Problem 1.
Let $f : X \to Y$ be a function. Prove that there exists a function $g : Y \to X$ such that $f \circ g = I_Y$ if and only if f is surjective. Here I_Y denotes the identity function on Y.

Sol.
($\Rightarrow$) Suppose that such a function $g : Y \to X$ exists. Let us prove then that f is surjective. Suppose that $y \in Y$. If we put $x = g(y)$, then $x \in X$ and $f(x) = f(g(y)) = I_Y(y) = y$. This shows that f is surjective.

($\Leftarrow$) Suppose that f is surjective. This means for any $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$. We can therefore define a function $g : Y \to X$ by the rule that, for each $y \in Y$, $g(y)$ is some element in X with $f(g(y)) = y$. Then by construction, $f \circ g = I_Y$.

Problem 2.
Let $f : X \to Y$ be a function, let A_1, A_2 be subsets of X and B_1, B_2 be subsets of Y.

(1) Prove that $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$, but that the converse is false in general. Is the converse true if f is injective? Explain.
(2) Prove that $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$, but that the converse is false in general. Is the converse true if f is surjective? Explain.
(3) Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Give an example for which equality does not hold.
(4) Prove that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Sol.
(1): Suppose that $A_1 \subseteq A_2$. If $f(x) \in f(A_1)$ where $x \in A_1$, then $x \in A_2$, so that $f(x) \in f(A_2)$, as required. To see that the converse is false in general, define a function f on $\{0, 1\}$ by $f(0) = f(1) = 0$, and put $A_1 = \{0\}$ and $A_2 = \{1\}$. Then $f(A_1) = f(\{0\}) = \{0\}$ and $f(A_2) = f(\{1\}) = \{0\} = f(A_1)$, so that in particular $f(A_1) \subseteq f(A_2)$, but $A_1 \nsubseteq A_2$.

The converse is true though if f is injective. Indeed, suppose that $f(A_1) \subseteq f(A_2)$ and let $x_0 \in A_1$. Then $f(x_0) \in f(A_1) \subseteq f(A_2)$ and thus there exists an element $x_1 \in A_2$ with $f(x_1) = f(x_0)$. By injectivity, we get $x_0 = x_1 \in A_2$. This shows that $A_1 \subseteq A_2$.

(2): Suppose that $B_1 \subseteq B_2$, and let $x_0 \in f^{-1}(B_1)$. Then $f(x_0) \in B_1 \subseteq B_2$, so that $x_0 \in f^{-1}(B_2)$ by definition. This shows that $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

To see that the converse is false in general, define a function $f : \{0\} \to \{0, 1\}$ by $f(0) = 0$, and take $B_1 = \{1\}$, $B_2 = \{0\}$. Then $f^{-1}(B_1) = \emptyset \subseteq \{0\} = f^{-1}(B_2)$ but clearly $B_1 \nsubseteq B_2$.

The converse is true though if f is surjective. Indeed, suppose that $f^{-1}(B_1) \subseteq f^{-1}(B_2)$ and let $y_0 \in B_1$. By surjectivity, there exists an element $x_0 \in X$ with $f(x_0) = y_0$. Then $x_0 \in f^{-1}(B_1) \subseteq f^{-1}(B_2)$, so that $y_0 = f(x_0) \in B_2$. This shows that $B_1 \subseteq B_2$.

(3): Let $f(x) \in f(A_1 \cap A_2)$ where $x \in A_1 \cap A_2$. Then $x \in A_1$ and $x \in A_2$, so that $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$, i.e. $f(x) \in f(A_1) \cap f(A_2)$, as required. To see that equality does not hold in general, define a function f on $\{0, 1\}$ by $f(0) = f(1) = 0$, and put $A_1 = \{0\}$ and $A_2 = \{1\}$. Then $f(A_1) = \{0\} = f(A_2)$ so that $f(A_1) \cap f(A_2) = \{0\}$, but $A_1 \cap A_2 = \emptyset$ so that $f(A_1 \cap A_2) = \emptyset$.

(4): Let $f(x) \in f(A_1 \cup A_2)$ where $x \in A_1 \cup A_2$. Then $x \in A_1$ or $x \in A_2$. In the first case, we get $f(x) \in f(A_1)$ and in the second case, we get $f(x) \in f(A_2)$. In both cases, $f(x) \in f(A_1) \cup f(A_2)$ and therefore $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.
For the reverse inclusion, assume that $y \in f(A_1) \cup f(A_2)$, so that $y \in f(A_1)$ or $y \in f(A_2)$. In both cases, there exists an element $x \in A_1 \cup A_2$ with $y = f(x)$, so that $y \in f(A_1 \cup A_2)$, as required.

Problem 3.
Suppose that there are 153 students enrolled in at least one of the three first year core Mathematics courses (Logic, Algebra and Calculus). If 100 of these students like Logic, 100 like Algebra, 100 like Calculus, 56 like Logic and Algebra, 60 like Logic and Calculus, 57 like Algebra and Calculus, and 25 like all three courses, how many of the students like none of the courses?

Sol. Let X be the set of all students who like Logic, Y the set of all students who like Algebra and Z the set of all students who like Calculus. Then the number we are looking for is $153 - |X \cup Y \cup Z|$. By the inclusion-exclusion principle, we have

$$\begin{aligned} |X \cup Y \cup Z| &= |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z| \\ &= 100 + 100 + 100 - 56 - 60 - 57 + 25 = 152. \end{aligned}$$

Thus the answer is $153 - 152 = 1$.

Problem 4.
At a Mathematics Conference of 100 participants, 75 speak English, 60 speak Spanish and 45 speak Italian, and everyone present speaks at least one of these languages.

(1) What is the maximum number of participants who can speak only one language?
(2) What is the maximum number of participants who speak only english?
(3) Prove that the greater the number of participants who speak all three languages, the greater the number of participants who speak only one language.

Sol. Let X be the set of all participants who speak English, Y the set of all participants who speak Spanish and Z the set of all participants who speak Italian. Then $|X \cup Y \cup Z| = 100, |X| = 75, |Y| = 60$ and $|Z| = 45$.

(1): To maximize the number of participants who can speak only one language, we need to minimize the number of people who speak more than one language. A Venn diagram shows that this quantity is

$$|X \cap Y| + |X \cap Z| + |Y \cap Z| - 2|X \cap Y \cap Z|.$$

By the inclusion-exclusion principle, we have

$$100 = 180 - (|X \cap Y| + |X \cap Z| + |Y \cap Z|) + |X \cap Y \cap Z|,$$

since $|X| + |Y| + |Z| = 180$, so that

$$(|X \cap Y| + |X \cap Z| + |Y \cap Z|) - |X \cap Y \cap Z| = 80$$

and thus we want to minimize

$$80 - |X \cap Y \cap Z|,$$

which is equivalent to maximizing $|X \cap Y \cap Z|$, which proves (3).
To maximize $|X \cap Y \cap Z|$, note that we have

$$|X \cap Y| + |X \cap Z| + |Y \cap Z| \geq 3|X \cap Y \cap Z|,$$

since the set on the right-hand side is a subset of each of the sets on the left-hand side. We get

$$80 = (|X \cap Y| + |X \cap Z| + |Y \cap Z|) - |X \cap Y \cap Z| \geq 2|X \cap Y \cap Z|,$$

so that the maximal value of $|X \cap Y \cap Z|$ is 40, which is attained when each of the double intersection is equal to $X \cap Y \cap Z$. By the first equation, the answer is $100 - (40 + 40 + 40 - 2(40)) = 100 - 40 = 60$.

(2): To maximize the number of participants who speak only English, we need to minimize the number of people who speak Spanish or Italian. This number is at least 60, since 60 people speak Spanish. Therefore the number of participants who speak only English is at most $100 - 60 = 40$.