NEWTON'S METHOD FOR ANALYTIC SYSTEMS OF EQUATIONS WITH CONSTANT RANK DERIVATIVES.

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Abstract. In this paper we study the convergence properties of Newton's sequence for analytic systems of equations with constant rank derivatives. Our main result is an alpha-theorem which insures the convergence of Newton's sequence to a least-square solution of this system.

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1. Introduction.

Newton's method is a classical numerical method to solve a system of nonlinear equations

$$f : \mathbf{E} \to \mathbf{F}$$

with $\mathbf{E}$ and $\mathbf{F}$ two Euclidean spaces or more generally two Banach spaces. If $x \in \mathbf{E}$ is an approximation of a zero of this system then, Newton's method updates this approximation by linearizing the equation $f(y) = 0$ around x so that

$$f(x) + Df(x)(y - x) = 0.$$

When $Df(x)$ is an isomorphism we obtain the classical Newton's iterate

$$y = N_f(x) = x - Df(x)^{-1} f(x).$$

When $\mathbf{E}$ and $\mathbf{F}$ are two Euclidean spaces and when $Df(x)$ is not an isomorphism we choose its Moore-Penrose inverse $Df(x)^\dagger$ instead of its classical inverse:

$$y = N_f(x) = x - Df(x)^\dagger f(x).$$

We recall that the Moore-Penrose inverse of a linear operator

$$A : \mathbf{E} \to \mathbf{F}$$

is the composition of two maps : $A^\dagger = B \circ \Pi_{\mathrm{Im}\ A}$ where $\Pi_{\mathrm{Im}\ A}$ is the orthogonal projection in $\mathbf{F}$ onto Im A and B is the right inverse of A whose image is the orthogonal complement of Ker A in $\mathbf{E}$ i.e. the inverse of the restriction

$$A|_{(\mathrm{Ker}\ A)^\perp} : (\mathrm{Ker}\ A)^\perp \to \mathrm{Im}\ A.$$

We have $A^\dagger = (A^*A)^{-1}A^*$ when A is injective, $A^\dagger = A^*(AA^*)^{-1}$ when A is surjective, where A^* denotes the adjoint of A. Notice that $A^\dagger A = \Pi_{(\mathrm{Ker}\ A)^\perp}$ and $AA^\dagger = \Pi_{\mathrm{Im}\ A}$.

For underdetermined systems, when $Df(x)$ is surjective, $Df(x)^\dagger$ is injective in $\mathbf{F}$ and hence the zeros of $f(x)$ corresponds to the fixed points of the Newton operator

$$N_f(x) = x - Df(x)^\dagger f(x).$$

The case of overdetermined systems is completely different. This iteration has been introduced for the first time by Gauss in 1809 [6] and, for this reason, it is called Newton-Gauss iteration. When $Df(x)$ is injective, the fixed points of $N_f(x)$ do not necessarily correspond to the zeros of f but to the least-square solutions of $f(x) = 0$, i.e. to the stationary points of $F(x) = \|f(x)\|^2$. In other words $N_f(x) = x$ if and only if $D(\|f(x)\|^2) = 0$.

In this paper, our aim is to study the properties of Newton's iteration for analytic systems of equations with constant rank derivatives. This case generalizes both the underdetermined case (Rank $Df(x) =$ Dim $\mathbf{F}$) and the overdetermined case of (Rank $Df(x) =$ Dim $\mathbf{E}$). It has been considered for the first time by Ben-Israel [2].

We consider an analytic function $f : \mathbf{E} \to \mathbf{F}$ between two Euclidean spaces. We let $n =$ Dim $\mathbf{E}$ and $m =$ Dim $\mathbf{F}$. We also consider the case of a function f defined in an open set $U \subset \mathbf{E}$ but by abuse of notation we continue to write $f : \mathbf{E} \to \mathbf{F}$.

As in the injective-overdetermined case, the fixed points of Newton's operator do not necessarily correspond to the zeros of f but to the least square solutions of this system:

Proposition 1. *The following statements are equivalent :*

1. $N_f(x) = x$,
2. $Df(x)^\dagger f(x) = 0$,
3. $Df(x)^* f(x) = 0$,
4. $f(x) \in \mathrm{Im}\ Df(x)^\perp$,
5. $DF(x) = 0$ *with* $F(x) = \|f(x)\|^2$.

The proof is easy and left to the reader. □

There are two points of view to analyze the convergence properties for Newton's method: Kantorovich like theorems and Smale's alpha-theory. Let $x \in \mathbf{E}$ be given. Under which hypothesis does the sequence

$$x_{k+1} = N_f(x_k),\ x_0 = x,$$

converges to a zero ξ of f?

Kantorovich gives an answer in terms of the behavior of f in a neighborhood of x with a weak regularity assumption, say f is C^2. See Ostrowski [12] or Ortega-Rheinboldt [11].

Alpha-theory, which was introduced by Kim in [8], [9] for one variable polynomial equations and by Smale for general systems of equations in [18], gives an answer in terms of three invariants.

$$\alpha(f,x) = \beta(f,x)\gamma(f,x)$$

$$\beta(f,x) = \|Df(x)^{-1}f(x)\|$$

$$\gamma(f,x) = \sup_{k\geq 2} \left\| Df(x)^{-1} \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}$$

which only depend on the derivatives $D^k f(x)$ at the given starting point x. Here a stronger regularity assumption is made: f is an analytic system of equations.

The main feature of Newton's iteration is its quadratic convergence to the zeros of f. Alpha-theory gives the size of the basin of attraction around these zeros in terms of the invariant $\gamma(f,x)$. We have :

Theorem 1. *(Smale) When ξ is a zero of f and $Df(\xi)$ is an isomorphism then, for any $x \in \mathbf{E}$ satisfying*

$$\|x - \xi\|\gamma(f,\xi) \leq \frac{3-\sqrt{7}}{2},$$

1. *the sequence $x_{k+1} = N_f(x_k),\ x_0 = x$ is well defined,*
2. *for any $k \geq 0$,*

$$\|x_k - \xi\| \leq \left(\frac{1}{2}\right)^{2^k-1} \beta(f,x).$$

This theorem is extended by Shub and Smale in [14] to the case of underdetermined systems of equations with surjective derivatives. They introduce the following invariants,

$$\alpha(f,x) = \beta(f,x)\gamma(f,x)$$

$$\beta(f,x) = \|Df(x)^{\dagger} f(x)\|$$

$$\gamma(f,x) = \sup_{k\geq 2} \left\| Df(x)^{\dagger} \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}},$$

when $Df(x)$ is onto and ∞ otherwise. They give the following:

Theorem 2. *(Shub-Smale) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ have zero as a regular value and define*

$$\gamma = \max_{\xi \in f^{-1}(0)} \gamma(f,\xi)$$

Then there is a universal constant C so that if $d(x, f^{-1}(0)) < \frac{c}{\gamma}$ then

1. *the sequence $x_{k+1} = N_f(x_k), x_0 = x$, is well defined,*
2. *it converges to a zero of ξ of f and*

$$\|x_k - \xi\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \beta(f,x).$$

The case of injective-overdetermined systems is slightly different. The main feature of Newton-Gauss iteration is a quadratic convergence to the zeros of f and a linear convergence to certain least-square solutions. Kantorovich like theorems are given in Ben-Israel [2], Dennis-Schnabel [5] and Seber-Wild [13]. Alpha-theory is studied by Dedieu-Shub in [4]. They introduce the following invariants,

$$\alpha_1(f,x) = \beta_1(f,x)\gamma_1(f,x)$$

$$\beta_1(f,x) = \|Df(x)^{\dagger}\| \; \|f(x)\|$$

$$\gamma_1(f,x) = \sup_{k\geq 2} \left(\|Df(x)^{\dagger}\| \; \left\| \frac{D^k f(x)}{k!} \right\| \right)^{\frac{1}{k-1}},$$

which differ slightly from α, β and γ introduced in the undetermined case. They prove the following theorems.

Theorem 3. *(Dedieu-Shub) Let x and $\xi \in \mathbf{E}$ be such that $f(\xi) = 0$, $Df(\xi)$ is injective and*

$$v = \|x - \xi\|\gamma_1(f,\xi) \leq \frac{3 - \sqrt{7}}{2}.$$

Then Newton's sequence $x_k = N_f^{(k)}(x)$ satisfies

$$\|x_k - \xi\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \xi\|.$$

Theorem 4. *(Dedieu-Shub) Let x and $\xi \in \mathbf{E}$ satisfying $Df(\xi)^\dagger f(\xi) = 0$, $Df(\xi)$ injective and*

$$v = \|x - \xi\|\gamma_1(f,\xi) < 1 - \frac{\sqrt{2}}{2}.$$

If

$$\lambda = \frac{v + \sqrt{2}(2-v)\alpha_1(f,\xi)}{1 - 4v + 2v^2} < 1$$

then Newton's sequence satisfies

$$\|x_k - \xi\| \le \lambda^k \|x - \xi\|.$$

Let us now come back to our problem: We recall that

$$f : \mathbf{E} \to \mathbf{F}$$

is an analytic function with Rank $Df(x) \le r$ for any $x \in \mathbf{E}$. We let

$$V = f^{-1}(0) = \{\xi \in \mathbf{E} \ : \ f(\xi) = 0\}$$

and

$$V_{ls} = \{\xi \in \mathbf{E} \ : Df(\xi)^\dagger f(\xi) = 0\}.$$

V is the set of zeros of f and V_{ls} the set of least square solutions. See Proposition 1. The following proposition describes the smooth part of V:

Proposition 2. *Let $\xi \in V$ with* Rank $Df(\xi) = r$. *Then*

1. *For any $x \in \mathbf{E}$ with $\|x - \xi\|\gamma_1(f,\xi) < 1 - \frac{\sqrt{2}}{2}$ one has* Rank $Df(x) = r$,
2. $V \cap B_{(1-\frac{\sqrt{2}}{2})/\gamma_1(f,\xi)}(\xi)$ *is a submanifold in $\mathbf{E}$ with* Dim $= n - r$.

Proof. The first assertion is proved in Lemma 1 below, the second assertion is a classical consequence of the first one, see Helgason [7], Chap. I, Sect. 15.2. □

We do not have a similar result for V_{ls}: if $\xi \in V_{ls}$ with Rank $Df(\xi) = r$ is V_{ls} a submanifold around ξ ?

In order to state our next result we introduce some more notation. Let $\psi(u) = 1 - 4u + 2u^2$. It is decreasing from 1 to 0 when $0 \le u \le 1 - \frac{\sqrt{2}}{2}$. $\Pi_{\mathbf{E}_1}$ denotes the orthogonal projection onto the subspace $\mathbf{E}_1 \subset \mathbf{E}$. For any linear operator $L : \mathbf{E} \to \mathbf{F}$,

$$K(L) = \|L\| \, \|L^\dagger\|$$

denotes its condition number and $\|L\|$ the operator norm. We also use the following function

$$A(v,K) = \frac{1}{\psi(v)} + \frac{2-v}{(1-v)^2} + \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2}\left(K + \frac{2v - v^2}{(1-v)^2}\right),$$

defined for $0 \le v < 1 - \frac{\sqrt{2}}{2}$ and $K \ge 0$ and

$$B(v,\alpha) = \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2} + \frac{\theta(\alpha)}{\alpha},$$

with

$$\theta(\alpha) = \alpha \left(2 + \frac{(1+\sqrt{5})(1+2\alpha)}{(1-2\alpha)^2} \right)$$

defined for $0 \le v < 1 - \frac{\sqrt{2}}{2}$ and $0 \le \alpha < \frac{1}{2}$. When ξ_0 is a zero of f with Rank $Df(\xi_0) = r$, then for any $x_0 \in \mathbf{E}$ in a neighborhood of ξ_0 Newton's sequence starting at x_0 converges quadratically to a zero of f, but not necessarily equal to ξ_0. More precisely we prove here the following: let

$$\gamma_R = \max_{\xi \in B_R(\xi_0) \cap V} \gamma_1(f, \xi)$$

$$A_R = \max_{\substack{\xi \in B_R(\xi_0) \cap V \\ x \in B_R(\xi_0)}} A(\|x - \xi\| \gamma_1(f, \xi), K(Df(\xi))).$$

Theorem 5. *Let $\xi_0 \in \mathbf{E}$, such that $f(\xi_0) = 0$ and* Rank $Df(\xi_0) = r$. *Let $R > 0$ satisfying the condition $RA_R\gamma_R \le \frac{1}{2}$, with γ_R and A_R as above. Let $x_0 \in B_{\frac{4}{3}R}(\xi_0)$ such that $\xi_0 = proj_V x_0$ i.e. ξ_0 is the point in V the closest to x_0. Then Newton's sequence $x_k = N^{(k)}(x_0)$ is contained in $B_R(\xi_0)$ and*

$$d(x_k, V) \le \left(\frac{1}{2}\right)^{2^k - 1} d(x_0, V).$$

As in the case of overdetermined systems with injective derivatives, the convergence of Newton's sequence to the set of least square solutions fails to be quadratic. We have

Theorem 6. *For $\xi_0 \in V_{ls}$ with* Rank $Df(\xi_0) = r$ *and $0 < R < 1 - \frac{\sqrt{2}}{2}$, define*

$$\Lambda = \max_{\substack{\xi \in B_R(\xi_0) \cap V_{ls} \\ x \in B_R(\xi_0)}} A(v, K(Df(\xi)))v + B(v, \alpha_1(f, \xi))\alpha_1(f, \xi),$$

with $v = \|x - \xi\| \gamma_1(f, \xi)$, and

$$\alpha_1 = \max_{\xi \in B_R(\xi_0) \cap V_{ls}} \alpha_1(f, \xi).$$

Let us suppose that $B_R(\xi_0) \cap V_{ls}$ is a smooth submanifold in $\mathbf{E}$, that $\Lambda < 1$ and $2\alpha_1 < 1$. Then, for any $x_0 \in \mathbf{E}$ such that

$$x_0 - \xi_0 \in (T_{\xi_0} V_{ls})^{\perp}, \text{ and } \|x_0 - \xi_0\| \le \frac{1 - \Lambda}{2\Lambda} R,$$

Newton's sequence $x_k = N^{(k)}(x_0)$ is contained in $B_R(\xi_0)$ and

$$d(x_k, V) \le \Lambda^k d(x_0, V).$$

Notice the following facts. The hypothesis in Theorem 6 is satisfied in a suitable neighborhood of $\xi_0 \in V_{ls}$ when V_{ls} is smooth around ξ_0 and $\alpha_1(f, \xi_0)$ small enough i.e. when $\lim_{R \to 0} \Lambda < 1$.

The invariant $\alpha_1(f, \xi_0)$ is small when the residue function $F(\xi_0) = \|f(\xi_0)\|^2$ is itself small.

The nonconvergence of Newton's sequence to least square solutions with large residues is a well known fact, see Dennis-Schnabel [5] and Dedieu-Shub [4].

When $\alpha_1(f, \xi_0)$ is small then ξ_0 is a strict local minimum for the residue function over $\xi_0 + (ker Df(\xi_0))^{\perp}$. More precisely

Proposition 3. *For any* $\xi \in V_{ls}$ *with* Rank $Df(\xi) = r$ *and* $\alpha_1(f,\xi) < \frac{1}{2}$ *we have* $DF(\xi) = 0$ *and* $D^2F(\xi)(\dot{x},\dot{x}) > 0$ *for any* $\dot{x} \in \text{Ker } Df(\xi)^{\perp}, \dot{x} \neq 0$.

In the following, under a simple assumption on f at x_0 we prove the existence of a least square solution ξ for f in a neighborhood of x_0 and the linear convergence of Newton's sequence $N_f^k(x_0)$ to ξ.

Theorem 7. *Suppose*

$$\alpha_1(f,x_0)K(Df(x_0)) \le \frac{1}{48}.$$

Then Newton's sequence $x_{k+1} = N_f(x_k)$ *satisfies*

$$\|x_{k+1} - x_k\| \le \left(\frac{1}{2}\right)^k \|x_1 - x_0\|.$$

This sequence converges to a least square solution ξ *of* f *:*

$$Df(\xi)^{\dagger} f(\xi) = 0 \text{ and } \|\xi - x_0\| \le 2\|x_1 - x_0\|.$$

We close this section with some examples. Examples of "constant rank" systems of equations are given by distance geometry problems: an important tool in determining the three-dimensional structure of a molecule. Distance geometry problems are concerned with finding positions $x_1, \dots, x_n$ of n atoms in $\mathbf{R}^3$ such that

$$\|x_i - x_j\| = \delta_{(i,j)},\ (i,j) \in S,$$

where S is a subset of the atom pairs and $\delta_{(i,j)}$ is the given distance between atoms i and j. When all these distances are given, this system has $3n$ unknowns and $n(n-1)/2$ equations. The dimension of the solution set, when it is nonempty, is at least 6 because these equations are invariant under translations and orthogonal transformations. Similar examples arise from the protein folding problem. For example the Lennard-Jones problem is to find the minimum energy structure of a cluster of n identical atoms using the Lennard-Jones potential energy:

$$\min_{\substack{x_i \in \mathbf{R}^3 \\ 1 \le i \le n}} \sum_{i<j} p(\|x_i - x_j\|)$$

with $p(r) = r^{-12} - 2r^{-6}$. Typically n can take large values: 10 000 for example. This global optimization problem is still unsolved. We can see this problem as a nonlinear least square problem related to the system of equations

$$(p(\|x_i - x_j\|) + 1)^{1/2} = 0,\ i < j.$$

Such a system enters in the category of "constant rank" systems. A good reference for such problems is the survey paper by A. Neumaier [10].

2. Proofs.

In this section we give the proofs of theorems 5, 6 and 7. We begin by a series of lemmas.

Lemma 1. *Let $x, y \in \mathbf{E}$ with* Rank $Df(y) \leq$ Rank $Df(x) = r$ *and* $u = \|x-y\|\gamma_1(f,x) < 1 - \frac{\sqrt{2}}{2}$. *Then*

1. $Df(y)$ *and* $\Pi_{\mathrm{Im}\ Df(x)} Df(y)$ *have rank* r,
2. $\Pi_{\mathrm{Ker}\ Df(x)} + Df(x)^\dagger Df(y)$ *is non-singular.*
3. $\left\| \left(\Pi_{\mathrm{Ker}\ Df(x)} + Df(x)^\dagger Df(y)\right)^{-1} \right\| \leq \frac{(1-u)^2}{\psi(u)}$.

Proof. $Df(x)^\dagger(Df(x) - Df(y)) = -Df(x)^\dagger \sum_{k\geq 2} k \frac{D^k f(x)}{k!}(y-x)^{k-1}$ so that

$$\|Df(x)^\dagger(Df(x) - Df(y))\| \leq \frac{1}{(1-u)^2} - 1 < 1.$$

By a classical linear algebra argument

$$id_{\mathbf{E}} - Df(x)^\dagger(Df(x) - Df(y)) = \Pi_{\mathrm{Ker}\ Df(x)} + Df(x)^\dagger Df(y)$$

is invertible and its inverse is bounded by

$$\frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1\right)} = \frac{(1-u)^2}{\psi(u)}.$$

This proves 2 and 3. Moreover

$$\Pi_{\mathrm{Im}\ Df(x)} Df(y) = Df(x)(\Pi_{\mathrm{Ker}\ Df(x)} + Df(x)^\dagger Df(y)) = (\mathrm{Rank}\ r) \circ (nonsingular)$$

has Rank r. Thus Rank $Df(y) \geq$ Rank $\Pi_{\mathrm{Im}\ Df(x)} Df(y) = r$ and we are done. □

The following linear algebra lemmas will be useful. Let A and B be $m \times n$ real or complex matrices with non-zero singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\tau_1 \geq \cdots \geq \tau_r > 0$. Thus Rank $A =$ Rank $B = r$. Let us denote by $\|A\|$ the usual spectral norm so that

$$\|A\| = \sigma_1 \text{ and } \|A^\dagger\| = \sigma_r^{-1}.$$

We have (see Stewart-Sun [19], Chap. IV, Theorem 4-11):

Lemma 2. *(Mirsky)*

$$\max |\sigma_i - \tau_i| \leq \|A - B\|$$

We also need bounds for $\|A^\dagger - B^\dagger\|$. The following lemma is valid in our context (see Stewart-Sun [19], Chap. III, Theorem 3.8):

Lemma 3. *(Wedin)*

$$\|A^\dagger - B^\dagger\| \leq \frac{1+\sqrt{5}}{2} \max(\|A^\dagger\|^2, \|B^\dagger\|^2)\|A - B\|.$$

The constant $(1+\sqrt{5})/2$ appearing in Lemma 3 may be improved according to the values of m, n and the ranks of A and B. The precise staatement is given in [19], Chapter III, Theorem 3. 9. The case of Frobenius norm and arbitrary matix norms are considered.

The following lemma generalizes a well-known result for square and non-singular matrices. It is probably well-known but we were not able to find it in the literature.

Lemma 4. *Let A and B two $m \times n$ matrices with* Rank $(A+B) \leq$ Rank $A = r$ *and* $\|A^\dagger\| \, \|B\| < 1$. *Then*

$$\text{Rank } (A+B) = r \text{ and } \|(A+B)^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \, \|B\|}.$$

Proof. Let us denote by $\sigma_1 \geq \cdots \geq \sigma_r > 0$ the non-zero singular values of A and by $\rho_1 \geq \cdots \geq \rho_p \geq 0$ ($p = \min(m,n)$) the singular values of $A+B$. By Lemma 2

$$\sigma_r^{-1} |\sigma_r - \rho_r| \leq \|A^\dagger\| \, \|B\| < 1$$

so that $\rho_r > 0$ and consequently Rank $(A+B) \geq r$. Since Rank $(A+B) \leq r$ by the hypothesis, we have proved the equality. The nonzero singular values of $A+B$ are

$$\rho_1 \geq \cdots \geq \rho_r > 0.$$

We have

$$\|(A+B)^\dagger\| = \rho_r^{-1} = \frac{\sigma_r^{-1}}{1 - \frac{\sigma_r - \rho_r}{\sigma_r}} \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \, \|B\|}$$

and we are done. □

Lemma 5. *Let $x, y \in \mathbf{E}$ with* Rank $Df(y) \leq$ Rank $Df(x) = r$ *and* $u = \|x-y\|\gamma_1(f,x) < 1 - \frac{\sqrt{2}}{2}$. *Then*

1. $\|Df(y) - Df(x)\| \leq \|Df(x)^\dagger\|^{-1} \frac{2u-u^2}{(1-u)^2}$,
2. $\|Df(y)\| \leq \|Df(x)^\dagger\|^{-1} \left(K(Df(x)) + \frac{2u-u^2}{(1-u)^2} \right)$,
3. $\|Df(y)^\dagger\| \leq \frac{(1-u)^2}{\psi(u)} \|Df(x)^\dagger\|$,
4. $\|Df(x)^\dagger - Df(y)^\dagger\| \leq \frac{1+\sqrt{5}}{2} \frac{(1-u)^2(2u-u^2)}{\psi(u)^2} \|Df(x)^\dagger\|$.

Proof. $Df(y) = Df(x) + \sum_{k \geq 2} k \frac{D^k f(x)}{k!} (y-x)^{k-1}$ so that

$$\|Df(y) - Df(x)\| \leq \|Df(x)^\dagger\|^{-1} \left(\frac{1}{(1-u)^2} - 1 \right)$$

and this proves 1) and 2). Assertion 3) comes from Lemma 4 with $A = Df(x)$ and $B = Df(y) - Df(x)$. We have Rank $(A+B) = r$ by Lemma 1

$$\|A^\dagger\| \, \|B\| \leq \|Df(x)^\dagger\| \times \|Df(x)^\dagger\|^{-1} \frac{2u-u^2}{(1-u)^2} \leq 1$$

by Lemma 5.1 and because $u \leq 1 - \frac{\sqrt{2}}{2}$. Thus, by Lemma 4,

$$\|Df(y)^\dagger\| \leq \frac{\|Df(x)^\dagger\|}{1 - \frac{2u-u^2}{(1-u)^2}} = \frac{(1-u)^2}{\psi(u)} \|Df(x)^\dagger\|.$$

The last assetion is a consequence of Lemma 3, Lemma 5.1 and Lemma 5.3.

$$\|Df(y)^\dagger - Df(x)^\dagger\| \leq \frac{1+\sqrt{5}}{2}\frac{(1-u)^4}{\psi(u)^2}\|Df(x)^\dagger\|^2\|Df(x)^\dagger\|^{-1}\frac{2u-u^2}{(1-u)^2}.$$

This achieves the proof of Lemma 5. $\square$

Lemma 6. *Let* ξ *and* $x \in \mathbf{E}$ *with* $Df(\xi)^\dagger f(\xi) = 0$, Rank $Df(x) \leq$ Rank $Df(\xi) = r$ *and* $v = \|x-\xi\|\gamma_1(f,\xi) < 1 - \frac{\sqrt{2}}{2}$. *Then*

$$\|Df(x)^\dagger f(\xi)\| \leq \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2}\|x-\xi\|\alpha_1(f,\xi).$$

Proof. It is a consequence of Lemma 5.4:

$$\|Df(x)^\dagger f(\xi)\| = \|(Df(x)^\dagger - Df(\xi)^\dagger)f(\xi)\| \leq \|Df(x)^\dagger - Df(\xi)^\dagger\|\ \|f(\xi)\|$$

$$\leq \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2}v\|Df(\xi)^\dagger\|\ \|f(\xi)\|.$$

$\square$

Lemma 7. *Under the hypothesis of Lemma 6, we have*

$$\|N_f(x)-\xi\| \leq \|\Pi_{\mathrm{Ker}\ Df(x)}(x-\xi)\| + \frac{v\|x-\xi\|}{\psi(v)} + \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2}\|x-\xi\|\alpha_1(f,\xi).$$

Proof. We have

$$\begin{aligned} N_f(x)-\xi &= x-\xi-Df(x)^\dagger f(x) \\ &= \Pi_{\mathrm{Ker}\ Df(x)}(x-\xi) + Df(x)^\dagger\left(Df(x)(x-\xi)-f(x)+f(\xi)\right) - Df(x)^\dagger f(\xi). \end{aligned}$$

Using Taylor's formula for both $f(x)$ and $Df(x)$ at ξ gives

$$Df(x)(x-\xi)-f(x)+f(\xi) = \sum_{k\geq 1}(k-1)\frac{D^k f(\xi)}{k!}(x-\xi)^k$$

so that

$$\begin{aligned} \|Df(x)(x-\xi)-f(x)+f(\xi)\| &\leq \|Df(\xi)^\dagger\|^{-1}\|x-\xi\|\sum_{k\geq 2}(k-1)v^{k-1} \\ &= \|Df(\xi)^\dagger\|^{-1}\|x-\xi\|\frac{v}{(1-v)^2}. \end{aligned}$$

By Lemma 5.3 we get

$$\|Df(x)^\dagger\left(Df(x)(x-\xi)-f(x)+f(\xi)\right)\| \leq \frac{(1-v)^2}{\psi(v)}\|x-\xi\|\frac{v}{(1-v)^2} = \frac{v\|x-\xi\|}{\psi(v)}.$$

The conclusion comes from Lemma 6:

$$\|N_f(x)-\xi\| \leq \|\Pi_{\mathrm{Ker}\ Df(x)}(x-\xi)\| + \frac{v\|x-\xi\|}{\psi(v)} + \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2}\|x-\xi\|\alpha_1(f,\xi)$$

$\square$

Lemma 8. *Under the hypothesis of Lemma 6, we have*

$$\|\Pi_{\text{Ker } Df(x)}(x-\xi)\| \leq \|\Pi_{\text{Ker } Df(\xi)}(x-\xi)\| +$$
$$v\|x-\xi\| \left(\frac{2-v}{(1-v)^2} + \frac{1+\sqrt{5}}{2} \frac{(1-v)^2(2-v)}{\psi(v)^2} (K(Df(\xi)) + \frac{2v-v^2}{(1-v)^2})) \right).$$

Proof.

$$\Pi_{\text{Ker } Df(x)}(x-\xi) = \left(id_{\mathbf{E}} - Df(x)^\dagger Df(x)\right)(x-\xi) =$$
$$\Pi_{\text{Ker } Df(\xi)}(x-\xi) + Df(\xi)^\dagger(Df(\xi) - Df(x))(x-\xi) + (Df(\xi)^\dagger - Df(x)^\dagger)Df(x)(x-\xi)$$
$$= a + b + c.$$

We give a bound for $\|b\|$ via Lemma 5.1:

$$\|b\| \leq \frac{2v-v^2}{(1-v)^2}\|x-\xi\|$$

and a bound for $\|c\|$ via Lemma 5.2 and 5.4:

$$\|c\| \leq \frac{1+\sqrt{5}}{2} \frac{(1-v)^2(2v-v^2)}{\psi(v)^2} \left(K(Df(\xi)) + \frac{2v-v^2}{(1-v)^2}\right)\|x-\xi\|.$$

□

Lemma 9. *Let ξ and $x \in \mathbf{E}$ with $f(\xi) = 0$,* Rank $Df(\xi) = r$ *and $v = \|x-\xi\|\gamma_1(f,\xi) \leq 1 - \frac{\sqrt{2}}{2}$. Then we have*

$$\|N_f(x) - \xi\| \leq \|\Pi_{\text{Ker } Df(\xi)}(x-\xi)\| + \|x-\xi\| v A\left(v, K(Df(\xi))\right)$$

with

$$A(v,K) = \frac{1}{\psi(v)} + \frac{2-v}{(1-v)^2} + \frac{1+\sqrt{5}}{2} \frac{(1-v)^2(2-v)}{\psi(v)^2} \left(K + \frac{2v-v^2}{(1-v)^2}\right)$$

and

$$K(Df(\xi)) = \|Df(\xi)\| \, \|Df(\xi)^\dagger\|.$$

Proof. It is an easy consequence of Lemma 7 and Lemma 8 with $f(\xi) = 0$. □

Proof of Theorem 5. Recall that $\|x_0 - \xi_0\| \leq \frac{3}{4}R$. We first notice that, for any $x \in B_R(\xi_0)$ we have

$$\|x - \xi_0\|\gamma(f,\xi_0) \leq R\gamma_{R,\xi_0} \leq \frac{1}{2A_{R,\xi_0}} < 1 - \frac{\sqrt{2}}{2}.$$

The last inequality is from the fact that $A(v,K) \geq 3$. Thus $V \cap B_R(\xi_0)$ is a smooth submanifold in $\mathbf{E}$ (Proposition 2). Since ξ_0 is the projection of x_0 onto V, and because $V \cap B_R(\xi_0)$ is smooth, the orthogonality relation

$$\Pi_{\text{Ker } Df(\xi_0)}(x_0 - \xi_0) = 0$$

holds. By Lemma 9, we get

$$\|N_f(x_0) - \xi_0\| \leq \|x_0 - \xi_0\|^2\gamma_1(f,\xi_0)A(v_0,K_0) \leq \|x_0 - \xi_0\|R\gamma_{R,\xi_0}A_{R,\xi_0} \leq \frac{1}{2}\|x_0 - \xi_0\|,$$

so that $x_1 = N_f(x_0)$ is in $B_{\frac{R}{2}}(\xi_0)$ and consequently projects on V in a point $\xi_1 \in B_R(\xi_0)$ because

$$\|\xi_1 - \xi_0\| \le \|x_1 - \xi_1\| + \|x_1 - \xi_0\| \le 2\|x_1 - \xi_0\| \le R.$$

Now we proceed by induction. Let $x_{k+1} = N_f(x_k)$ and ξ_k be the projection of x_k onto V. Then

$$\begin{aligned}
\|x_{k+1} - \xi_{k+1}\| &\le \|x_{k+1} - \xi_k\| \le \|x_k - \xi_k\|^2 \gamma_1(f, \xi_k) A(v_k, K_k) \\
&\le \left((\frac{1}{2})^{2^k - 1} \|x_0 - \xi_0\| \right)^2 \gamma_1(f, \xi_k) A(v_k, K_k) \\
&\le \left((\frac{1}{2})^{2^k - 1} \right)^2 \|x_0 - \xi_0\| \; \|x_0 - \xi_0\| \gamma_{R,\xi_0} A_{R,\xi_0} \\
&\le \frac{1}{2} (\frac{1}{2})^{2^{k+1} - 2} \|x - \xi_0\| = (\frac{1}{2})^{2^{k+1} - 1} \|x - \xi_0\|.
\end{aligned}$$

Here $K_k = K(f, \xi_k)$, $v_k = \|x_k - \xi_k\| \gamma_1(f, \xi_k)$. Further we have $\xi_{k+1} \in B_R(\xi_0)$ by noting that

$$\begin{aligned}
\|\xi_{k+1} - \xi_k\| &\le \|x_{k+1} - \xi_{k+1}\| + \|x_{k+1} - \xi_k\| \le 2\|x_{k+1} - \xi_k\|, \\
\|\xi_{k+1} - \xi_0\| &\le \sum_{j=0}^{k} \|\xi_{j+1} - \xi_j\| \le 2 \sum_{j=0}^{k} \|x_{j+1} - \xi_j\| \\
&\le 2 \sum_{j=0}^{k} \left(\frac{1}{2} \right)^{2^{j+1} - 1} \|x_0 - \xi_0\| \le 2 \frac{1/2}{1 - 1/4} \|x_0 - \xi_0\| \\
&\le \frac{4}{3} \|x_0 - \xi_0\| \le R,
\end{aligned}$$

which compeletes the induction. □

The following lemmas will be used to prove Proposition 3 and to compute the tangent space $T_{\xi_0} V_{ls}$ for $\xi_0 \in V_{ls}$ as required in Theorem 5. We begin with an identity given in Stewart-Sun [19] Chapter III, §3.4.

Lemma 10. *Let A and B be $m \times n$ matrices with* Rank $A =$ Rank $B = r$. *Then*

$$B^\dagger = A^\dagger - A^\dagger (B - A) A^\dagger + (A^* A)^\dagger (B - A)^* \Pi_{(\mathrm{Im}\ A)^\perp} - \Pi_{\mathrm{Ker}\ A} (B - A)^* (A A^*)^\dagger + O(\|B - A\|^2).$$

Lemma 11. *When* Rank $Df(x) = r$, *the derivative of $Df(x)^\dagger f(x)$ is given by*

$$\begin{aligned}
D(Df(x)^\dagger f(x))(\dot{x}) &= \Pi_{(\mathrm{Ker}\ Df(x))^\perp}(\dot{x}) - Df(x)^\dagger (D^2 f(x)(\dot{x})) Df(x)^\dagger f(x) \\
&\quad + (Df(x)^* Df(x))^\dagger (D^2 f(x)(\dot{x}))^* \Pi_{(\mathrm{Im}\ Df(x))^\perp} f(x) \\
&\quad - \Pi_{\mathrm{Ker}\ Df(x)} (D^2 f(x)(\dot{x}))^* (Df(x) Df(x)^*)^\dagger f(x).
\end{aligned}$$

Proof. Note that

$$D(Df(x)^\dagger f(x))(\dot{x}) = D(Df(x)^\dagger)(\dot{x}) f(x) + Df(x)^\dagger Df(x)(\dot{x}).$$

Now use Lemma 10 with $A = Df(x)$ and the chain rule to $\dagger \circ Df$. Notice that $Df(y)$ has rank r in a neighborhood of x. □

Lemma 12. *When* $Df(\xi)^\dagger f(\xi) = 0$ *and* Rank $Df(\xi) = r$, *we have*

$$D(Df(\xi)^\dagger f(\xi))\dot{x} = \Pi_{(\mathrm{Ker}\ Df(\xi))^\perp}\dot{x} + (Df(\xi)^* Df(\xi))^\dagger (D^2 f(\xi)\dot{x})^* f(\xi).$$

When V_{ls} *is smooth around* ξ, *its tangent space is the kernel in* $\mathbf{E}$ *of this linear operator.*

Proof. In Lemma 11, use the fact $f(\xi) \in \mathrm{Im}\ Df(\xi)^\perp$; This gives us $\Pi_{(\mathrm{Im}\ Df(\xi))^\perp} f(\xi) = f(\xi)$ which simplifies the third term, and that $(Df(\xi)Df(\xi)^*)^\dagger f(\xi) = 0$ which annihilates the last term in Lemma 11. This is because $\mathrm{Ker}\ (AA^*) = \mathrm{Ker}\ (A^*) = \mathrm{Im}\ A^\perp$, for any matrix A. □

Lemma 13. *When* $Df(\xi)^\dagger f(\xi) = 0$, Rank $f(\xi) = r$ *and* $\alpha_1(f,\xi) < \frac{1}{2}$, *then*

$$\|\Pi_{\mathrm{Ker}\ Df(\xi)}(x-\xi)\| \le \|\Pi_{T_\xi V_{ls}}(x-\xi)\| + \theta(\alpha_1(f,\xi))\|x-\xi\|$$

with

$$\theta(\alpha) = \alpha\left(2 + \frac{(1+\sqrt{5})(1+2\alpha)}{(1-2\alpha)^2}\right),\ 0 \le \alpha < \frac{1}{2}.$$

Proof. We first notice that $D(Df(\xi)^\dagger f(\xi))\dot{x}$ is always in $\mathrm{Ker}\ Df(\xi)^\perp$ so that the rank of this operator is $\le r$. Let us write $A = \Pi_{(\mathrm{Ker}\ Df(\xi))^\perp}$ and $B\dot{x} = (Df(\xi)^* Df(\xi))^\dagger (D^2 f(\xi)(\dot{x}))^* f(\xi)$. We have

$$D(Df(\xi)^\dagger f(\xi)) = A + B,\ \Pi_{\mathrm{Ker}\ Df(\xi)} = \Pi_{\mathrm{Ker}\ A} \text{ and } \Pi_{T_\xi V_{ls}} = \Pi_{\mathrm{Ker}\ (A+B)}.$$

We also can notice that $\|A\| = \|A^\dagger\| = 1$ and $\|B\| \le 2\alpha_1(f,\xi) < 1$, by the definition of α_1. By Lemma 4 we get Rank $(A+B) = r$ and

$$\|(A+B)^\dagger\| \le \frac{1}{1-\|B\|} \le \frac{1}{1-2\alpha_1(f,\xi)}.$$

We have

$$\Pi_{\mathrm{Ker}\ A} - \Pi_{\mathrm{Ker}\ (A+B)} = (A+B)^\dagger(A+B) - A^\dagger A = ((A+B)^\dagger - A^\dagger)(A+B) + A^\dagger B,$$

so that, by Lemma 3

$$\begin{aligned}\|\Pi_{\mathrm{Ker}\ A} - \Pi_{\mathrm{Ker}\ (A+B)}\| &\le \frac{1+\sqrt{5}}{2}\max\left(\|(A+B)^\dagger\|^2, \|A^\dagger\|^2\right)\|B\|(\|A\| + \|B\|) + \|A^\dagger\|\ \|B\| \\ &\le \frac{1+\sqrt{5}}{2}\frac{2\alpha_1(f,\xi)}{(1-2\alpha_1(f,\xi))^2}(1+2\alpha_1(f,\xi)) + 2\alpha_1(f,\xi) = \theta(\alpha_1(f,\xi)).\end{aligned}$$

The conclusion is now easy. □

Lemma 14. *Let* ξ *be given as in Lemma 13 and* $x \in \mathbf{E}$ *with* $v = \|x-\xi\|\gamma_1(f,\xi) < 1 - \frac{\sqrt{2}}{2}$. *Then*

$$\|N_f(x) - \xi\| \le \|\Pi_{T_\xi V_{ls}}(x-\xi)\| + A(v, K(Df(\xi)))v\|x-\xi\| + B(v, \alpha_1(f,\xi))\alpha_1(f,\xi)\|x-\xi\|$$

with

$$B(v,\alpha) = \frac{1+\sqrt{5}}{2}\frac{(1-v)^2(2-v)}{\psi(v)^2} + \frac{\theta(\alpha)}{\alpha}.$$

Proof of Theorem 6. The proof of Theorem 6 is similar to the proof of Theorem 5 but uses Lemma 14 instead of Lemma 9. We define $x_{k+1} = N_f(x_k)$ inductively and let $\xi_k = proj_{V_{ls}} x_k$. Inductively by Lemma 14,

$$\|x_1 - \xi_1\| \leq \|x_1 - \xi_0\| \leq \Lambda \|x_0 - \xi_0\| \leq \frac{(1-\Lambda)R}{2},$$

recalling that $\|x_0 - \xi_0\| \leq \frac{1-\Lambda}{2\Lambda} R$ and $\Lambda < 1$. Moreover because

$$\|\xi_1 - \xi_0\| \leq \|\xi_1 - x_1\| + \|x_1 - \xi_0\| \leq 2\|x_1 - \xi_0\| < 2\Lambda \|x_0 - \xi_0\| \leq (1-\Lambda)R < R$$

so that $\xi_1 \in B_R(\xi_0)$. Inducitively by Lemma 14 with $x = x_{k-1}$, we have

$$\|x_k - \xi_k\| \leq \|x_k - \xi_{k-1}\| \leq \Lambda^k \|x_0 - \xi_0\|,$$

Note that $\|\xi_k - \xi_{k-1}\| \leq \|x_k - \xi_k\| + \|x_k - \xi_{k-1}\| \leq 2\|x_k - \xi_{k-1}\|$. Moreover $\xi_k \in B_R(\xi_0)$, becuase

$$\|\xi_k - \xi_0\| \leq \sum_{j=1}^{k} \|\xi_j - \xi_{j-1}\| \leq \sum_{j=1}^{k} 2\|x_j - \xi_{j-1}\| \leq 2\sum_{j=1}^{k} \Lambda^j \|x_0 - \xi_0\| \leq 2\frac{\Lambda}{1-\Lambda}\|x_0 - \xi_0\| \leq R,$$

which compeletes the proof. $\square$

Proof of Proposition 3. We first notice that

$$\|Df(\xi)^\dagger\| = \mu^{-1} \text{ with } \mu = \min_{\substack{\|\dot{x}\|=1 \\ \dot{x} \in (\text{Ker } Df(\xi))^\perp}} \|Df(\xi)\dot{x}\|.$$

We also have

$$\frac{1}{2} D^2 F(\xi)\dot{x} = (D^2 f(\xi)\dot{x})^* f(\xi) + (Df(\xi)^* Df(\xi))\dot{x}$$

so that

$$\frac{1}{2} D^2 F(\xi)(\dot{x}, \dot{x}) = < f(\xi), D^2 f(\xi)(\dot{x}, \dot{x}) > + \|Df(\xi)\dot{x}\|^2.$$

If we take $\dot{x} \in (\text{Ker } Df(\xi))^\perp$, $\|\dot{x}\| = 1$, then

$$\frac{1}{2} D^2 F(\xi)(\dot{x}, \dot{x}) \geq \mu^2 - \|f(\xi)\| \, \|D^2 f(\xi)\| = \mu^2(1 - \|Df(\xi)^\dagger\|^2 \|f(\xi)\| \, \|D^2 f(\xi)\|) \geq \mu^2(1 - 2\alpha_1(f, \xi)) > 0.$$

$\square$

Lemma 15. *Let $x, y \in \mathbf{E}$ and $u = \|y - x\|\gamma_1(f, x) < 1 - \frac{\sqrt{2}}{2}$ as in Lemma 5. Then*

1. $\beta_1(f, y) \leq \dfrac{(1-u)^2}{\psi(u)} (\beta_1(f, x) + \dfrac{u}{1-u}\|y - x\| + K(Df(x))\|y - x\|)$,
2. $\gamma_1(f, y) \leq \dfrac{\gamma_1(f, x)}{(1-u)\psi(u)}$,
3. $\alpha_1(f, y) \leq \dfrac{1-u}{\psi(u)^2} \left(\alpha_1(f, x) + \dfrac{u^2}{1-u} + K(Df(x))u \right)$.

Proof. 3) is a consequence of 1) and 2). 1) goes as follows: Recall that $\gamma_1 = \sup \left(\|Df(x)^\dagger\| \, \|\frac{D^k f(x)}{k!}\| \right)^{1/k-1}$ and $u = \|y - x\|\gamma_1(f, x)$. We have

$$f(y) = f(x) + Df(x)(y - x) + \sum_{k \geq 2} \frac{D^k f(x)}{k!}(y - x)^k$$

so that

$$\|f(y)\| \le \|f(x)\| + \|Df(x)\| \, \|y-x\| + \|Df(x)^\dagger\|^{-1}\|y-x\|\frac{u}{1-u}$$

and we conclude by Lemma 5.3. To prove 2) we start from

$$D^k f(y) = \sum_{\ell=0}^{\infty} \frac{(k+\ell)!}{\ell!}\frac{D^{k+l}f(x)}{(k+\ell)!}(y-x)^\ell.$$

This gives

$$\begin{aligned} \|\frac{D^k f(y)}{k!}\| &\le \sum_{\ell=0} \begin{pmatrix} k+\ell \\ \ell \end{pmatrix} \frac{D^{k+\ell}f(x)}{(k+\ell)!}\|y-x\|^\ell \\ &\le \sum_{\ell} \begin{pmatrix} k+\ell \\ \ell \end{pmatrix} \gamma_1^{k+\ell-1}\|y-x\|^\ell\|Df(x)^\dagger\|^{-1} = \frac{\gamma_1^{k-1}}{(1-u)^{k+1}}\|Df(x)^\dagger\|^{-1}, \end{aligned}$$

noting that $(\frac{1}{1-u})^{(k)} = \frac{1}{(1-u)^{k+1}} = \sum_{\ell=0} \begin{pmatrix} k+\ell \\ \ell \end{pmatrix} u^\ell$. By Lemma 5.3, we obtain

$$\|Df(y)^\dagger\|\frac{\|D^k f(y)\|}{k!} \le \frac{(1-u)^2}{\psi(u)}\frac{\gamma_1^{k-1}}{(1-u)^{k+1}} = \frac{1}{\psi(u)}\frac{\gamma_1^{k-1}}{(1-u)^{k-1}}$$

thus

$$\gamma_1(f,y) \le \frac{\gamma_1(f,x)}{(1-u)\psi(u)}.$$

□

In the following Lemmas we consider $x_0, x \in \mathbf{E}$ with Rank $Df(x_0) = r$ and such that

$$u = \|x-x_0\|\gamma_1(f,x_0) \le 2\alpha_1(f,x_0) < 1 - \frac{\sqrt{2}}{2}.$$

We also introduce $y = N_f(x)$. Our objective is to give an estimate for $\|N_f(y) - N_f(x)\|$ in terms of $\|y-x\|$. We begin a series of Lemmas. We often use the notations $\alpha_0 = \alpha_1(f,x_0)$ and $K_0 = K(Df(x_0))$.

Lemma 16. *Suppose that* $u = \|x-x_0\|\gamma_1(f,x_0) \le 2\alpha_1(f,x_0) \le \frac{1}{24}$. *Then*

1. $\alpha_1(f,x) \le 4.2\alpha_1(f,x_0)K(f(x_0))$,
2. $K(f(x)) \le 1.25K(f(x_0))$.

Proof. From Lemma 15.3 with x and x_0 instead of y and x, we have

$$\begin{aligned} \alpha_1(f,x) &\le \frac{1-u}{\psi(u)^2}(\alpha_0 + \frac{u^2}{1-u} + K_0 u) \le \frac{1-u}{\psi(u)^2}(3K_0\alpha_0 + \frac{2\alpha_0 u}{1-u}) \\ &\le \frac{1-u}{\psi(u)^2}\alpha_0(3K_0 + \frac{2u}{(1-u)}) \le (1.37)\alpha_0(3K_0 + 0.03) \le 4.2\alpha_0 K_0, \end{aligned}$$

for $u \le 2\alpha_0 \le \frac{1}{24}$. A bound for $K(Df(x))$ is given by Lemma 5.2 and 5.3.

$$K(Df(x)) \le \frac{(1-u)^2}{\psi(u)}(K(Df(x_0)) + \frac{2u-u^2}{\psi(u)}) \le (1.122)(K_0 + 0.11) \le 1.25K_0,$$

for $u \le \frac{1}{24}$.

□

Lemma 17. *When $y = N_f(x)$ then*

$$\begin{aligned} N_f(y) - N_f(x) &= Df(x)^\dagger(Df(x)(y-x) + f(x) - f(y)) \\ &\quad +(Df(x)^\dagger - Df(y)^\dagger)f(x) + (Df(x)^\dagger - Df(y)^\dagger)(f(y) - f(x)). \end{aligned}$$

Proof. Just note that $y - x = Df(x)^\dagger Df(x)(y-x)$, because $N_f(x) - x \in \mathrm{Im}\ Df(x)^\dagger$.

□

In Lemma 17, $N_f(y) - N_f(x)$ appears as the sum of the three quantities. We will use the notation

$$\|N_f(y) - N_f(x)\| \le A + B + C,$$

for the norm of each of these expressions.

Lemma 18. *Let $u_x = \|y-x\|\gamma_1(f,x)$.*

1. $A \le \|x-y\|\dfrac{u_x}{1-u_x}$.
2. $B \le \dfrac{1+\sqrt{5}}{2}\dfrac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2}\alpha_1(f,x)\|y-x\|$.
3. $C \le \dfrac{1+\sqrt{5}}{2}\dfrac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2}u_x(K(Df(x)) + \dfrac{u_x}{1-u_x})\|x-y\|$.

Proof. By using the Taylor series of $f(y)$ around x and the definition of $\gamma_1(f,x)$ we obtain

$$\begin{aligned} A &\le \|Df(x)^\dagger(Df(x)(y-x) + f(x) - f(y))\| \\ &\le \|Df(x)^\dagger\| \sum_{k=2} \left\| \frac{D^k f(x)}{k!} \right\| \|y-x\|^k = \|y-x\|\frac{u_x}{1-u_x}. \end{aligned}$$

¿From Lemma 5.4, we have

$$\|Df(x)^\dagger - Df(y)^\dagger\| \le \frac{1+\sqrt{5}}{2}\frac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2}u_x\|Df(x)^\dagger\|,$$

so that

$$B \le \frac{1+\sqrt{5}}{2}\frac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2}\alpha_1(f,x)\|y-x\|.$$

The Taylor expansion of $f(y)$ at x gives

$$\|f(y) - f(x)\| \le \|Df(x)^\dagger\|^{-1}(K(Df(x)) + \frac{u_x}{1-u_x})\|x-y\|.$$

This yields, using Lemma 5.4,

$$C \le \frac{1+\sqrt{5}}{2}\frac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2}u_x(K(Df(x)) + \frac{u_x}{1-u_x})\|x-y\|$$

□

Proof of Theorem 7. Let us denote $y = N_f(x)$ and $u = \|x - x_0\|\gamma_1(f,x_0)$. Under the hypothesis $\|y-x\| \le \|x_1 - x_0\|$ and $u \le \frac{1}{24}$ we will prove that

$$\|N_f(y) - N_f(x)\| \le \frac{1}{2}\|y-x\|.$$

First notice that, using Lemma 15.2,

$$u_x = \|y-x\|\gamma_1(f,x) \leq \|x_1 - x_0\| \frac{\gamma_1(f,x_0)}{(1-u)\psi(u)} \leq 1.25\alpha_0 \leq \frac{1}{38},$$

for $u \leq \frac{1}{24}$. Hence we have

$$A \leq \|y-x\| \frac{u_x}{1-u_x} \leq \|y-x\|(1.25)\alpha_0 \frac{1}{1-u_x} \leq (1.25)(1.03)\alpha_0\|y-x\| \leq 1.3\alpha_0\|y-x\|.$$

It is convenient to have the following estimate:

$$E_x \leq \frac{1+\sqrt{5}}{2} \frac{(1-u_x)^2(2-u_x)}{\psi(u_x)^2} \leq 3.78$$

for $u_x \leq \frac{1}{38}$. For B, by Lemma 16.1, we have

$$B \leq E_x \alpha_x \|y-x\| \leq (3.78)(4.2\alpha_0 K_0)\|y-x\| \leq 15.9\alpha_0 K_0\|y-x\|.$$

Using Lemma 16.2, we have

$$\begin{aligned} C &\leq E_x u_x (K_x + \frac{u_x}{1-u_x})\|y-x\| \leq E_x u_x (1.25K_0 + 0.03)\|y-x\| \\ &\leq (3.78)(1.25)\alpha_0(1.28)K_0\|y-x\| \leq 6.1\alpha_0 K_0\|y-x\|. \end{aligned}$$

Hence we have

$$\|N_f(y) - N_f(x)\| \leq A + B + C \leq (1.3 + 15.9 + 6.1)\alpha_0 K_0\|y-x\| \leq 24\alpha_0 K_0\|y-x\| \leq \frac{1}{2}\|y-x\|,$$

because $\alpha_0 K_0 \leq \frac{1}{48}$. Now it is easy to prove, by induction over k, that

$$\|x_{k+1} - x_k\| \leq \left(\frac{1}{2}\right)^k \|x_1 - x_0\|$$

This completes the proof.

$\square$

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