PRACTICE FINAL FOR MAT 341

In problems (1)-(4) below we consider a cylindrical rod centered along the x-axis in 3-dimensional space, from $x = 0$ to $x = a$; for each $0 \leq x \leq a$ the intersection of this rod with the plane containing $(x, 0, 0)$ and perpendicular to the x-axis is a disc D_x of area A. We assume that the physical properties of this rod are the same at each of its points: in particular its density function is a constant function ρ, and the heat capacity capicity per unit mass for the rod is also a constant function c (see page 36). We assume that for each time $t \geq 0$ and for each $0 \leq x \leq a$ the temperature at each point of D_x is equal to the same value $u(x, t)$. Finally we assume that the cylindrical surface of the rod is insulated (the ends of the rod are not necessarily insulated).

(1) What does the *heat flux* function $q(x, t)$ measure? State *Fourier's law of heat conduction* for this rod.

(2) Suppose that the rod is also insulated at its right hand end D_a and is kept at a constant temperature of 2 degrees celius at its left hand end D_0.

(a) Give a mathematical description (some equations) of all these conditions placed on $u(x, t)$, $0 \leq x \leq a$ and $0 \leq t$.
(b) Find a general solution to the equations in part (a).

(3) Let $H(x, t)$ denote the total heat contained within the portion of the rod between D_0 and D_x. Recall that $H(x, t) = \int_0^x \rho c A u(y, t) dy$ (see page 136 of text).

Suppose that the rod is insulated at its right hand end D_a and that $H(a, 2) < H(a, 0)$. Then show that $\frac{\partial u}{\partial x}(0, t) > 0$ holds for some $0 \leq t \leq 2$.

(4) Suppose that $u(x, t)$ satisfies $u(x, 0) = 3$ in addition to the properties of problem (2)(a) above.

(a) Compute $H(a, 0)$ and $limit_{t \to \infty} H(a, t)$.
(b) Verify that $\frac{\partial u}{\partial x}(0, t) > 0$ for all $t > 0$. (**Hint:** Write $u(x, t)$ as an infinite series and compute its x-derivative term by term.)
(c) Use part (b) to verify that $H(a, t)$ is a decreasing function in t.

(5) For all $0 \leq x \leq \pi$ and $0 \leq t$ suppose that the following equations hold for the function $u(x, t)$:

$$(i) \quad \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, t)$$

$$(ii) \quad u(0, t) = 0, u(\pi, t) = 0$$

$$(iii) \quad u(x,0) = sin(x), \frac{\partial u}{\partial t}(x,0) = sin(x)$$

(a) find the d'Alembert solution to these equations.
(b) find the Fourier type solution to these equations.
(c) does this vibrating string ever return to its original position?

(6) Show that if $u_1(x,t)$ and $u_2(x,t)$ both satisfy equations (i),(ii) in problem (5), then $u(x,t) = \alpha_1 u_1(x,t) + \alpha_2 u_2(x,t)$ also satisfies (i),(ii) in problem (5) for any real numbers α_1, α_2.

(7) Do problem (11) on page 232 of the text.

(8) Suppose that $u(x,t)$, $0 \le x, t$, satisfies

$$\frac{\partial u}{\partial x}(x,t) = -\frac{1}{c}\frac{\partial u}{\partial t}(x,t)$$
$$u(0,t) = 0$$
$$u(x,0) = f(x)$$

for some given differentiable function $f(x)$.

(a) Show that $u(x,t)$ also satisfies

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t)$$
$$\frac{\partial u}{\partial t}(x,0) = -c\frac{df}{dx}(x).$$

(b) Solve for $u(x,t)$ in terms of the function f.
(c) Give a physical description of the solution of part (b).

(9) A real valued function $u(x,y)$ of the two real variables x, y is *harmonic* if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on its domain.

(a) If $u(x,y) = \sum_{0 \le i+j \le 3} a_{i,j} x^i y^j$, and u is harmonic in a disc of radius 2 centered at (-3,4), then prove that u is harmonic on the whole plane.
(b) It is a fact that if u is harmonic on a finite rectangle $\mathbb{R} = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, then it takes on neither a maximum value nor a minimum value in the interior of this rectangle $\{(x,y) \mid a < x < b, c < y < d\}$. Prove this fact under the additional hypothesis that $\frac{\partial^2 u}{\partial x^2}$ does not vanish in the interior of the rectangle.

(10) Consider the following 2-dimensional heat problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(x,0,t) = 0, \quad u(x,b,t) = 0$$

$$u(0,y,t) = 3sin(\frac{2\pi}{b}y), \quad u(a,y,t) = -sin(\frac{5\pi}{b}y)$$

$$u(x,y,0) = x + y$$

Find the steady state solution for this problem.