Math 131 **Name (Print):** ______________
Summer 2015
Exam 2
6/17/15
Time Limit: 120 Minutes ID number ______________

This exam contains 10 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	5	
3	15	
4	40	
5	10	
6	10	
Total:	100	

1. (Find the limit)

(a) (5 points)

$$\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$$

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right) &= \lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right)\\ &= \lim_{x\to 1}\frac{1}{x-2}\\ &= -1\end{aligned}$$

(b) (5 points)

$$\lim_{x\to +\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$$

$$\begin{aligned}\lim_{x\to +\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right) &= \lim_{x\to +\infty}\frac{(\sqrt{x^2+x+1}-\sqrt{x^2-x})\cdot(\sqrt{x^2+x+1}+\sqrt{x^2-x})}{(\sqrt{x^2+x+1}+\sqrt{x^2-x})}\\ &= \lim_{x\to +\infty}\frac{(x^2+x+1)-(x^2-x)}{(\sqrt{x^2+x+1}+\sqrt{x^2-x})}\\ &= \lim_{x\to +\infty}\frac{2x+1}{(\sqrt{x^2+x+1}+\sqrt{x^2-x})}\\ &= \lim_{x\to +\infty}\frac{\frac{2x+1}{x}}{(\sqrt{\frac{x^2+x+1}{x^2}}+\sqrt{\frac{x^2-x}{x^2}})}\\ &= 1\end{aligned}$$

(c) (5 points)

$$\lim_{x\to+\infty} \frac{3x^3+5x+8}{6x^3+3x^2+2x+1}$$

$$\begin{aligned}\lim_{x\to+\infty} \frac{3x^3+5x+8}{6x^3+3x^2+2x+1} &= \lim_{x\to+\infty} \frac{\frac{3x^3+5x+8}{x^3}}{\frac{6x^3+3x^2+2x+1}{x^3}} \\ &= \lim_{x\to+\infty} \frac{3+\frac{5}{x^2}+\frac{8}{x^3}}{6+\frac{3}{x}+\frac{2}{x^2}+\frac{1}{x^3}} \\ &= \frac{3}{6} = \frac{1}{2}\end{aligned}$$

(d) (5 points)

$$\lim_{x\to 0} x^2 \cdot \cos\frac{2}{x^2}$$

Since

$$-1 \le \cos\frac{2}{x^2} \le 1$$

So

$$-x^2 \le x^2 \cdot \cos\frac{2}{x^2} \le x^2$$

And

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$$

Therefore by Squeeze Theorem we have:

$$\lim_{x\to 0} x^2 \cdot \cos\frac{2}{x^2} = 0$$

2. (Each limit represents the derivative of some function f at some number a. State such an f and a in each case. Do **NOT** use L'Hospital's Rule)

   (a) (5 points)

$$\lim_{x \to \frac{\pi}{6}} \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}}$$

Solution

$f(x) = \sin x$, $a = \frac{\pi}{6}$

$$\lim_{x \to \frac{\pi}{6}} \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

3. If $f(x) = x^3 - 2x^2 + x + 1$.

   (a) (5 points) On what intervals are f increasing and on what intervals are f decreasing?
   Solution
   $$f^{'}(x) = 3x^2 - 4x + 1$$
   So
   $$f^{'}(x) \leq 0 \Rightarrow x \in [\frac{1}{3}, 1]$$
   That is, $f(x)$ is decreasing on $[\frac{1}{3}, 1]$. likewise
   $$f^{'}(x) \geq 0 \Rightarrow x \in (-\infty, \frac{1}{3}] \cup [1, +\infty)$$
   So $f(x)$ is increasing on $(-\infty, \frac{1}{3}]$,$[1, +\infty)$.
   Remark: Do not write the intervals of increasing like a Union. $f(x)$ is increasing on each interval but not on the union!

   (b) (5 points) On what intervals are f concave upward and on what intervals are f concave downward?
   $$f^{''}(x) = 6x - 4$$
   Thus
   $$f^{''}(x) \leq 0 \Leftrightarrow x \in (\infty, \frac{2}{3})$$
   So we have $f(x)$ concave downward on $(-\infty, \frac{2}{3})$, concave upward on $(\frac{2}{3}, +\infty)$

(c) (5 points) Sketch the graph of the function f.(Indicate monotonicity, concavity)

4. Find the derivatives

(a) (10 points)

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

$$f'(x) = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot (1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot (1 + \frac{1}{2\sqrt{x}}))$$

(b) (10 points)

$$f(x) = \arctan(\sqrt{\ln x - 1})$$

$$\begin{aligned} f'(x) &= \frac{1}{1 + (\sqrt{\ln x - 1})^2} \cdot \frac{1}{2\sqrt{\ln x - 1}} \cdot \frac{1}{x} \\ &= \frac{1}{2\ln x} \cdot \frac{1}{\sqrt{\ln x - 1}} \cdot \frac{1}{x} \end{aligned}$$

(c) (10 points)

$$f(x) = \frac{x^{\frac{5}{3}} \cdot \sqrt{x+2}}{(x^2+x+1)^5 \cdot \sqrt{x+1}}$$

$$y = \frac{x^{\frac{5}{3}} \cdot \sqrt{x+2}}{(x^2+x+1)^5 \cdot \sqrt{x+1}}$$

$$\ln y = \frac{5}{3}\ln x + \frac{1}{2}\ln(x+2) - 5\ln(x^2+x+1) - \frac{1}{2}\ln(x+1)$$

$$\frac{y^{'}}{y} = \frac{5}{3x} + \frac{1}{2(x+2)} - \frac{10x+5}{x^2+x+1} - \frac{1}{2(x+1)}$$

$$f^{'} = \frac{x^{\frac{5}{3}} \cdot \sqrt{x+2}}{(x^2+x+1)^5 \cdot \sqrt{x+1}} \cdot (\frac{5}{3x} + \frac{1}{2(x+2)} - \frac{10x+5}{x^2+x+1} - \frac{1}{2(x+1)})$$

(d) (10 points)

$$f(x) = (\sin x)^{\sin x}$$

$$y = (\sin x)^{\sin x}$$

$$\ln y = \sin x \ln \sin x$$

$$\frac{y^{'}}{y} = \cos x \ln \sin x + \sin x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$f^{'}(x) = (\sin x)^{\sin x} \cdot (1 + \ln \sin x)\cos x$$

5. Find the tangent line at given point:

   (a) (10 points)

$$xy^4 + x^2y = x + 3y$$

   at point$(\sqrt{3}, 1)$.

$$y^4 + x \cdot 4y^3 \cdot y' + 2xy + x^2y' = 1 + 3y'$$

   Plug in point$(\sqrt{3}, 1)$,

$$1 + \sqrt{3} \cdot 4 \cdot 1^3 \cdot y' + 2 \cdot \sqrt{3} \cdot 1 + 3y' = 1 + 3y'$$

   $\Rightarrow y' = -\frac{1}{2}$. So the tangent line is

$$y - 1 = -\frac{1}{2} \cdot (x - \sqrt{3})$$

6. (a) (10 points) Use linear approximation of function $f(x) = \sqrt{x}$ at $x = 16$ to estimate $\sqrt{15.99}$.
Since $f^{'}(x) = \frac{1}{2\sqrt{x}}$, we have the linear approximation to be

$$y - f(16) = f^{'}(16)(x - 16)$$

So we have

$$\sqrt{15.99} \approx f(16) + f^{'}(16) \cdot (15.99 - 16) = 4 - \frac{1}{800}$$