ &= \frac{1}{2}x^2+2x+3\ln|x+1|+C.\end{aligned}$$

Visit this **website (http://www.openstaxcollege.org/l/20_polylongdiv)** for a review of long division of polynomials.

3.17 Evaluate $\int \frac{x-3}{x+2}dx.$

To integrate $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}.$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example 3.29

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}dx.$

Solution

Since $\deg(3x+2) < \deg\left(x^3-x^2-2x\right)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2). \qquad (3.8)$$

There are two different strategies for finding the coefficients $A, \quad B,$ and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Rule: Method of Equating Coefficients

Rewrite Equation 3.8 in the form

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x + (-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1, \quad B = \frac{4}{3},$ and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Rule: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of $A, \quad B,$ and C that satisfy Equation 3.8 for *all* values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find $A, \quad B,$ and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of $A,\ B,$ and $C,$ we rewrite the original integral:

$$\int \frac{3x+2}{x^3 - x^2 - 2x}dx = \int \left(-\frac{1}{x} + \frac{4}{3} \cdot \frac{1}{(x-2)} - \frac{1}{3} \cdot \frac{1}{(x+1)}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3 - x^2 - 2x}dx = -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 3.30

Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2 + 3x + 1}{x^2 - 4}dx.$

Solution

Since $\text{degree}(x^2 + 3x + 1) \geq \text{degree}(x^2 - 4),$ we must perform long division of polynomials. This results in

$$\frac{x^2 + 3x + 1}{x^2 - 4} = 1 + \frac{3x+5}{x^2 - 4}.$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2 - 4} = \frac{3x+5}{(x+2)(x-2)}.$ We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x + 5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4.$

Rewriting the original integral, we have

$$\int \frac{x^2 + 3x + 1}{x^2 - 4}dx = \int \left(1 + \frac{11}{4} \cdot \frac{1}{x-2} + \frac{1}{4} \cdot \frac{1}{x+2}\right)dx.$$

Evaluating the integral produces

$$\int \frac{x^2 + 3x + 1}{x^2 - 4}dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example 3.31

Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x} dx.$

Solution

Let's begin by letting $u = \sin x.$ Consequently, $du = \cos x\, dx.$ After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x} dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $1/u(u-1)$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}.$

Thus,

$$\begin{aligned}\int \frac{\cos x}{\sin^2 x - \sin x} dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin x| + \ln|\sin x - 1| + C.\end{aligned}$$

3.18 Evaluate $\int \frac{x+1}{(x+3)(x-2)} dx.$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n,$ where n is a positive integer greater than or equal to $2.$ If the denominator contains the repeated linear factor $(ax+b)^n,$ then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

Example 3.32

Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx.$

Solution

We have $\text{degree}(x-2) < \text{degree}\left((2x-1)^2(x-1)\right),$ so we can proceed with the decomposition. Since

$(2x-1)^2$ is a repeated linear factor, include $\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$ in the decomposition. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x-2=A(2x-1)(x-1)+B(x-1)+C(2x-1)^2. \quad (3.9)$$

We then use the method of equating coefficients to find the values of $A,\ B,$ and C.

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0,\ -3A+B-4C=1,$ and $A-B+C=-2.$ Solving this system yields $A=2,\ B=3,$ and $C=-1.$

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into **Equation 3.9** easily produces the values $B=3$ and $C=-1.$ At this point, it may seem that we have run out of good choices for $x,$ however, since we already have values for B and $C,$ we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2.$

Now that we have the values for $A,\ B,$ and $C,$ we rewrite the original integral and evaluate it:

$$\begin{aligned}\int\frac{x-2}{(2x-1)^2(x-1)}dx &= \int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx\\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$

3.19 Set up the partial fraction decomposition for $\int\frac{x+2}{(x+3)^3(x-4)^2}dx.$ (Do not solve for the coefficients or complete the integration.)

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x),$ use the following steps:

1. Make sure that $\text{degree}(P(x))<\text{degree}(Q(x)).$ If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x))<\deg(Q(x)),$ the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x).$
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n),$ where each linear factor is distinct,

then it is possible to find constants $A_1, A_2, ...A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}.$$

b. If $Q(x)$ contains the repeated linear factor $(ax + b)^n$, then the decomposition must contain

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}.$$

c. For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include

$$\frac{Ax + B}{ax^2 + bx + c}.$$

d. For each repeated irreducible quadratic factor $\left(ax^2 + bx + c\right)^n$, the decomposition must include

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

Example 3.33

Rational Expressions with an Irreducible Quadratic Factor

Evaluate $\int \frac{2x - 3}{x^3 + x} dx$.

Solution

Since $\deg(2x - 3) < \deg(x^3 + x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3 + x = x(x^2 + 1)$ contains the irreducible quadratic factor $x^2 + 1$, include $\frac{Ax + B}{x^2 + 1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x - 3}{x(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x - 3 = (Ax + B)x + C\left(x^2 + 1\right).$$

Solving for $A, B,$ and $C,$ we get $A = 3,$ $B = 2,$ and $C = -3.$

Thus,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int \frac{2x-3}{x^3+x}dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right)dx \\ &= 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx - 3\int \frac{1}{x}dx & \text{Split up the integral.} \\ &= \frac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - 3\ln|x| + C. & \text{Evaluate each integral.}\end{aligned}$$

Note: We may rewrite $\ln\left|x^2+1\right| = \ln(x^2+1)$, if we wish to do so, since $x^2+1>0$.

Example 3.34

Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution

We can start by factoring $x^3-8=(x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4(1)(4)=-12<0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1 = A\left(x^2+2x+4\right) + (Bx+C)(x-2).$$

Applying either method, we get $A=\frac{1}{12}$, $B=-\frac{1}{12}$, and $C=-\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

We can see that

$\int \frac{1}{x-2}dx = \ln|x-2| + C$, but $\int \frac{x+4}{x^2+2x+4}dx$ requires a bit more effort. Let's begin by completing the square on x^2+2x+4 to obtain

$$x^2+2x+4 = (x+1)^2+3.$$

By letting $u = x + 1$ and consequently $du = dx,$ we see that

$$\begin{aligned}\int \frac{x+4}{x^2+2x+4}dx &= \int \frac{x+4}{(x+1)^2+3}dx && \text{Complete the square on the denominator.}\\ &= \int \frac{u+3}{u^2+3}du && \text{Substitute } u = x+1,\ x = u-1, \text{ and } du = dx.\\ &= \int \frac{u}{u^2+3}du + \int \frac{3}{u^2+3}du && \text{Split the numerator apart.}\\ &= \frac{1}{2}\ln\left|u^2+3\right| + \frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C && \text{Evaluate each integral.}\\ &= \frac{1}{2}\ln\left|x^2+2x+4\right| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C. && \text{Rewrite in terms of } x \text{ and simplify.}\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln\left|x^2+2x+4\right| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x.

Example 3.35

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{\left(x^2+1\right)^2}$ and the x-axis over the interval $[0,\ 1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved (see **Figure 3.11**). From the sketch, we see that the shell method is a good choice for solving this problem.

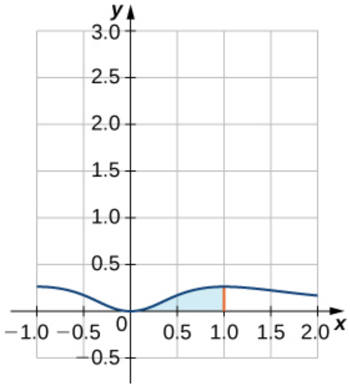


Figure 3.11 We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V = 2\pi\int_0^1 x \cdot \frac{x^2}{\left(x^2+1\right)^2}dx = 2\pi\int_0^1 \frac{x^3}{(x^2+1)^2}dx.$$

Since $\deg\left(\left(x^2+1\right)^2\right) = 4 > 3 = \deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3 = (Ax+B)\left(x^2+1\right) + Cx + D.$$

Solving, we obtain $A = 1$, $B = 0$, $C = -1$, and $D = 0$. Substituting back into the integral, we have

$$\begin{aligned} V &= 2\pi\int_0^1 \frac{x^3}{(x^2+1)^2}dx \\ &= 2\pi\int_0^1 \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx \\ &= 2\pi\left(\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Big|_0^1 \\ &= \pi\left(\ln 2 - \frac{1}{2}\right). \end{aligned}$$

3.20 Set up the partial fraction decomposition for $\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}dx.$

3.4 EXERCISES

Express the rational function as a sum or difference of two simpler rational expressions.

182. $\frac{1}{(x-3)(x-2)}$

183. $\frac{x^2+1}{x(x+1)(x+2)}$

184. $\frac{1}{x^3-x}$

185. $\frac{3x+1}{x^2}$

186. $\frac{3x^2}{x^2+1}$ (*Hint:* Use long division first.)

187. $\frac{2x^4}{x^2-2x}$

188. $\frac{1}{(x-1)(x^2+1)}$

189. $\frac{1}{x^2(x-1)}$

190. $\frac{x}{x^2-4}$

191. $\frac{1}{x(x-1)(x-2)(x-3)}$

192. $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)\left(x^2+1\right)}$

193. $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

194. $\frac{2x}{(x+2)^2}$

195. $\frac{3x^4+x^3+20x^2+3x+31}{(x+1)\left(x^2+4\right)^2}$

Use the method of partial fractions to evaluate each of the following integrals.

196. $\int \frac{dx}{(x-3)(x-2)}$

197. $\int \frac{3x}{x^2+2x-8}dx$

198. $\int \frac{dx}{x^3-x}$

199. $\int \frac{x}{x^2-4}dx$

200. $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

201. $\int \frac{2x^2+4x+22}{x^2+2x+10}dx$

202. $\int \frac{dx}{x^2-5x+6}$

203. $\int \frac{2-x}{x^2+x}dx$

204. $\int \frac{2}{x^2-x-6}dx$

205. $\int \frac{dx}{x^3-2x^2-4x+8}$

206. $\int \frac{dx}{x^4-10x^2+9}$

Evaluate the following integrals, which have irreducible quadratic factors.

207. $\int \frac{2}{(x-4)\left(x^2+2x+6\right)}dx$

208. $\int \frac{x^2}{x^3-x^2+4x-4}dx$

209. $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$

210. $\int \frac{x}{(x-1)\left(x^2+2x+2\right)^2}dx$

Use the method of partial fractions to evaluate the following integrals.

211. $\int \frac{3x+4}{\left(x^2+4\right)(3-x)}dx$

212. $\int \frac{2}{(x+2)^2(2-x)}dx$

213. $\int \frac{3x+4}{x^3-2x-4}dx$ (*Hint:* Use the rational root theorem.)

Use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

214. $\int_0^1 \frac{e^x}{36-e^{2x}}dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

215. $\int \frac{e^x dx}{e^{2x}-e^x}dx$

216. $\int \frac{\sin x\,dx}{1-\cos^2 x}$

217. $\int \frac{\sin x}{\cos^2 x+\cos x-6}dx$

218. $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx$

219. $\int \frac{dt}{\left(e^t-e^{-t}\right)^2}$

220. $\int \frac{1+e^x}{1-e^x}dx$

221. $\int \frac{dx}{1+\sqrt{x+1}}$

222. $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

223. $\int \frac{\cos x}{\sin x(1-\sin x)}dx$

224. $\int \frac{e^x}{\left(e^{2x}-4\right)^2}dx$

225. $\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}dx$

226. $\int \frac{1}{2+e^{-x}}dx$

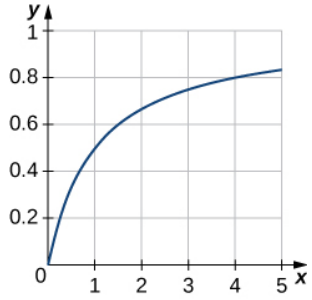
227. $\int \frac{1}{1+e^x}dx$

Use the given substitution to convert the integral to an integral of a rational function, then evaluate.

228. $\int \frac{1}{t-\sqrt[3]{t}}dt\, t=x^3$

229. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx;\ x=u^6$

230. Graph the curve $y=\frac{x}{1+x}$ over the interval $[0, 5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x=4$.



231. Find the volume of the solid generated when the region bounded by $y=1/\sqrt{x(3-x)}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

232. The velocity of a particle moving along a line is a function of time given by $v(t)=\frac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t=5$ sec.

Solve the initial-value problem for x as a function of t.

233. $\left(t^2-7t+12\right)\frac{dx}{dt}=1,\ (t>4,\ x(5)=0)$

234. $(t+5)\frac{dx}{dt}=x^2+1,\ t>-5,\ x(1)=\tan 1$

235. $\left(2t^3-2t^2+t-1\right)\frac{dx}{dt}=3,\ x(2)=0$

236. Find the x-coordinate of the centroid of the area bounded by $y\left(x^2-9\right)=1$, $y=0$, $x=4$, and $x=5$. (Round the answer to two decimal places.)

237. Find the volume generated by revolving the area bounded by $y=\frac{1}{x^3+7x^2+6x}x=1,\ x=7$, and $y=0$ about the y-axis.

238. Find the area bounded by $y=\frac{x-12}{x^2-8x-20}$, $y=0$, $x=2$, and $x=4$. (Round the answer to the nearest hundredth.)

239. Evaluate the integral $\int \frac{dx}{x^3 + 1}$.

For the following problems, use the substitutions $\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{2}{1 + t^2}dt$, $\sin x = \frac{2t}{1 + t^2}$, and $\cos x = \frac{1 - t^2}{1 + t^2}$.

240. $\int \frac{dx}{3 - 5\sin x}$

241. Find the area under the curve $y = \frac{1}{1 + \sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

242. Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \frac{2}{1 + t^2}dt$, $\sin x = \frac{2t}{1 + t^2}$, and $\cos x = \frac{1 - t^2}{1 + t^2}$.

243. Evaluate $\int \frac{\sqrt[3]{x - 8}}{x}dx$.

3.5 | Other Strategies for Integration

Learning Objectives

3.5.1 Use a table of integrals to solve integration problems.
3.5.2 Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta,$ we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

However, using $x = \sinh\theta,$ we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$

In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 3.36

Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2} du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\int \frac{\sqrt{16-e^{2x}}}{e^x} dx.$

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}.$ This expression is actually similar to $\sqrt{16-e^{2x}},$ where $a = 4$ and $u = e^x.$ Keep in mind that we must also have $du = e^x.$ Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x} dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}} e^x\,dx.$$

Substituting $u = e^x$ and $du = e^x$ produces $\int \frac{\sqrt{a^2 - u^2}}{u^2} du$. From the integration table (#88 in **Appendix A**),

$$\int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned} \int \frac{\sqrt{16 - e^{2x}}}{e^x} dx &= \int \frac{\sqrt{16 - e^{2x}}}{e^{2x}} e^x \, dx && \text{Substitute } u = e^x \text{ and } du = e^x \, dx. \\ &= \int \frac{\sqrt{4^2 - u^2}}{u^2} du && \text{Apply the formula using } a = 4. \\ &= -\frac{\sqrt{4^2 - u^2}}{u} - \sin^{-1}\frac{u}{4} + C && \text{Substitute } u = e^x. \\ &= -\frac{\sqrt{16 - e^{2x}}}{u} - \sin^{-1}\left(\frac{e^x}{4}\right) + C. \end{aligned}$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 3.37

Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2 - 4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2 - 4}} = \ln\left|\sqrt{x^2 - 4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2 - 4} + x}{2}\right| + C = \ln\left|\sqrt{x^2 - 4} + x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an **integral calculator (http://www.openstaxcollege.org/l/20_intcalc)** for more examples.

Example 3.38

Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\,dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C,$ the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\,dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C.$ To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x + 2x) - 9\cos x)\\ &= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x)\\ &= \frac{1}{12}(\cos x\left(2\cos^2 x - 1\right) - \sin x(2\sin x\cos x) - 9\cos x)\\ &= \frac{1}{12}(2\cos x - \cos x - 2\cos x\left(1 - \cos^2 x\right) - 9\cos x)\\ &= \frac{1}{12}(4\cos x - 12\cos x)\\ &= \frac{1}{3}\cos x - \cos x.\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

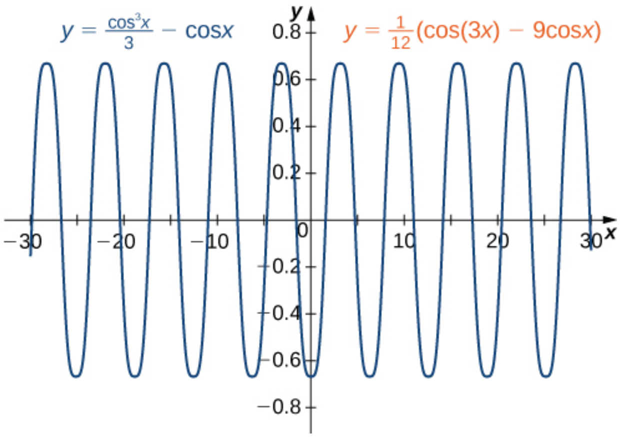


Figure 3.12 The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

3.21 Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2 + 4}}.$

3.5 EXERCISES

Use a table of integrals to evaluate the following integrals.

244. $\int_0^4 \frac{x}{\sqrt{1+2x}}dx$

245. $\int \frac{x+3}{x^2+2x+2}dx$

246. $\int x^3\sqrt{1+2x^2}dx$

247. $\int \frac{1}{\sqrt{x^2+6x}}dx$

248. $\int \frac{x}{x+1}dx$

249. $\int x \cdot 2^{x^2}dx$

250. $\int \frac{1}{4x^2+25}dx$

251. $\int \frac{dy}{\sqrt{4-y^2}}$

252. $\int \sin^3(2x)\cos(2x)dx$

253. $\int \csc(2w)\cot(2w)dw$

254. $\int 2^y dy$

255. $\int_0^1 \frac{3x\,dx}{\sqrt{x^2+8}}$

256. $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)dx$

257. $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)dx$

258. $\int \cos^3 x\,dx$

259. $\int \tan^5(3x)dx$

260. $\int \sin^2 y \cos^3 y\,dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

261. **[T]** $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

262. **[T]** $\int \frac{dw}{1-\cos(7w)}$

263. **[T]** $\int_0^t \frac{dt}{4\cos t + 3\sin t}$

264. **[T]** $\int \frac{\sqrt{x^2-9}}{3x}dx$

265. **[T]** $\int \frac{dx}{x^{1/2}+x^{1/3}}$

266. **[T]** $\int \frac{dx}{x\sqrt{x-1}}$

267. **[T]** $\int x^3 \sin x\,dx$

268. **[T]** $\int x\sqrt{x^4-9}\,dx$

269. **[T]** $\int \frac{x}{1+e^{-x^2}}dx$

270. **[T]** $\int \frac{\sqrt{3-5x}}{2x}dx$

271. **[T]** $\int \frac{dx}{x\sqrt{x-1}}$

272. **[T]** $\int e^x \cos^{-1}(e^x)dx$

Use a calculator or CAS to evaluate the following integrals.

273. **[T]** $\int_0^{\pi/4} \cos(2x)dx$

274. **[T]** $\int_0^1 x \cdot e^{-x^2}dx$

275. **[T]** $\int_0^8 \frac{2x}{\sqrt{x^2+36}}dx$

276. **[T]** $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}dx$

277. **[T]** $\int \frac{dx}{x^2 + 4x + 13}$

278. **[T]** $\int \frac{dx}{1 + \sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

279. $\int \frac{dx}{x^2 + 2x + 10}$

280. $\int \frac{dx}{\sqrt{x^2 - 6x}}$

281. $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

282. $\int \frac{\cos x}{\sin^2 x + 2\sin x} dx$

283. $\int \frac{\arctan\left(x^3\right)}{x^4} dx$

284. $\int \frac{\ln|x|\arcsin(\ln|x|)}{x} dx$

Use tables to perform the integration.

285. $\int \frac{dx}{\sqrt{x^2 + 16}}$

286. $\int \frac{3x}{2x + 7} dx$

287. $\int \frac{dx}{1 - \cos(4x)}$

288. $\int \frac{dx}{\sqrt{4x + 1}}$

289. Find the area bounded by $y\left(4 + 25x^2\right) = 5,\ x = 0,\ y = 0,$ and $x = 4.$ Use a table of integrals or a CAS.

290. The region bounded between the curve $y = \frac{1}{\sqrt{1 + \cos x}},\ 0.3 \le x \le 1.1,$ and the x-axis is revolved about the x-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

291. Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y = e^x,\ 0 \le x \le 3,$ about the x-axis. (Round the answer to two decimal places.)

292. **[T]** Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y = \frac{x^2}{2},\ 0 \le x \le 1,$ about the x-axis. (Round the answer to two decimal places.)

293. **[T]** Use a CAS or tables to find the area of the surface generated by revolving the curve $y = \cos x,\ 0 \le x \le \frac{\pi}{2},$ about the x-axis. (Round the answer to two decimal places.)

294. Find the length of the curve $y = \frac{x^2}{4}$ over $[0,\ 8].$

295. Find the length of the curve $y = e^x$ over $[0,\ \ln(2)].$

296. Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0,\ 9]$ about the x-axis.

297. Find the average value of the function $f(x) = \frac{1}{x^2 + 1}$ over the interval $[-3,\ 3].$

298. Approximate the arc length of the curve $y = \tan(\pi x)$ over the interval $\left[0,\ \frac{1}{4}\right].$ (Round the answer to three decimal places.)

3.6 | Numerical Integration

Learning Objectives

3.6.1 Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
3.6.2 Determine the absolute and relative error in using a numerical integration technique.
3.6.3 Estimate the absolute and relative error using an error-bound formula.
3.6.4 Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
3.6.5 Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of Riemann sums. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}, \text{ where } a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\}, \text{ where } x_{i-1} \le x_i^* \le x_i \text{ for all } i.$$

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, the length of the ith subinterval.

The **midpoint rule** for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

Theorem 3.3: The Midpoint Rule

Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length Δx, and m_i is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x. \tag{3.10}$$

Then $\lim_{n \to \infty} M_n = \int_a^b f(x)dx$.

As we can see in **Figure 3.13**, if $f(x) \ge 0$ over $[a, b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.

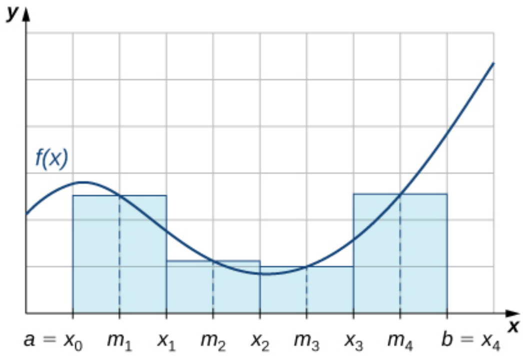


Figure 3.13 The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example 3.39

Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution

Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and} \left[\frac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$M_4 = \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \frac{1}{4}f\left(\frac{5}{8}\right) + \frac{1}{4}f\left(\frac{7}{8}\right) = \frac{1}{4}\cdot\frac{1}{64} + \frac{1}{4}\cdot\frac{9}{64} + \frac{1}{4}\cdot\frac{25}{64} + \frac{1}{4}\cdot\frac{21}{64} = \frac{21}{64}.$$

Since

$$\int_0^1 x^2\,dx = \frac{1}{3} \text{ and } \left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 3.40

Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

$$\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Since $\frac{dy}{dx} = x,$ this integral becomes $\int_1^4 \sqrt{1 + x^2} dx.$

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4 - 1}{6} = \frac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$. If we set $f(x) = \sqrt{1 + x^2},$

$$\begin{aligned} M_6 &= \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) + \frac{1}{2}f\left(\frac{9}{4}\right) + \frac{1}{2}f\left(\frac{11}{4}\right) + \frac{1}{2}f\left(\frac{13}{4}\right) + \frac{1}{2}f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431. \end{aligned}$$

3.22 Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x} dx.$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In **Figure 3.14**, the area beneath the curve is approximated by trapezoids rather than by rectangles.

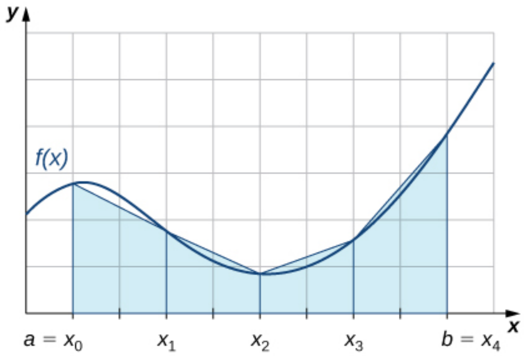


Figure 3.14 Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in **Figure 3.14**. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in **Figure 3.14** is

$$\frac{1}{2}\Delta x(f(x_0) + f(x_1)).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x(f(x_1)+f(x_2)),\ \frac{1}{2}\Delta x(f(x_2)+f(x_3)),\text{ and }\frac{1}{2}\Delta x(f(x_3)+f(x_4)).$$

Consequently,

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x(f(x_0)+f(x_1))+\frac{1}{2}\Delta x(f(x_1)+f(x_2))+\frac{1}{2}\Delta x(f(x_2)+f(x_3))+\frac{1}{2}\Delta x(f(x_3)+f(x_4)).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x\big(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\big).$$

Generalizing, we formally state the following rule.

Theorem 3.4: The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2 \ldots, x_n\}$. Set

$$T_n = \frac{1}{2}\Delta x\big(f(x_0)+2f(x_1)+2f(x_2)+\ \cdots\ +2f(x_{n-1})+f(x_n)\big). \tag{3.11}$$

Then, $\lim\limits_{n \to +\infty} T_n = \int_a^b f(x)dx.$

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n + R_n)\text{ where }L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x\text{ and }R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of **Figure 3.15** leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

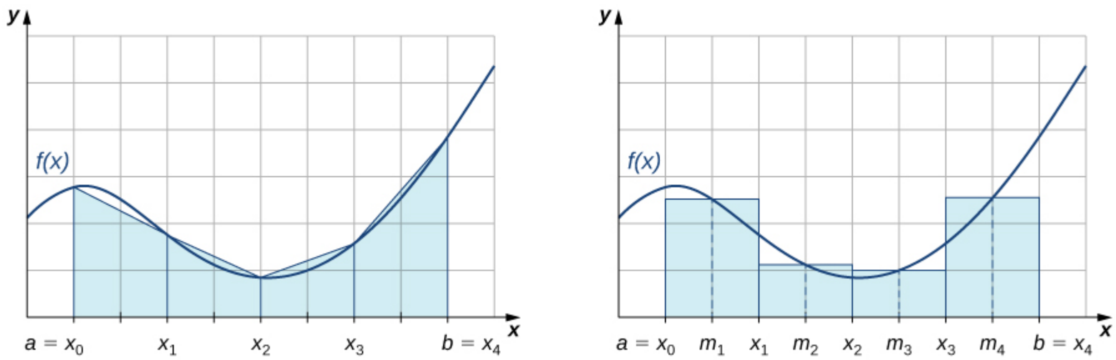


Figure 3.15 The trapezoidal rule tends to be less accurate than the midpoint rule.

Example 3.41

Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2\,dx$ using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$.

Thus,

$$\begin{aligned}\int_0^1 x^2\,dx &\approx \frac{1}{2}\cdot\frac{1}{4}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right)\\ &= \frac{1}{8}\left(0 + 2\cdot\frac{1}{16} + 2\cdot\frac{1}{4} + 2\cdot\frac{9}{16} + 1\right)\\ &= \frac{11}{32}.\end{aligned}$$

3.23 Use the trapezoidal rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}dx$.

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define **absolute error** and **relative error**.

Definition

If B is our estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$. The relative error is the error as a percentage of the absolute value and is given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right|\cdot 100\%$.

Example 3.42

Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in Example 3.39.

Solution

The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\left(\frac{1}{3}\right) - \left(\frac{21}{64}\right)\right| = \frac{1}{192} \approx 0.0052$. The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 3.43

Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in **Example 3.41**.

Solution

The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$. The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

3.24 In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}dx$ to be $\frac{24}{35}$ using T_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Theorem 3.5: Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)dx$ are

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \tag{3.12}$$

and

$$\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}. \tag{3.13}$$

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example 3.44

Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2}dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0, 1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}\left(1 + 2x^2\right) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound **Equation 3.12**, we have

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that $\left|\int_0^1 e^{x^2}dx - M_n\right| < 0.01.$

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

3.25 Use **Equation 3.13** to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2 dx$.

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule**, we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)dx$ with $\int_{x_0}^{x_2} p(x)dx$, where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ (**Figure 3.16**). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.

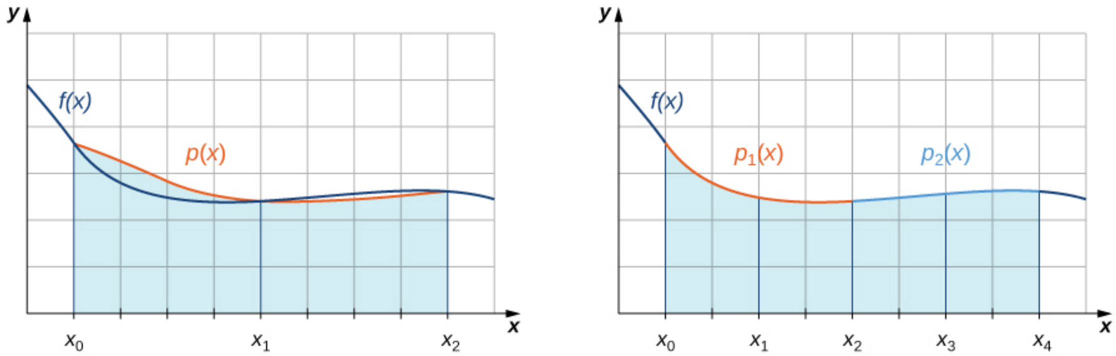


Figure 3.16 With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0{}^2 + Bx_0 + C$$
$$f(x_1) = p(x_1) = Ax_1{}^2 + Bx_1 + C$$
$$f(x_2) = p(x_2) = Ax_2{}^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &\approx \int_{x_0}^{x_2} p(x)dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C)dx \\
&= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{x_0}^{x_2} && \text{Find the antiderivative.} \\
&= \frac{A}{3}\left(x_2{}^3 - x_0{}^3\right) + \frac{B}{2}\left(x_2{}^2 - x_0{}^2\right) + C(x_2 - x_0) && \text{Evaluate the antiderivative.} \\
&= \frac{A}{3}(x_2 - x_0)\left(x_2{}^2 + x_2 x_0 + x_0{}^2\right) \\
&+ \frac{B}{2}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6}\left(2A\left(x_2{}^2 + x_2 x_0 + x_0{}^2\right) + 3B(x_2 + x_0) + 6C\right) && \text{Factor out } \frac{x_2 - x_0}{6}. \\
&= \frac{\Delta x}{3}\left(\left(Ax_2{}^2 + Bx_2 + C\right) + (Ax_0{}^2 + Bx_0 + C\right) \\
&+ A\left(x_2{}^2 + 2x_2 x_0 + x_0{}^2\right) + 2B(x_2 + x_0) + 4C) \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C\right) && \text{Rearrange the terms.} \\
& && \text{Factor and substitute.} \\
& && f(x_2) = Ax_0{}^2 + Bx_0 + C \text{ and} \\
& && f(x_0) = Ax_0{}^2 + Bx_0 + C. \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C\right) && \text{Substitute } x_2 + x_0 = 2x_1. \\
&= \frac{\Delta x}{3}(f(x_2) + 4f(x_1) + f(x_0)). && \text{Expand and substitute} \\
& && f(x_1) = Ax_1{}^2 + Bx_1 + .
\end{aligned}$$

If we approximate $\int_{x_2}^{x_4} f(x)dx$ using the same method, we see that we have

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{\Delta x}{3}(f(x_4) + 4f(x_3) + f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)dx = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Theorem 3.6: Simpson's Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b - a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)). \tag{3.14}$$

Then,

$$\lim_{n \to +\infty} S_n = \int_a^b f(x)dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right)M_n + \left(\frac{1}{3}\right)T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)dx$ is given by

$$\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}. \qquad (3.15)$$

Example 3.45

Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3\,dx$. Estimate a bound for the error in S_2.

Solution

Since $[0, 1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$S_4 = \frac{1}{3} \cdot \frac{1}{2}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{6}\left(0 + 4 \cdot \frac{1}{8} + 1\right) = \frac{1}{4}$. Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$\text{Error in } S_2 \le \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}$.

Example 3.46

Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1 + x^2}\,dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4 - 1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$. Setting $f(x) = \sqrt{1 + x^2}$,

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right).$$

After substituting, we have

$$\begin{aligned} S_6 &= \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \\ &\approx 8.14594. \end{aligned}$$

3.26 Use S_2 to estimate $\int_1^2 \frac{1}{x}\,dx$.

3.6 EXERCISES

Approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

299. $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

300. $\int_0^3 \sqrt{4 + x^3}\,dx$; trapezoidal rule; $n = 6$

301. $\int_0^3 \sqrt{4 + x^3}\,dx$; Simpson's rule; $n = 3$

302. $\int_0^{12} x^2\,dx$; midpoint rule; $n = 6$

303. $\int_0^1 \sin^2(\pi x)dx$; midpoint rule; $n = 3$

304. Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2\,dx.$

305. Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2\,dx.$

306. Find the exact value of $\int_2^4 x^2\,dx.$ Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to three decimal places using the indicated rule.

307. $\int_0^1 \sin^2(\pi x)dx$; trapezoidal rule; $n = 6$

308. $\int_0^3 \frac{1}{1 + x^3}dx$; trapezoidal rule; $n = 6$

309. $\int_0^3 \frac{1}{1 + x^3}dx$; Simpson's rule; $n = 3$

310. $\int_0^{0.8} e^{-x^2}\,dx$; trapezoidal rule; $n = 4$

311. $\int_0^{0.8} e^{-x^2}\,dx$; Simpson's rule; $n = 4$

312. $\int_0^{0.4} \sin(x^2)dx$; trapezoidal rule; $n = 4$

313. $\int_0^{0.4} \sin(x^2)dx$; Simpson's rule; $n = 4$

314. $\int_{0.1}^{0.5} \frac{\cos x}{x}dx$; trapezoidal rule; $n = 4$

315. $\int_{0.1}^{0.5} \frac{\cos x}{x}dx$; Simpson's rule; $n = 4$

316. Evaluate $\int_0^1 \frac{dx}{1 + x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π.

317. Approximate $\int_2^4 \frac{1}{\ln x}dx$ using the midpoint rule with four subdivisions to four decimal places.

318. Approximate $\int_2^4 \frac{1}{\ln x}dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

319. Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$ to four decimal places.

320. Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx.$ Compare this value with the exact value and find the error estimate.

321. Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)dx.$

322. Show that the exact value of $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}.$ Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

323. Given $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e},$ use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

324. Find an upper bound for the error in estimating $\int_0^3 (5x+4)dx$ using the trapezoidal rule with six steps.

325. Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}dx$ using the trapezoidal rule with seven subdivisions.

326. Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)dx$ using Simpson's rule with $n=10$ steps.

327. Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}dx$ using Simpson's rule with $n=10$ steps.

328. Find an upper bound for the error in estimating $\int_0^\pi 2x\cos(x)dx$ using Simpson's rule with four steps.

329. Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 \left(5x^2+8\right)dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

330. Determine a value of *n* such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}dx$ with an error of no more than 0.01.

331. Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 \left(2x^3+4x\right)dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

332. Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

333. Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ from $x=0$ to $x=0.4$.

334. Use Simpson's rule with $n=14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y=0,\quad x=0,$ and $x=\pi/2$.

335. The length of one arch of the curve $y=3\sin(2x)$ is given by $L=\int_0^{\pi/2}\sqrt{1+36\cos^2(2x)}\,dx$. Estimate L using the trapezoidal rule with $n=6$.

336. The length of the ellipse $x=a\cos(t),\ y=b\sin(t),\ 0\le t\le 2\pi$ is given by $L=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2(t)}\,dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n=6$ subdivisions to estimate the length of the ellipse when $a=2$ and $e=1/3$.

337. Estimate the area of the surface generated by revolving the curve $y=\cos(2x),\ 0\le x\le\frac{\pi}{4}$ about the *x*-axis. Use the trapezoidal rule with six subdivisions.

338. Estimate the area of the surface generated by revolving the curve $y=2x^2,\quad 0\le x\le 3$ about the *x*-axis. Use Simpson's rule with $n=6$.

339. The growth rate of a certain tree (in feet) is given by $y=\frac{2}{t+1}+e^{-t^2/2}$, where *t* is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

340. **[T]** Use a calculator to approximate $\int_0^1 \sin(\pi x)dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

341. **[T]** Given $\int_1^5 \left(3x^2-2x\right)dx=100,$ approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

342. Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

343. The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	*y*	*x*	*y*
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

344. Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number
b. odd number
c. either an even or an odd number
d. a multiple of 4

345. The "Simpson" sum is based on the area under a ____.

346. The error formula for Simpson's rule depends on___.

a. $f(x)$
b. $f'(x)$
c. $f^{(4)}(x)$
d. the number of steps

3.7 | Improper Integrals

Learning Objectives

3.7.1 Evaluate an integral over an infinite interval.
3.7.2 Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
3.7.3 Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the x-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^{+\infty} f(x)dx$? We can integrate $\int_a^t f(x)dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. Figure 3.17 shows that $\int_a^t f(x)dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

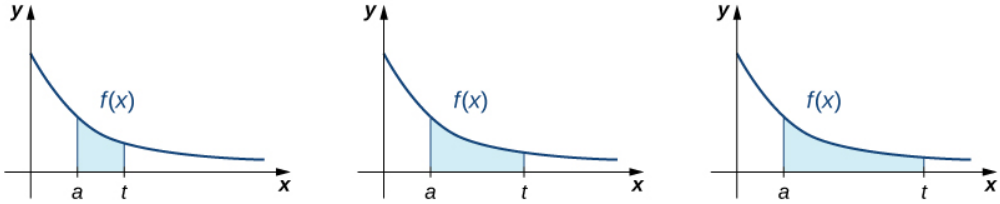


Figure 3.17 To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx, \tag{3.16}$$

provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx, \tag{3.17}$$

provided this limit exists.
In each case, if the limit exists, then the **improper integral** is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{+\infty} f(x)dx, \tag{3.18}$$

provided that $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$ for any value of a.)

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

Example 3.47

Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in the following graph.

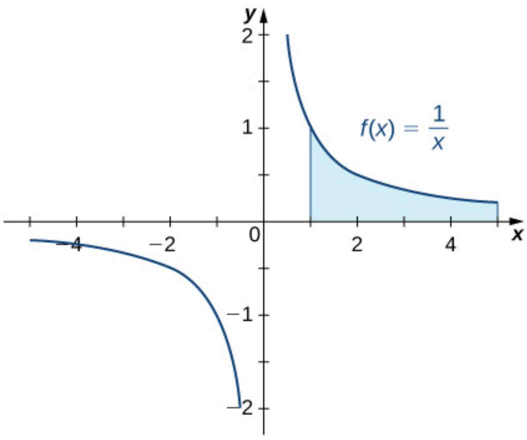


Figure 3.18 We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by $A = \int_{1}^{\infty} \frac{1}{x}dx$. Then we have

$$\begin{aligned} A &= \int_1^{\infty} \frac{1}{x} dx \\ &= \lim_{t \to +\infty} \int_1^t \frac{1}{x} dx && \text{Rewrite the improper integral as a limit.} \\ &= \lim_{t \to +\infty} \ln|x| \Big|_1^t && \text{Find the antiderivative.} \\ &= \lim_{t \to +\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative.} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example 3.48

Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution

The solid is shown in **Figure 3.19**. Using the disk method, we see that the volume V is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2} dx.$$

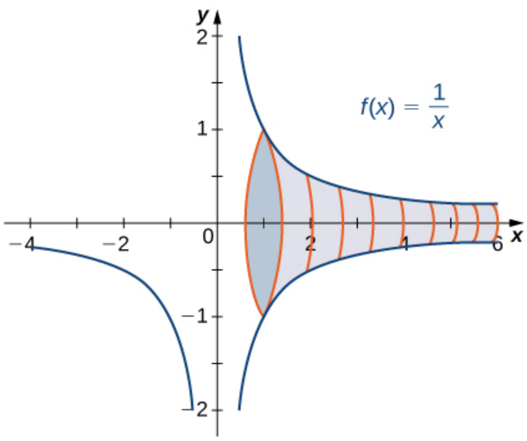


Figure 3.19 The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$
\begin{aligned}
V &= \pi \int_1^{+\infty} \frac{1}{x^2} dx \\
&= \pi \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx && \text{Rewrite as a limit.} \\
&= \pi \lim_{t \to +\infty} -\frac{1}{x}\Big|_1^t && \text{Find the antiderivative.} \\
&= \pi \lim_{t \to +\infty} \left(-\frac{1}{t} + 1\right) && \text{Evaluate the antiderivative.} \\
&= \pi.
\end{aligned}
$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the *x*-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the *x*-axis is finite. The solid generated is known as *Gabriel's Horn*.

Visit this **website (http://www.openstaxcollege.org/l/20_GabrielsHorn)** to read more about Gabriel's Horn.

Example 3.49

Chapter Opener: Traffic Accidents in a City

Figure 3.20 (credit: modification of work by David McKelvey, Flickr)

In the chapter opener, we stated the following problem: Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?

Probability theory tells us that if the average time between events is k, the probability that X, the time between events, is between a and b is given by

$$P(a \le x \le b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 \text{ if } x < 0 \\ ke^{-kx} \text{ if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

$$P(a \le x \le b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 3e^{-3x} \text{ if } x \ge 0 \end{cases}.$$

To answer the question, we must compute $P(X \ge 8) = \int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \ge 8) &= \int_8^{+\infty} 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} -e^{-3x}\Big|_8^t \\ &= \lim_{t \to +\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example 3.50

Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}dx$. State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}dx$ as a limit using **Equation 3.17** from the definition. Thus,

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{x^2+4}dx &= \lim_{t \to -\infty} \int_t^0 \frac{1}{x^2+4}dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to -\infty} \tan^{-1}\frac{x}{2}\Big|_t^0 && \text{Find the antiderivative.} \\ &= \lim_{t \to -\infty} \left(\tan^{-1} 0 - \tan^{-1}\frac{t}{2}\right) && \text{Evaluate the antiderivative.} \\ &= \frac{\pi}{2}. && \text{Evaluate the limit and simplify.} \end{aligned}$$

The improper integral converges to $\frac{\pi}{2}$.

Example 3.51

Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges. Compute each integral separately.

For the first integral,

$\int_{-\infty}^{0} xe^x\,dx = \lim\limits_{t \to -\infty} \int_{t}^{0} xe^x\,dx$	Rewrite as a limit.
$= \lim\limits_{t \to -\infty} (xe^x - e^x)\Big\vert_t^0$	Use integration by parts to find he antiderivative. (Here $u = x$ and $dv = e^x$.)
$= \lim\limits_{t \to -\infty} \left(-1 - te^t + e^t\right)$	Evaluate the antiderivative.
$= -1.$	Evaluate the limit. *Note:* $\lim\limits_{t \to -\infty} te^t$ is indeterminate of the form $0 \cdot \infty$. Thus, $\lim\limits_{t \to -\infty} te^t = \lim\limits_{t \to -\infty} \frac{t}{e^{-t}} = \lim\limits_{t \to -\infty} \frac{-1}{e^{-t}} = \lim\limits_{t \to -\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$\int_{0}^{+\infty} xe^x\,dx = \lim\limits_{t \to +\infty} \int_{0}^{t} xe^x\,dx$	Rewrite as a limit.
$= \lim\limits_{t \to +\infty} (xe^x - e^x)\Big\vert_0^t$	Find the antiderivative.
$= \lim\limits_{t \to +\infty} \left(te^t - e^t + 1\right)$	Evaluate the antiderivative.
$= \lim\limits_{t \to +\infty} \left((t-1)e^t + 1\right)$	Rewrite. ($te^t - e^t$ is indeterminate.)
$= +\infty.$	Evaluate the limit.

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

3.27 Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)dx,$ where $f(x)$ is continuous over $[a, b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a, t]$ for all values of t satisfying $a < t < b,$ the integral $\int_a^t f(x)dx$ is defined for all such values of t. Thus, it makes sense to consider the values of $\int_a^t f(x)dx$ as t approaches b for $a < t < b$. That is, we define $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx,$ provided this limit exists. **Figure 3.21** illustrates $\int_a^t f(x)dx$ as areas of regions for values of t approaching b.

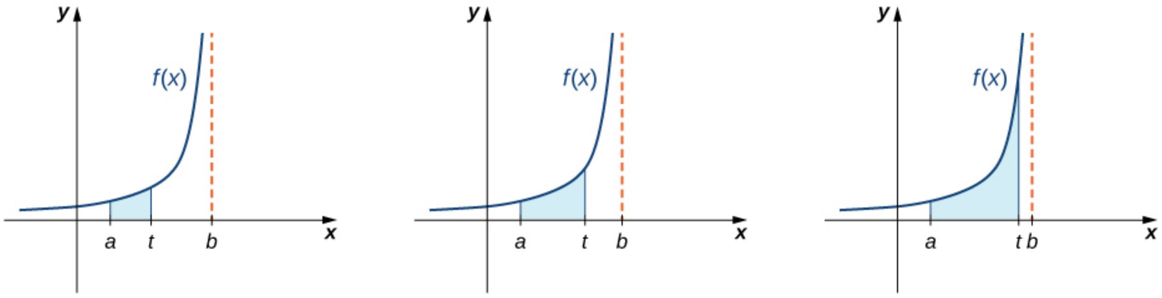


Figure 3.21 As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)dx,$ where $f(x)$ is continuous over $(a, b]$ and discontinuous at a. We now proceed with a formal definition.

Definition

1. Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx. \tag{3.19}$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx. \tag{3.20}$$

   In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in $(a, b),$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{3.21}$$

   provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

The following examples demonstrate the application of this definition.

Example 3.52

Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}dx,$ if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using **Equation 3.19** from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}dx$ as a limit:

$$\begin{aligned} \int_0^4 \frac{1}{\sqrt{4-x}}dx &= \lim_{t \to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.} \\ &= \lim_{t \to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.} \\ &= 4. && \text{Evaluate the limit.} \end{aligned}$$

The improper integral converges.

Example 3.53

Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x \ln x\,dx.$ State whether the integral converges or diverges.

Solution

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using **Equation 3.20**:

$$\begin{aligned} \int_0^2 x \ln x\,dx &= \lim_{t \to 0^+} \int_t^2 x \ln x\,dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to 0^+} \left(\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\right)\Big|_t^2 && \text{Evaluate } \int x \ln x\,dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x. \\ &= \lim_{t \to 0^+} \left(2\ln 2 - 1 - \tfrac{1}{2}t^2 \ln t + \tfrac{1}{4}t^2\right). && \text{Evaluate the antiderivative.} \\ &= 2\ln 2 - 1. && \text{Evaluate the limit. } \lim_{t \to 0^+} t^2 \ln t \text{ is indeterminate. To evaluate it, rewrite as a quotient and apply L'Hôpital's rule.} \end{aligned}$$

The improper integral converges.

Example 3.54

Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^{1} \frac{1}{x^3} dx$. State whether the improper integral converges or diverges.

Solution

Since $f(x) = 1/x^3$ is discontinuous at zero, using **Equation 3.21**, we can write

$$\int_{-1}^{1} \frac{1}{x^3} dx = \int_{-1}^{0} \frac{1}{x^3} dx + \int_{0}^{1} \frac{1}{x^3} dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^{0} \frac{1}{x^3} dx$:

$$\begin{aligned} \int_{-1}^{0} \frac{1}{x^3} dx &= \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^3} dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-1}^{t} && \text{Find the antiderivative.} \\ &= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) && \text{Evaluate the antiderivative.} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Therefore, $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges. Since $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges, $\int_{-1}^{1} \frac{1}{x^3} dx$ diverges.

3.28 Evaluate $\int_{0}^{2} \frac{1}{x} dx$. State whether the integral converges or diverges.

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (**Figure 3.22**). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_{a}^{t} f(x) dx \le \int_{a}^{t} g(x) dx \text{ for } t \ge a.$$

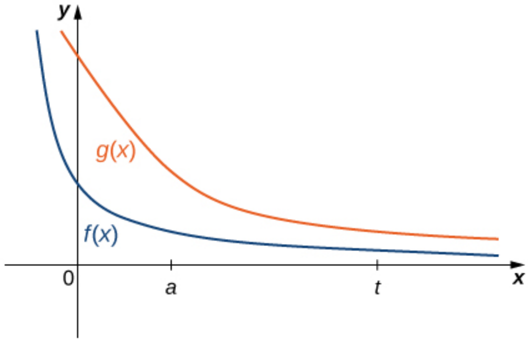


Figure 3.22 If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)dx \le \int_a^t g(x)dx$.

Thus, if

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = +\infty,$$

then

$\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = +\infty$ as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = L$ for some real number L, then

$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$ must converge to some value less than or equal to L, since $\int_a^t f(x)dx$ increases as t increases and $\int_a^t f(x)dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

Theorem 3.7: A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = +\infty$, then $\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = +\infty$.

ii. If $\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = L$, where L is a real number, then $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = M$ for some real number $M \le L$.

Example 3.55

Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x} dx$ converges.

Solution

We can see that

$$0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x},$$

so if $\int_1^{+\infty} e^{-x} dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x} dx$. To evaluate $\int_1^{+\infty} e^{-x} dx$, first rewrite it as a limit:

$$\begin{aligned} \int_1^{+\infty} e^{-x} dx &= \lim_{t \to +\infty} \int_1^t e^{-x} dx \\ &= \lim_{t \to +\infty} (-e^{-x})\Big|_1^t \\ &= \lim_{t \to +\infty} \left(-e^{-t} + e^1\right) \\ &= e^1. \end{aligned}$$

Since $\int_1^{+\infty} e^{-x} dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x} dx$.

Example 3.56

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Solution

For $p < 1$, $1/x \le 1/(x^p)$ over $[1, +\infty)$. In **Example 3.47**, we showed that $\int_1^{+\infty} \frac{1}{x} dx = +\infty$. Therefore, $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

3.29 Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Student PROJECT

Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$L\{t\} = \int_0^{\infty} te^{-st}dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$L\{t\} = \int_0^{\infty} te^{-st}dt = \lim_{z \to \infty} \int_0^z te^{-st}dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to *t*, so we treat the variable *s* as a constant. We have

$$\begin{aligned} u &= t & dv &= e^{-st}dt \\ du &= dt & v &= -\frac{1}{s}e^{-st}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \lim_{z \to \infty} \int_0^z te^{-st}dt &= \lim_{z \to \infty} \left[\left[-\frac{t}{s}e^{-st} \right]\Big|_0^z + \frac{1}{s}\int_0^z e^{-st}dt \right] \\ &= \lim_{z \to \infty} \left[\left[-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s} \right] + \frac{1}{s}\int_0^z e^{-st}dt \right] \\ &= \lim_{z \to \infty} \left[\left[-\frac{z}{s}e^{-sz} + 0 \right] - \frac{1}{s}\left[\frac{e^{-st}}{s} \right]\Big|_0^z \right] \\ &= \lim_{z \to \infty} \left[\left[-\frac{z}{s}e^{-sz} \right] - \frac{1}{s^2}\left[e^{-sz} - 1 \right] \right] \\ &= \lim_{z \to \infty} \left[-\frac{z}{se^{sz}} \right] - \lim_{z \to \infty} \left[\frac{1}{s^2 e^{sz}} \right] + \lim_{z \to \infty} \frac{1}{s^2} \\ &= 0 - 0 + \frac{1}{s^2} \\ &= \frac{1}{s^2}. \end{aligned}$$

1. Calculate the Laplace transform of $f(t) = 1$.

2. Calculate the Laplace transform of $f(t) = e^{-3t}$.

3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \lim_{z \to \infty} \int_0^z e^{-st} f(t)dt.$$

4. Use integration by parts to evaluate $\lim_{z \to \infty} \int_0^z e^{-st} f(t)dt$. (Let $u = f(t)$ and $dv = e^{-st} dt$.)

After integrating by parts and evaluating the limit, you should see that

$$L\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s}[L\{f'(t)\}].$$

Then,

$$L\{f'(t)\} = sL\{f(t)\} - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^t f(u)du$. Then,

$$L\{g(t)\} = \int_0^{\infty} e^{-st} g(t)dt = \lim_{z \to \infty} \int_0^z e^{-st} g(t)dt.$$

5. Use integration by parts to evaluate $\lim_{z \to \infty} \int_0^z e^{-st} g(t)dt$. (Let $u = g(t)$ and $dv = e^{-st} dt$. Note, by the way, that we have defined $g(t)$, $du = f(t)dt$.)

As you might expect, you should see that

$$L\{g(t)\} = \frac{1}{s} \cdot L\{f(t)\}.$$

Integration in the time domain simplifies to division by s in the frequency domain.

3.7 EXERCISES

Evaluate the following integrals. If the integral is not convergent, answer “divergent.”

347. $\int_2^4 \frac{dx}{(x-3)^2}$

348. $\int_0^\infty \frac{1}{4+x^2}dx$

349. $\int_0^2 \frac{1}{\sqrt{4-x^2}}dx$

350. $\int_1^\infty \frac{1}{x\ln x}dx$

351. $\int_1^\infty xe^{-x}\,dx$

352. $\int_{-\infty}^\infty \frac{x}{x^2+1}dx$

353. Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

354. Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x+1}}dx$ converges or diverges.

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

355. $\int_0^\infty e^{-x}\cos x\,dx$

356. $\int_1^\infty \frac{\ln x}{x}dx$

357. $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$

358. $\int_0^1 \ln x\,dx$

359. $\int_{-\infty}^\infty \frac{1}{x^2+1}dx$

360. $\int_1^5 \frac{dx}{\sqrt{x-1}}$

361. $\int_{-2}^2 \frac{dx}{(1+x)^2}$

362. $\int_0^\infty e^{-x}\,dx$

363. $\int_0^\infty \sin x\,dx$

364. $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}dx$

365. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

366. $\int_0^2 \frac{dx}{x^3}$

367. $\int_{-1}^2 \frac{dx}{x^3}$

368. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

369. $\int_0^3 \frac{1}{x-1}dx$

370. $\int_1^\infty \frac{5}{x^3}dx$

371. $\int_3^5 \frac{5}{(x-4)^2}dx$

Determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

372. $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

373. $\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^\infty \frac{dx}{2\sqrt{x}}$.

Evaluate the integrals. If the integral diverges, answer “diverges.”

374. $\int_1^{\infty} \frac{dx}{x^e}$

375. $\int_0^1 \frac{dx}{x^{\pi}}$

376. $\int_0^1 \frac{dx}{\sqrt{1-x}}$

377. $\int_0^1 \frac{dx}{1-x}$

378. $\int_{-\infty}^0 \frac{dx}{x^2+1}$

379. $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

380. $\int_0^1 \frac{\ln x}{x} dx$

381. $\int_0^e \ln(x) dx$

382. $\int_0^{\infty} xe^{-x} dx$

383. $\int_{-\infty}^{\infty} \frac{x}{\left(x^2+1\right)^2} dx$

384. $\int_0^{\infty} e^{-x} dx$

Evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

385. $\int_0^9 \frac{dx}{\sqrt{9-x}}$

386. $\int_{-27}^1 \frac{dx}{x^{2/3}}$

387. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

388. $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

389. $\int_0^4 x\ln(4x) dx$

390. $\int_0^3 \frac{x}{\sqrt{9-x^2}} dx$

391. Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

392. Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

393. Evaluate $\int_2^{\infty} \frac{dx}{(x^2-1)^{3/2}}$.

394. Find the area of the region in the first quadrant between the curve $y = e^{-6x}$ and the x-axis.

395. Find the area of the region bounded by the curve $y = \frac{7}{x^2}$, the x-axis, and on the left by $x = 1$.

396. Find the area under the curve $y = \frac{1}{(x+1)^{3/2}}$, bounded on the left by $x = 3$.

397. Find the area under $y = \frac{5}{1+x^2}$ in the first quadrant.

398. Find the volume of the solid generated by revolving about the x-axis the region under the curve $y = \frac{3}{x}$ from $x = 1$ to $x = \infty$.

399. Find the volume of the solid generated by revolving about the y-axis the region under the curve $y = 6e^{-2x}$ in the first quadrant.

400. Find the volume of the solid generated by revolving about the x-axis the area under the curve $y = 3e^{-x}$ in the first quadrant.

The Laplace transform of a continuous function over the interval $[0, \infty)$ is defined by $F(s) = \int_0^{\infty} e^{-sx} f(x) dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

401. $f(x) = 1$

402. $f(x) = x$

403. $f(x) = \cos(2x)$

404. $f(x) = e^{ax}$

405. Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

A function is a probability density function if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)dt = 1$. The probability that a random variable *x* lies between *a* and *b* is given by $P(a \le x \le b) = \int_a^b f(t)dt$.

406. Show that $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 7e^{-7x} \text{ if } x \ge 0 \end{cases}$ is a probability density function.

407. Find the probability that *x* is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

CHAPTER 3 REVIEW

KEY TERMS

absolute error if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$

computer algebra system (CAS) technology used to perform many mathematical tasks, including integration

improper integral an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

integration by parts a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

integration table a table that lists integration formulas

midpoint rule a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)dx$

numerical integration the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

partial fraction decomposition a technique used to break down a rational function into the sum of simple rational functions

power reduction formula a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

relative error error as a percentage of the absolute value, given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right| \cdot 100\%$

Simpson's rule a rule that approximates $\int_a^b f(x)dx$ using the integrals of a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)dx$ is given by $S_n = \frac{\Delta x}{3}\begin{pmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{pmatrix}$

trapezoidal rule a rule that approximates $\int_a^b f(x)dx$ using trapezoids

trigonometric integral an integral involving powers and products of trigonometric functions

trigonometric substitution an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

KEY EQUATIONS

- **Integration by parts formula**
  $\int u\,dv = uv - \int v\,du$
- **Integration by parts for definite integrals**
  $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**
  $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- **Sine and Cosine Products**
  $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$
- **Cosine Products**
  $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$
- **Power Reduction Formula**
  $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-1} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$
- **Power Reduction Formula**
  $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$
- **Midpoint rule**
  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$
- **Trapezoidal rule**
  $T_n = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$
- **Simpson's rule**
  $S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$
- **Error bound for midpoint rule**
  Error in $M_n \le \frac{M(b-a)^3}{24n^2}$
- **Error bound for trapezoidal rule**
  Error in $T_n \le \frac{M(b-a)^3}{12n^2}$
- **Error bound for Simpson's rule**
  Error in $S_n \le \frac{M(b-a)^5}{180n^4}$
- **Improper integrals**
  $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$
  $\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$
  $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{+\infty} f(x)dx$

KEY CONCEPTS

3.1 Integration by Parts

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

3.2 Trigonometric Integrals

- Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include
  1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
  2. Using integration by parts
  3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
  4. Applying reduction formulas

3.3 Trigonometric Substitution

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$.

3.4 Partial Fractions

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

3.5 Other Strategies for Integration

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

3.6 Numerical Integration

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson’s rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson’s rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

3.7 Improper Integrals

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.

- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

CHAPTER 3 REVIEW EXERCISES

For the following exercises, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

408. $\int e^x \sin(x)dx$ cannot be integrated by parts.

409. $\int \frac{1}{x^4+1}dx$ cannot be integrated using partial fractions.

410. In numerical integration, increasing the number of points decreases the error.

411. Integration by parts can always yield the integral.

For the following exercises, evaluate the integral using the specified method.

412. $\int x^2 \sin(4x)dx$ using integration by parts

413. $\int \frac{1}{x^2\sqrt{x^2+16}}dx$ using trigonometric substitution

414. $\int \sqrt{x}\ln(x)dx$ using integration by parts

415. $\int \frac{3x}{x^3+2x^2-5x-6}dx$ using partial fractions

416. $\int \frac{x^5}{\left(4x^2+4\right)^{5/2}}dx$ using trigonometric substitution

417. $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)dx$ using a table of integrals or a CAS

For the following exercises, integrate using whatever method you choose.

418. $\int \sin^2(x)\cos^2(x)dx$

419. $\int x^3\sqrt{x^2+2}\,dx$

420. $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}dx$

421. $\int \frac{1}{x^4+4}dx$

422. $\int \frac{\sqrt{3+16x^4}}{x^4}dx$

For the following exercises, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

423. **[T]** $\int_1^2 \sqrt{x^5+2}\,dx$

424. **[T]** $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}dx$

425. **[T]** $\int_1^4 \frac{\ln(1/x)}{x}dx$

For the following exercises, evaluate the integrals, if possible.

426. $\int_1^\infty \frac{1}{x^n}dx,$ for what values of n does this integral converge or diverge?

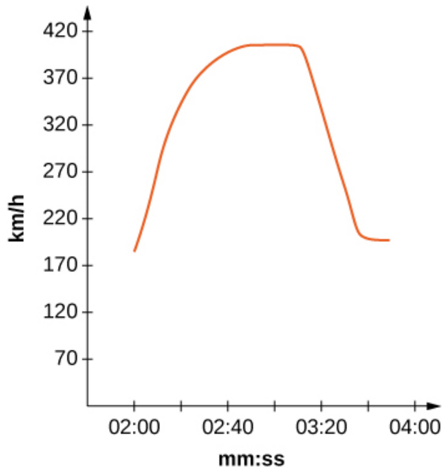
427. $\int_1^\infty \frac{e^{-x}}{x}dx$

For the following exercises, consider the gamma function given by $\Gamma(a)=\int_0^\infty e^{-y}y^{a-1}dy.$

428. Show that $\Gamma(a)=(a-1)\Gamma(a-1).$

429. Extend to show that $\Gamma(a) = (a-1)!,$ assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



430. **[T]** Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = a\exp^{bx}\sin(cx) + d.$ (*Hint:* Consider the time units.)

431. **[T]** Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.