Problem 1.

Suppose that X and Y are non-empty finite sets with $|X| = |Y|$, and let $f : X \to Y$ be a function. Show that f is injective if and only if it is surjective.

Sol. Assume that f is injective. Then f can be viewed as an injection $X \to \operatorname{Im} f$, so that $|Y| = |X| \leq |\operatorname{Im} f|$, by Corollary 11.1.1. On the other hand, $\operatorname{Im} f$ is a subset of the finite set Y, so that $|\operatorname{Im} f| \leq |Y|$ by Corollary 11.1.5. Therefore, we get that the subset $\operatorname{Im} f$ of Y has the same cardinality as Y, which clearly implies that $\operatorname{Im} f = Y$, so that f is surjective, as required.

Conversely, assume that $f : X \to Y$ is surjective. By Homework 6, Problem 1, f has a right-inverse $g : Y \to X$, meaning that $f \circ g = I_Y$. It easily follows from this that g is injective. By the first part of the proof, g is also surjective, i.e. g is a bijection. This implies that f is injective. Indeed, let $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Since $g : Y \to X$ is surjective, there exist $y_1, y_2 \in Y$ with $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Then the equality $f(x_1) = f(x_2)$ becomes $f(g(y_1)) = f(g(y_2))$, i.e. $y_1 = y_2$ since $f \circ g = I_Y$. Therefore, $x_1 = g(y_1) = g(y_2) = x_2$, as required.

Problem 2.

Let X be a set and suppose that $f : \mathbb{N} \to X$ is injective. Prove that X is an infinite set.

Sol. Assume for a contradiction that X is finite, say $|X| = n$ for some $n \in \mathbb{N}$. Since $f : \mathbb{N} \to X$ is injective, the restriction $f|_{\mathbb{N}_{n+1}} : \mathbb{N}_{n+1} \to X$ also is, so that $n + 1 = |\mathbb{N}_{n+1}| \leq |X| = n$ by Corollary 11.1.1, a contradiction. Hence X is infinite.

Problem 3.

Prove that if seven distinct numbers are chosen from the set $\{1, 2, \ldots, 11\}$, then there are two of these numbers which sum to 12.

Sol. Consider the following six sets : $\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}$. By the Pigeonhole Principle, one of these sets must contain two of our chosen numbers. These two numbers must add to 12.

Problem 4.

Let A be a set containing 51 positive integers, each less than or equal to 100. Prove that there are two elements of A which are consecutive numbers.

Sol. Consider the following 50 sets : $\{1,2\},\{3,4\},\{5,6\},\ldots,\{99,100\}$. By the Pigeonhole principle, one of these sets must contain two elements of A. These two elements of A are then consecutive numbers.

Problem 5.

Prove that if five points are chosen inside a square of side length 2, then there are two of them which are at distance no greater than $\sqrt{2}$ from each other.

Sol. Divide the square into four squares of side length 1 by joining the middle points of opposite sides. By the Piegeonhole principle, one of these square must contain two of the points. These two points are contained in a square of side length 1, hence the distance between them is less than or equal to the length of the diagonal of the square, which is $\sqrt{2}$.

Problem 6.

Let A be a set containing ten positive integers, each less than or equal to 100. Prove that there exist two disjoint non-empty subsets of A which have the same sum of elements.

Sol. Consider the function f on $\mathcal{P}(A)\setminus\{\emptyset\}$ which maps a subset B of A to the sum of its elements. Each sum is certainly less than

$$91+92+93+94+95+96+97+98+99+100=955,$$

so that the image of f is contained in $\mathbb{N}_{955}$. Now, by Proposition 12.2.1, there are $2^{10}-1=1023$ non-empty subsets of A. By the Pigeonhole Principle, the function f is not injective, which is equivalent to saying that two non-empty subsets of A have the same sum of elements. If these two sets are disjoint, we are done. If not, note that any of the two sets is not a subset of the other, since otherwise the sum of their elements would be different. By removing from each of the sets their common element, we obtain two disjoint non-empty sets which still have the same sum of elements.