Advanced Calculus, Final Test, December 2003

1. (a) Show that the solution space for the following problem is a subspace of R^3.
$$3x - 2y + z = 0$$
   (b) Find the basis of Z.
   (c) Find the orthonormal basis of Z by going through Gram-Schmidt process.

2. Determine if (1,1) and (0,1) are linearly independent.

3. Find the distance between (1,3) and (2,2) in 1-norm, 2-norm, max norm.

4. Find the matrix associated with the following linear maps
   (a) Projection map. $L : R^4 \to R^4$ and $L(x_1, x_2, x_3, x_4) = (x_3, x_4)$.
   (b) $L : R^3 \to R^3$ and $L(x_1, x_2, x_3) = (2x_1, 2x_2, 5x_3)$
   (c) Permutation $L : R^4 \to R^4$ and $L(x_1, x_2, x_3, x_4) = (x_2, x_4, x_1, x_3)$

5. Find the determinant of the following 4x4 matrix.
$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

6. Prove the following statements by using $\epsilon - \delta$ method.
   (a) $\lim_{x \to 2} x^2 - 3x = -2$
   (b) $\lim_{n \to \infty} \frac{n-1}{n} = 1$

7. Let $f(x) = -1$ for $x \leq 0$ and $f(x) = x^2$ if $x > 0$. Find a point c where f(x) is discontinuous. Show that c is a discontinuity point of f using $\epsilon - \delta$ method. In other words, find an ϵ for which there is no such δ that
$$\text{If } |x - c| < \delta \text{ then } then |f(x) - f(c)| < \epsilon$$

8. Let $f : R^4 \to R^2$ defined by $f(x, y) = (x^2, y, xy, y^2)$.
   (a) Find $F'(c)$ where $c = (2, 1)$.
   (b) Find the basis of the tangent plane at c and find the tangent plane at c.

9. Let $f(x, y) = \frac{xy^2}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ if $(x, y) = (0, 0)$
   (a) Find the directional derivative $D_v f(0, 0)$ for $v = (1, 2)$.
   (b) Show that $D_v f(0, 0) = f(v)$ for all v.
   (c) Show that f is not differentiable at (0,0).