MAT211 Fall 2007 Midterm 1 Review Sheet

The topics tested on Midterm 1 will be among the following.

(i) Finding the coefficient and augmented matrix of a system of linear equations.

(ii) Using Gauss-Jordan elimination or other elementary row operations to find the reduced row echelon form of a matrix.

(iii) Determining when a system of linear equations has no solution, a unique solution, or infinitely many solutions.

(iv) Finding the general solution of a system of linear equations.

(v) Finding the matrix representative of a linear transformation, i.e., finding A given T.

(vi) Adding and scaling matrices, determining when two matrices can be multiplied, and computing the matrix product when it exists.

(vii) Determining whether a matrix is or is not invertible, and computing the matrix inverse when it exists.

(viii) Given an ordered sequence $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of elements in $\mathbb{R}^n$, determining which are redundant and irredundant, and expressing each redundant element as a linear combination of irredundant elements.

(ix) Finding an ordered basis for the image of a linear transformation.

(x) Finding an ordered basis for the kernel of a linear transformation (which is roughly the same as Item (iv) above).

You should know the meaning of all of the following words and phrases: linear equation, system of linear equations, consistent, inconsistent, zero matrix, identity matrix, coefficient matrix, augmented matrix, entries, column vector, vector, row vector, elementary row operation, reduced row echelon form, rank, leading 1, leading column, leading variable, free column, free variable, matrix product, linear combination, linear transformation, domain, codomain = target, inverse transformation, invertible, rotation, reflection, scaling, projection, shear, associativity, image, kernel, span, linear relation, trivial linear relation, linear subspace, linearly independent, linearly dependent, redundant term, irredundant term, ordered basis.

Problem 1 For the following linear system, find the augmented matrix and the reduced row echelon form of the augmented matrix. Say whether the solution is consistent or inconsistent. If the system is consistent, write down the general form of the solution.

$$\left\{\begin{array}{rcrcrcrcrcrcr} 4x_1 & + & 2x_3 & + & 22x_4 & + & x_5 & + & -14x_6 & = & 14 \\ -x_1 & - & x_3 & - & 8x_4 & + & & & 4x_6 & = & -4 \\ & & 3x_3 & + & 15x_4 & - & 7x_5 & + & 8x_6 & = & 3 \\ 3x_1 & + & & & 9x_4 & + & 2x_5 & - & 10x_6 & = & 9 \end{array}\right.$$

Solution to Problem 1 The augmented matrix is

$$\widetilde{A} = \left(\begin{array}{rrrrrr|r} 4 & 0 & 2 & 22 & 1 & -14 & 14 \\ -1 & 0 & -1 & -8 & 0 & 4 & -4 \\ 0 & 0 & 3 & 15 & -7 & 8 & 3 \\ 3 & 0 & 0 & 9 & 2 & -10 & 9 \end{array}\right)$$

By Gauss-Jordan elimination or some other sequences of elementary row operations, $\mathrm{rref}(\widetilde{A})$ equals

$$\mathrm{rref}(\widetilde{A}) = \left(\begin{array}{rrrrrr|r} 1 & 0 & 0 & 3 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

From the reduced row echelon form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 2 \\ 1 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \boxed{\begin{bmatrix} 3 - 3c_2 + 2c_3 \\ c_1 \\ 1 - 5c_2 + 2c_3 \\ c_2 \\ 2c_3 \\ c_3 \end{bmatrix}}$$

for arbitrary real numbers c_1, c_2 and c_3.

Problem 2 For the following augmented matrix $\widetilde{A} = [A|\vec{b}]$, find the reduced row echelon form of $\widetilde{A}$ and say whether or not the system $A\vec{x} = \vec{b}$ is consistent. If the system is consistent, write down the general form of the solution.

$$\left(\begin{array}{ccc|c} 3 & -2 & 7 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right).$$

Solution to Problem 2 The reduced row echelon form is

$$\text{rref}(\widetilde{A}) = \boxed{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)}.$$

Since there is a leading 1 in the "constant column", i.e., the column for $\vec{b}$, the system is inconsistent.

Problem 3 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for the unique 4-tuple of real numbers b_1, b_2, b_3, b_4 such that

$$y'' + 2y = b_1 e^{-t}\cos(t) + b_2 e^{-t}\sin(t) + b_3 e^t \cos(t) + b_4 e^t \sin(t)$$

where

$$y(t) = a_1 e^{-t}\cos(t) + a_2 e^{-t}\sin(t) + a_3 e^{t}\cos(t) + a_4 e^{t}\sin(t).$$

Find the unique matrix A such that for every 4-vector $\vec{x}$

$$T(\vec{x}) = A\vec{x}.$$

Solution to Problem 3 The matrix is

$$A = \boxed{\left(\begin{array}{cc|cc} 2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right)}$$

(the vertical and horizontal lines are just for effect).

Problem 4 Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$S(\vec{x}) = (\vec{e}_2 \cdot \vec{x})\vec{e}_1 - (\vec{e}_1 \cdot \vec{x})\vec{e}_2 + \vec{e}_3 \times \vec{x}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Is the function S a linear transformation? If so, write down the matrix of S.

Solution to Problem 4 Notice that for the vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

the following identities hold

$$\vec{e}_1 \cdot \vec{x} = x_1, \;\; \vec{e}_2 \cdot \vec{x} = x_2,$$

and

$$\vec{e}_3 \times \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}.$$

Substituting this in gives

$$S(\vec{x}) = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-x_1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In other words, for every $\vec{x}$ in $\mathbb{R}^3$,

$$S(\vec{x}) = \vec{0}.$$

This is indeed a linear transformation with matrix

$$A_S = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}.$$

Problem 5 Reflection in $\mathbb{R}^3$ about a plane containing the z-axis is a linear transformation with matrix

$$X = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real numbers a and b satisfying $a^2 + b^2 = 1$.

Reflection in $\mathbb{R}^3$ about a plane containing the y-axis is a linear transformation with matrix

$$Y = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix}$$

for some real numbers c and d satisfying $c^2 + d^2 = 1$.

Compute the product matrices $Z = XY$ and $W = YX$. Does XY equal YX? Also compute ZW and WZ. Explain your answer.

Solution to Problem 5 The matrix products are

$$Z = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix} = \boxed{\begin{pmatrix} ac & b & ad \\ bc & -a & bd \\ d & 0 & -c \end{pmatrix}}$$

and

$$W = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix} \cdot \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} ac & bc & d \\ b & -a & 0 \\ ad & bd & -c \end{pmatrix}}.$$

The matrices Z and W are not equal, except in a small number of exceptional cases:

(i) if $(a, b) = (1, 0)$ and (c, d) is arbitrary,

(ii) if (a, b) is arbitrary and $(c, d) = (1, 0)$,

(iii) and if $(a, b) = ((-1)^m, 0)$ and $(c, d) = ((-1)^n, 0)$ for integers m and n, i.e., a is ± 1 and c is ± 1 independent of each other.

Because ZW equals $(XY)(YX)$, by associativity this is $X(YY)X$. Since Y is a reflection, YY equals I_3. Thus ZW equals $X(I_3)X$, i.e., XX. Since X is a reflection, XX equals I_3. Thus ZW equals I_3. By a similar argument, $WZ = (YX)(XY) = Y(XX)Y = Y(I_3)Y = YY$ also equals I_3. Thus

$$\boxed{ZW = WZ = I_3}$$

Problem 6 For which values of k is the following system consistent? For which values are there infinitely many solutions? Whenever the solution is unique, compute its value.

$$\begin{cases} 2x_1 + k^2x_2 + (4k-4)x_3 = k+10 \\ 3x_2 + k^2x_2 = k+9 \\ -2x_2 + k^2x_2 + (5k-5)x_3 = k-1 \end{cases}$$

Solution to Problem 6 The augmented matrix of this linear system is

$$\widetilde{A} = \left(\begin{array}{ccc|c} 2 & k^2 & 4k-4 & k+10 \\ 3 & k^2 & 0 & k+9 \\ -2 & k^2 & 5k-5 & k-1 \end{array}\right).$$

Elementary row operations transform this to the row equivalent matrix

$$\widetilde{B} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & k^2 & 0 & k \\ 0 & 0 & k-1 & 1 \end{array}\right).$$

There are 3 cases depending on whether $k = 0$, $k = 1$ or neither. First of all, if $k = 0$ then

$$\widetilde{B}|_{k=0} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

whose reduced row echelon form is

$$\mathrm{rref}(\widetilde{B}|_{k=0}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So when $k = 0$ the system has infinitely many solutions,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ c \\ -1 \end{bmatrix}$$

where c is an arbitrary real number.

Problem 7 For which values of t is the following matrix invertible? Whenever it is invertible, write down the inverse. Whenever it is not invertible, write down an element in the kernel.

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}.$$

The next case is when $k = 1$. In this case,

$$\widetilde{B}|_{k=1} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

whose reduced row echelon form is

$$\mathrm{rref}(\widetilde{B}|_{k=1}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

So when $k = 1$ the system is inconsistent.

Finally, when k is neither 0 nor 1, the reduced row echelon form is

$$\widetilde{B} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1/k \\ 0 & 0 & 1 & 1/(k-1) \end{array}\right).$$

So when k is neither 0 nor 1 the system has a unique solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/k \\ 1/(k-1) \end{bmatrix}.$$

Problem 8 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution to Problem 8 The inverse of the matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Problem 9 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & 1 \\ -3 & 1 & 0 \end{pmatrix}$$

Solution to Problem 9 The inverse of the matrix is

$$\frac{1}{19}\begin{pmatrix} -1 & 2 & -7 \\ -3 & 6 & -2 \\ 15 & -11 & 10 \end{pmatrix}.$$

Problem 10 For the following matrix A, determine which column vectors are irredundant and which are redundant. Then write down an ordered basis for the image of T_A and an ordered basis for the kernel of T_A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & -1 & 1 & 5 \end{pmatrix}.$$

Solution to Problem 10 The reduced row echelon form of A is

$$\text{rref}A = \begin{pmatrix} 1 & 0 & -7/5 & 0 & 0 & 0 \\ 0 & 1 & 11/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The leading columns of the reduced row echelon matrix are the first, second, fourth and sixth columns. Thus the irredundant columns of A are the 1st, 2nd, 4th, and 6th columns. And the redundant columns of A are the 3rd and 5th columns. Therefore an ordered basis for the image of T_A is

$$\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 5 \end{bmatrix} \right).$$

Finally, an ordered basis for the kernel of T_A is

$$\left(\begin{bmatrix} 7/5 \\ -11/5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

Problem 11 From the following ordered sequence of vectors, determine which terms are redundant and which are irredundant. Write every redundant term as a linear combination of the irredundant ones.

$$(2\vec{e}_1 + 2\vec{e}_2, \vec{0}, \vec{e}_2 + \vec{e}_3, -2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3, \vec{e}_1 + \vec{e}_4, -\vec{e}_1 - \vec{e}_2 + \vec{e}_3 + \vec{e}_4)$$

Solution to Problem 11 The ordered sequence of vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6)$ is the same as the ordered sequence of column vectors of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & -2 & 1 & -1 \\ 2 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The reduced row echelon form is

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the irredundant terms are $\vec{v}_1, \vec{v}_3, \vec{v}_5$ and the redundant terms are $\vec{v}_2, \vec{v}_4, \vec{v}_6$. Moreover, the redundant terms are linear combinations of the irredundant terms as follows

$$\begin{bmatrix} \vec{v}_2 & \vec{v}_4 & \vec{v}_6 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_5 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 12 Let A be a 4×4 matrix such that Image(T_A) equals Ker(T_A). What can you say about A^2? Write down an example of such a matrix.

Solution to Problem 12 If Image(T_A) equals Ker(T_A), or even if Image(T_A) is just contained in Kern(T_A), then A^2 equals the zero matrix. A typical example is

$$A = \left(\begin{array}{r|r} -XR & -XRX \\ \hline R & RX \end{array}\right)$$

where X is an arbitrary 2×2 matrix and R is an invertible 2×2 matrix, i.e.,

$$A = \begin{pmatrix} -x & -y \\ -z & -w \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r & s \\ t & u \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{pmatrix}$$

for arbitrary real numbers x, y, z, w and for real numbers r, s, t, u for which $ru - st$ is nonzero. Just to be very specific, one such example is when $X = 0$ and $R = I_2$, i.e.,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$