Problem 1.

Suppose that X and Y are disjoint sets. One can prove that the function

$$\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y) \to \mathcal{P}_k(X \cup Y)$$

defined by

$$(A, B) \mapsto A \cup B$$

is a bijection. Taking this for granted, deduce that

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}. \tag{1}$$

Sol. Let $m, n \in \mathbb{N}$ and let X and Y be disjoint finite sets with $|X| = m, |Y| = n$. Then we have

$$\left|\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)\right| = |\mathcal{P}_k(X \cup Y)| \tag{2}$$

since there is a bijection between these two finite sets. By a theorem proved in class, the cardinality in the right-hand side is equal to $\binom{m+n}{k}$, since

$$|X \cup Y| = |X| + |Y| = m + n$$

(recall that X and Y are disjoint). On the other hand, the sets in the union in the left-hand side are clearly pairwise disjoint so that we have, by the addition and multiplication principles,

$$\begin{aligned} \left|\bigcup_{j=0}^{k} \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)\right| &= \sum_{j=0}^{k} |\mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)| \\ &= \sum_{j=0}^{k} |\mathcal{P}_j(X)||\mathcal{P}_{k-j}(Y)| = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}. \end{aligned}$$

Combining this with equation (2) gives the result.

Problem 2.

Prove that for $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0.$$

Sol. We have, by the binomial theorem with $a = 1$ and $b = -1$,

$$0 = (1 + (-1))^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (-1)^j = \sum_{j=0}^{n} (-1)^j \binom{n}{j}.$$

Problem 3.

What is the sum of the coefficients of the polynomial $(1 + x)^{53}$?

Sol. By the binomial theorem, we have

$$(1 + x)^{53} = \sum_{j=0}^{53} \binom{53}{j} 1^{53-j} x^j = \sum_{j=0}^{53} \binom{53}{j} x^j$$

hence the sum of the coefficients is

$$\sum_{j=0}^{53} \binom{53}{j} = 2^{53}.$$