MAT211 Fall 2007 Midterm 1 Review Sheet

The topics tested on Midterm 1 will be among the following.

(i) Finding the coefficient and augmented matrix of a system of linear equations.

(ii) Using Gauss-Jordan elimination or other elementary row operations to find the reduced row echelon form of a matrix.

(iii) Determining when a system of linear equations has no solution, a unique solution, or infinitely many solutions.

(iv) Finding the general solution of a system of linear equations.

(v) Finding the matrix representative of a linear transformation, i.e., finding A given T.

(vi) Adding and scaling matrices, determining when two matrices can be multiplied, and computing the matrix product when it exists.

(vii) Determining whether a matrix is or is not invertible, and computing the matrix inverse when it exists.

(viii) Given an ordered sequence $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ of elements in $\mathbb{R}^n$, determining which are redundant and irredundant, and expressing each redundant element as a linear combination of irredundant elements.

(ix) Finding an ordered basis for the image of a linear transformation.

(x) Finding an ordered basis for the kernel of a linear transformation (which is roughly the same as Item (iv) above).

You should know the meaning of all of the following words and phrases: linear equation, system of linear equations, consistent, inconsistent, zero matrix, identity matrix, coefficient matrix, augmented matrix, entries, column vector, vector, row vector, elementary row operation, reduced row echelon form, rank, leading 1, leading column, leading variable, free column, free variable, matrix product, linear combination, linear transformation, domain, codomain = target, inverse transformation, invertible, rotation, reflection, scaling, projection, shear, associativity, image, kernel, span, linear relation, trivial linear relation, linear subspace, linearly independent, linearly dependent, redundant term, irredundant term, ordered basis.

Problem 1 For the following linear system, find the augmented matrix and the reduced row echelon form of the augmented matrix. Say whether the solution is consistent or inconsistent. If the system is consistent, write down the general form of the solution.

$$\left\{\begin{array}{rcrcrcrcrcr} 4x_1 & + & 2x_3 & + & 22x_4 & + & x_5 & + & -14x_6 & = & 14 \\ -x_1 & - & x_3 & - & 8x_4 & + & & & 4x_6 & = & -4 \\ & & 3x_3 & + & 15x_4 & - & 7x_5 & + & 8x_6 & = & 3 \\ 3x_1 & + & & & 9x_4 & + & 2x_5 & - & 10x_6 & = & 9 \end{array}\right.$$

Problem 2 For the following augmented matrix $\widetilde{A} = [A|\vec{b}]$, find the reduced row echelon form of $\widetilde{A}$ and say whether or not the system $A\vec{x} = \vec{b}$ is consistent. If the system is consistent, write down the general form of the solution.

$$\left(\begin{array}{rrr|r} 3 & -2 & 7 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right).$$

Problem 3 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for the unique 4-tuple of real numbers b_1, b_2, b_3, b_4 such that

$$y'' + 2y = b_1 e^{-t}\cos(t) + b_2 e^{-t}\sin(t) + b_3 e^{t}\cos(t) + b_4 e^{t}\sin(t)$$

where

$$y(t) = a_1 e^{-t}\cos(t) + a_2 e^{-t}\sin(t) + a_3 e^{t}\cos(t) + a_4 e^{t}\sin(t).$$

Find the unique matrix A such that for every 4-vector $\vec{x}$

$$T(\vec{x}) = A\vec{x}.$$

Problem 4 Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$S(\vec{x}) = (\vec{e}_2 \cdot \vec{x})\vec{e}_1 - (\vec{e}_1 \cdot \vec{x})\vec{e}_2 + \vec{e}_3 \times \vec{x}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Is the function S a linear transformation? If so, write down the matrix of S.

Problem 5 Reflection in $\mathbb{R}^3$ about a plane containing the z-axis is a linear transformation with matrix

$$X = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real numbers a and b satisfying $a^2 + b^2 = 1$.

Reflection in $\mathbb{R}^3$ about a plane containing the y-axis is a linear transformation with matrix

$$Y = \begin{pmatrix} c & 0 & d \\ 0 & 1 & 0 \\ d & 0 & -c \end{pmatrix}$$

for some real numbers c and d satisfying $c^2 + d^2 = 1$.

Compute the product matrices $Z = XY$ and $W = YX$. Does XY equal YX? Also compute ZW and WZ. Explain your answer.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 6 For which values of k is the following system consistent? For which values are there infinitely many solutions? Whenever the solution is unique, compute its value.

$$\left\{\begin{array}{rcrcrcr} 2x_1 & + & k^2x_2 & + & (4k-4)x_3 & = & k+10 \\ 3x_2 & + & k^2x_2 & & & = & k+9 \\ -2x_2 & + & k^2x_2 & + & (5k-5)x_3 & = & k-1 \end{array}\right.$$

Problem 7 For which values of t is the following matrix invertible? Whenever it is invertible, write down the inverse. Whenever it is not invertible, write down an element in the kernel.

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}.$$

Problem 8 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Problem 9 Find the inverse of the following invertible 3×3 matrix.

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & 1 \\ -3 & 1 & 0 \end{pmatrix}$$

Problem 10 For the following matrix A, determine which column vectors are irredundant and which are redundant. Then write down an ordered basis for the image of T_A and an ordered basis for the kernel of T_A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ -2 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & -1 & 1 & 5 \end{pmatrix}.$$

Problem 11 From the following ordered sequence of vectors, determine which terms are redundant and which are irredundant. Write every redundant term as a linear combination of the irredundant ones.

$$(2\vec{e}_1 + 2\vec{e}_2, \vec{0}, \vec{e}_2 + \vec{e}_3, -2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3, \vec{e}_1 + \vec{e}_4, -\vec{e}_1 - \vec{e}_2 + \vec{e}_3 + \vec{e}_4)$$

Problem 12 Let A be a 4×4 matrix such that Image(T_A) equals Ker(T_A). What can you say about A^2? Write down an example of such a matrix.