Problem 1.

Prove by induction on n that $n! > 2^n$ for all integers $n \geq 4$.

Sol. Let us prove by induction on $n \geq 4$ the statement $n! > 2^n$, which we denote by $P(n)$.

Base case: If $n = 4$, then $n! = 4! = 24$ and $2^n = 2^4 = 16$. Since $24 > 16$, $P(4)$ is true.

Inductive step: Suppose that $k! > 2^k$ for some integer $k \geq 4$ (inductive hypothesis). We have

$$(k+1)! = (k+1)k! > (k+1)2^k > (2)2^k = 2^{k+1},$$

by the induction hypothesis and by the fact that $k+1 > 2$. This shows that $P(k+1)$ is true, as required.

Conclusion: Therefore, by the induction principle, $n! > 2^n$ for all $n \geq 4$.

Problem 2.

Prove by induction on n that

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

for all positive integers n.

Sol. We prove by induction on $n \geq 1$ the equality, which we denote by $P(n)$.

Base case: For $n = 1$, the sum on the left-hand side is equal to the first term, which is $1/2$. The right-hand side is also equal to $1/2$, which shows that $P(1)$ is true.

Inductive step: Assume that

$$\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}$$

for some integer $k \geq 1$ (inductive hypothesis). We have

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)},$$

by the induction hypothesis. This last expression is equal to

$$\frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Hence $P(k+1)$ is true.

Conclusion: Therefore, by the induction principle, the equality

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

holds for all positive integers n.

Problem 3.

For a positive integer n, the number a_n is defined inductively by

$$a_1 = 1$$

and

$$a_{k+1} = \frac{6a_k + 5}{a_k + 2}$$

for k a positive integer. Prove by induction on $n \geq 1$ that $0 < a_n < 5$.

Sol. Let us prove by induction on $n \geq 1$ the statement $0 < a_n < 5$, which we denote by $P(n)$.

Base case : For $n = 1$, $a_1 = 1$ and so clearly $0 < a_1 < 5$. This shows that $P(1)$ is true.

Inductive step : Suppose that $0 < a_k < 5$ for some positive integer k (inductive hypothesis). First, since $a_k > 0$, we have $6a_k + 5 > 0$ and $a_k + 2 > 0$, from which it follows that

$$0 < \frac{6a_k + 5}{a_k + 2} = a_{k+1}.$$

Secondly, we have

$$5 > a_{k+1} = \frac{6a_k + 5}{a_k + 2}$$

if and only if

$$5a_k + 10 > 6a_k + 5$$

if and only if $5 > a_k$, which is true. This shows that $0 < a_{k+1} < 5$ and thus $P(k+1)$ is true.

Conclusion : Therefore, by the induction principle, $0 < a_n < 5$ for all positive integers n.

Problem 4.

Prove by induction on n that

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$$

for integers $n \geq 2$. See Problem 18 p.55 for the inductive definition of the product.

Sol. Let us prove by induction on $n \geq 2$ the equality, which we denote by $P(n)$.

Base case : For $n = 2$, the product is equal to the first factor, which is $1 - 1/4 = 3/4$. The right-hand side of the equality is also equal to $3/4$, so that $P(2)$ is true.

Inductive step : Suppose that

$$\prod_{j=2}^{k}\left(1-\frac{1}{j^2}\right)=\frac{k+1}{2k}$$

for some integer $k \geq 2$ (inductive hypothesis). We have

$$\begin{aligned}\prod_{j=2}^{k+1}\left(1-\frac{1}{j^2}\right)=\prod_{j=2}^{k}\left(1-\frac{1}{j^2}\right)\left(1-\frac{1}{(k+1)^2}\right) &= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)\\ &= \frac{k+1}{2k}\frac{(k+1)^2-1}{(k+1)^2}\\ &= \frac{k^2+2k}{2k(k+1)}=\frac{k+2}{2(k+1)},\end{aligned}$$

which proves $P(k+1)$.

Conclusion : Therefore, by the induction principle,

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$$

for integers $n \geq 2$.