

Problem 1.

Prove the following statements for x a real number.

- (1) $x^2 - x - 2 = 0 \Leftrightarrow x = -1$ or $x = 2$.
- (2) $x^2 - x - 2 > 0 \Leftrightarrow x < -1$ or $x > 2$.

Sol. To prove (1), first note that $x^2 - x - 2 = (x + 1)(x - 2)$. To prove the first implication (\Rightarrow), assume that $(x + 1)(x - 2) = 0$. We want to prove that $x = -1$ or $x = 2$. If $x \neq -1$, then $x + 1 \neq 0$ and we can divide through by $x + 1$ to obtain $x - 2 = 0$, i.e. $x = 2$. This shows that $x = -1$ or $x = 2$. The converse easily follows from the fact that $0 \times a = 0 = a \times 0$ for all real numbers a .

Let us now prove (2). For the first implication (\Rightarrow), assume that $(x + 1)(x - 2) > 0$. We want to prove that $x < -1$ or $x > 2$. Suppose that $x \geq -1$. If $x = -1$, then $x + 1 = 0$ and $(x + 1)(x - 2) = 0$, which is a contradiction. Hence $x > -1$, i.e. $x + 1 > 0$. This, together with $(x + 1)(x - 2) > 0$, implies that $x - 2 > 0$, by the multiplicative law for inequalities. Thus $x < -1$ or $x > 2$.

For the converse (\Leftarrow), we proceed by cases. If $x < -1$ (i.e. $x + 1 < 0$), then $x < 2$ (i.e. $x - 2 < 0$) and, by the multiplicative law for inequalities, we get $x^2 - x - 2 = (x + 1)(x - 2) > 0$. Likewise, if $x > 2$ (i.e. $x - 2 > 0$), then $x > -1$ (i.e. $x + 1 > 0$) and we also get $x^2 - x - 2 = (x + 1)(x - 2) > 0$. Therefore, $x < -1$ or $x > 2$ implies $x^2 - x - 2 > 0$, which is what we wanted to prove.

Problem 2.

Prove by contradiction that there does not exist a largest integer.

Sol. Suppose, for a contradiction, that there exists a largest integer, say n . But then $n + 1$ is an integer which is larger than n (since $n + 1 > n$ by the addition law for inequalities), a contradiction. Hence there does not exist a largest integer.

Problem 3.

Prove by contradiction that there does not exist a smallest positive real number.

Sol. Suppose, for a contradiction, that there exists a smallest positive real number, say $x > 0$. But then $x/2$ is a positive real number which is smaller

than x (since $x/2 < x$ by the multiplicative law for inequalities), a contradiction. Therefore, there does not exist a smallest positive real number.

Problem 4.

Use induction to prove Bernoulli's inequality :

$$(1 + x)^n \geq 1 + nx$$

for all non-negative integers n and real numbers $x > -1$.

Sol. Fix some real number $x > -1$. We use induction on $n \geq 0$ to prove the statement $(1 + x)^n \geq 1 + nx$, which we denote by $P(n)$.

Base case: If $n = 0$, then $(1 + x)^n = (1 + x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. Since $1 \geq 1$, we get that $P(0)$ is true.

Inductive step: Suppose that $P(k)$ is true for some integer k , that is $(1 + x)^k \geq 1 + kx$ (inductive hypothesis). We have to show that $P(k + 1)$ is true, i.e. $(1 + x)^{k+1} \geq 1 + (k + 1)x$. We have

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)^k(1 + x) \geq (1 + kx)(1 + x) \\ &= 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x, \end{aligned}$$

where we used the inductive hypothesis and the multiplicative law for inequalities, which is valid since $1 + x > 0$. Thus $P(k + 1)$ is true.

Conclusion: Therefore, by the induction principle, $P(n)$ is true for all $n \geq 0$, that is $(1 + x)^n \geq 1 + nx$ for all $n \geq 0$.

Problem 5.

Let us prove by induction on $n \geq 1$ the statement 'In any group of n people, all people have the same sex', which we denote by $P(n)$.

Base case: Clearly, in any group of one people, all people have the same sex. Thus $P(1)$ is true.

Inductive step : Suppose that $P(k)$ is true for some integer k , that is, in any group of k people, all people have the same sex (inductive hypothesis). We have to prove that $P(k + 1)$ is true. To do this, consider a group of $k + 1$ people.

First, exclude the last person and consider only the first k people. Then all these people have the same sex, by the induction hypothesis. Likewise, exclude the first person and consider only the last k people. Then these too must have the same sex. Therefore, the first person in the group is of the same sex as the people in the middle, who in turn are of the same sex as

the last person. Hence everyone in the group is of the same sex.

This shows that $P(k + 1)$ is true.

Conclusion Therefore, by the induction principle, $P(n)$ is true for all integers $n \geq 1$. In other words, in any group of n people, all people have the same sex.

But this is clearly false! Explain what is wrong in the above proof.

Sol. The problem is that the inductive step breaks down in the case $k = 1$. In other words, the implication $P(1) \Rightarrow P(2)$ is false. Indeed, in this case, the argument doesn't work since the middle group is empty.