MAT 200, Logic, Language and Proof, Fall 2015 — Final Exam

Name :

Student ID :

Problem 1 (20 pts).

a. Let A and B be two sets. Give the *mathematical* definitions of $A \cup B$, $A \cap B$ and $A \setminus B$, and illustrate each of these three sets by a Venn diagram.

Sol. We have $A \cup B = \{x : x \in A \text{ or } x \in B\}$, $A \cap B = \{x : x \in A \text{ and } x \in B\}$ and $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. The Venn diagrams can be found in the book, p.69.

b. Let X be a set. Give the definition of $\mathcal{P}(X)$, the power set of X.

Sol. $\mathcal{P}(X)$ is the set of all subsets of X :

$$\mathcal{P}(X) = \{A : A \subseteq X\}.$$

c. If $X = \{1, 2, 3\}$, what is $\mathcal{P}(X)$?

Sol. We have

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

d. If A and B are two subsets of some universal set U, write $(A \cup B)^c$ and $(A \cap B)^c$ in terms of A^c and B^c. In other words, state the De Morgan laws. No justification is needed here.

Sol. We have $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Problem 2 (20 pts).

a. Give the definition of the binomial coefficient $\binom{n}{r}$, where r and n are two non-negative integers.

Sol. The binomial coefficient $\binom{n}{r}$ is the cardinality of $\mathcal{P}_r(X)$ when $|X| = n$, i.e. it is the number of subsets of a set of cardinality n which have cardinality r.

b. What is the relationship between the two binomial coefficients $\binom{132}{37}$ and $\binom{132}{95}$? Give a short justification.

Sol. They are equal, since

$$\binom{n}{r} = \binom{n}{n-r}$$

for any integers n, r with $0 \leq r \leq n$.

c. State the Binomial Theorem.

Sol. The theorem states that for all real numbers a and b and non-negative integers n, we have

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.$$

Problem 3 (20 pts).

a. Let a and b be positive integers, and suppose that there exist integers m and n such that

$$ma + nb = 1.$$

Prove that $(a, b) = 1$.

Sol. If $d = (a, b)$, then d divides both a and b, so that it must divide $ma + nb = 1$. It follows that $d = 1$.

b. Let a and b be positive integers, and let m and n be integers such that

$$ma + nb = d,$$

where $d = (a, b)$. Prove that $(m, n) = 1$.

Sol. From $ma + nb = d$ we get

$$m\frac{a}{d} + n\frac{b}{d} = 1.$$

Since a/d and b/d are integers (d is a common divisor of a and b), it follows from part **a.** that $(m, n) = 1$.

c. Find all solutions m, n to the diophantine equation

$$133m + 56n = 35.$$

Sol. With the euclidean algorithm, we find that $(133, 56) = 7$. Since 7 divides 35, the equation has solutions. Proceeding backwards in the euclidean algorithm, we find that

$$3 \times 133 - 7 \times 56 = 7,$$

so that $m_0 = 15$ and $n_0 = -35$ is a solution to

$$133m_0 + 56n_0 = 35.$$

It follows from a theorem proved in class that all solutions m and n are given by

$$m = m_0 + \frac{b}{(a, b)}q = 15 + 8q$$

and

$$n = n_0 - \frac{a}{(a, b)}q = -35 - 19q$$

for some $q \in \mathbb{Z}$.

Problem 4 (20 pts).

a. Prove that 7 divides $3 \cdot 2^{101} + 9$.

Sol. Since $2^3 \equiv 1 \mod 7$, we have that

$$3 \cdot 2^{101} + 9 = 3 \cdot (2^3)^{33} \cdot 2^2 + 9 \equiv 3 \cdot 2^2 + 9 \mod 7 \equiv 21 \mod 7 \equiv 0 \mod 7,$$

which proves the result.

b. Prove that a positive integer n is divisible by 9 if and only if the sum of its digits is divisible by 9.
(*Hint* : Write $n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$, where $a_0, a_1, \ldots, a_{k-1}, a_k$ are the digits of n.)

Sol. Since $10 \equiv 1 \mod 9$, we have

$$n \equiv 1^k a_k + 1^{k-1} a_{k-1} + \cdots + 1a_1 + a_0 \mod 9 \equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \mod 9,$$

so that n is divisible by 9 if and only if $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 9, as required.

Problem 5 (20 pts).

a. Define what is a prime number, and state the Fundamental Theorem of Arithmetic.

Sol. A positive integer n is prime if $n > 1$ and the only positive divisors of n are 1 and n. The Fundamental Theorem of Arithmetic states that every positive integer greater than 1 can be written uniquely as a product of prime numbers, with the prime factors in the product written in non-decreasing order.

b. Prove by contradiction that there are infinitely many prime numbers. (*Hint* : Suppose that $p_1, p_2, \ldots, p_n$ are the only prime numbers, and consider the number $m = p_1 p_2 \cdots p_n + 1$.)

Sol. See Theorem 23.5.1, p.285