MAT 341 – Applied Real Analysis

SPRING 2015

Midterm 1 – March 10, 2015

SOLUTIONS

NAME: ____________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
TOTAL	

Problem 1: (22 points) Suppose that the Fourier cosine series of a given function $f(x)$ is

$$f(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3} \cos\left(\frac{n\pi x}{4}\right).$$

a) Show that $f(x) = f(x+8)$.

SOLUTION. Clearly $\cos\left(\frac{n\pi x}{4}\right)$ is periodic of period 8 for each integer n. The sum of periodic functions of the same period (in this case 8) is again a periodic function of the same period. So $f(x)$ is periodic of period 8. □

b) Does this Fourier cosine series converge uniformly? Explain.

SOLUTION. The Fourier cosine series converges uniformly because

$$\sum_{n=1}^{\infty} |a_n| + |b_n| = \sum_{n=1}^{\infty} \frac{|1+2(-1)^n|}{2015n^3} < \sum_{n=1}^{\infty} \frac{3}{2015n^3} = \frac{3}{2015} \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$$

We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-integral test for $p = 3 > 1$). □

c) Find the Fourier cosine series of $1 - 5f(x)$.

SOLUTION. The Fourier cosine series of 1 is just 1. Note that the function $1 - 5f(x)$ is even if f is even, so it has a Fourier cosine series which is

$$1 - 5f(x) = 1 - 5\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3} \cos\left(\frac{n\pi x}{4}\right) = 1 - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1+2(-1)^n}{403n^3} \cos\left(\frac{n\pi x}{4}\right)$$

□

d) Find the Fourier cosine series of $f'(x)$ if it exists. If it does not exist, explain why it does not exist.

SOLUTION. We have

$$\sum_{n=1}^{\infty} |na_n| + |nb_n| = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{|1+2(-1)^n|}{2015n^2} < \frac{3}{2015\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

since we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This means that the differentiated Fourier series converges uniformly to the derivative $f'(x)$. Therefore the Fourier sine series of $f'(x)$ is the following

$$f'(x) = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1+2(-1)^n}{2015n^3} \frac{n\pi}{4} \sin\left(\frac{n\pi x}{4}\right) = -\frac{1}{8060\pi} \sum_{n=1}^{\infty} \frac{1+2(-1)^n}{n^2} \sin\left(\frac{n\pi x}{4}\right).$$

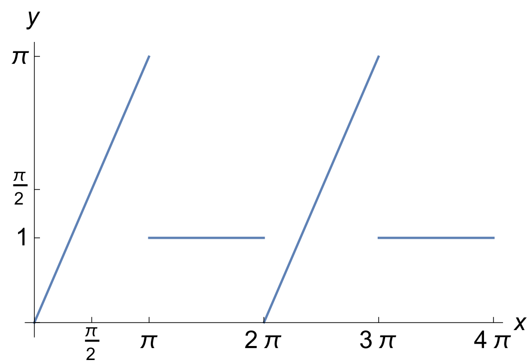
However, there are no cosine terms in this series (and the Fourier series is unique) so there is no cosine Fourier series for $f'(x)$. Alternatively, since we are given the Fourier cosine series of f, we can assume that f is even (or work with it's even extension). But $f(x) = f(-x)$ gives $f'(x) = -f'(-x)$ so the derivative f' is odd, so it has a sine series, rather than a cosine series. □

Problem 2: (30 points) Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi \le x < 0, \\ x & \text{if} \quad 0 \le x < \pi; \end{cases} \qquad f(x+2\pi) = f(x).$$

a) Sketch the graph of f on the interval $[0, 4\pi]$.

SOLUTION.



□

b) Find the Fourier series for f.

SOLUTION. The function is periodic of period 2π so $a = \pi$. The Fourier series of f is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \\ &= \left.\frac{\sin(nx)}{n\pi}\right|_{-\pi}^{0} + \left.\frac{\cos(nx)}{n^2\pi}\right|_{0}^{\pi} + \left.\frac{x\sin(nx)}{n\pi}\right|_{0}^{\pi} = \frac{\cos(n\pi)-1}{n^2\pi} = \frac{(-1)^n - 1}{n^2\pi}. \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx \\ &= \left.\frac{-\cos(nx)}{n\pi}\right|_{-\pi}^{0} + \left.\frac{\sin(nx)}{n^2\pi}\right|_{0}^{\pi} - \left.\frac{x\cos(nx)}{n\pi}\right|_{0}^{\pi} = \frac{-1+\cos(n\pi)}{n\pi} - \frac{\pi\cos(n\pi)}{n\pi} \\ &= \frac{-1+(-1)^n}{n\pi} - \frac{(-1)^n}{n}. \end{aligned}$$

Also $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,dx + \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{2} + \frac{\pi}{4}$. The Fourier series is

$$f(x) = \frac{1}{2} + \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \left(\frac{-1+(-1)^n}{n\pi} - \frac{(-1)^n}{n}\right)\sin(nx)\right).$$

In computing the coefficients a_n and b_n we have used the formulas provided on the last page of the exam. □

c) To what value does the Fourier series converge at:

i) $x = 0$; ii) $x = \frac{\pi}{2}$; iii) $x = 3\pi$? Explain.

SOLUTION. Clearly the function is piecewise continuous and has a piecewise continuous derivative. The function is discontinuous (it has a jump) at $x = 0$ and $x = 3\pi$, as seen from the graph. So the Fourier series converges to $\frac{f(0+)+f(0-)}{2} = \frac{1}{2}$ at $x = 0$ and to $\frac{f(3\pi+)+f(3\pi-)}{2} = \frac{\pi+1}{2}$ at $x = 3\pi$. The function is continuous at $x = \frac{\pi}{2}$ so the Fourier series converges to $f(x) = \frac{\pi}{2}$ in this case. □

d) Does the Fourier series of f converges uniformly on the interval $[0, \pi]$? Does it converge uniformly on the interval $[0, 4\pi]$? Explain.

SOLUTION. Both intervals $[0, \pi]$ and $[0, 4\pi]$ contain the points $x = 0$ and $x = \pi$. At $x = 0$ the Fourier series converges to $\frac{1}{2}$ as shown above. However, if we take x arbitrarily close to 0 then the function is continuous on $(0, \pi)$ and the Fourier series converges to $f(x) = x$. For example, for $x = 0.01$, the Fourier series converges to 0.01, which is far from $\frac{1}{2} = 0.5$. So in both cases the Fourier series of f does not converge uniformly. □

Problem 3: (24 points) Consider the heat conduction problem in a bar that is in thermal contact with an external heat source. Then the modified heat conduction equation is

$$\frac{\partial^2 u}{\partial x^2} + s(x) = \frac{1}{k}\frac{\partial u}{\partial t}$$

where the term $s(x)$ describes the effect of the external agency; $s(x)$ is positive for a source. Suppose that the boundary conditions are

$$u(0,t) = T_0, \quad u(a,t) = T_1$$

and the initial condition is $u(x,0) = f(x)$.

a) Write $u(x,t) = w(x,t) + v(x)$, where $w(x,t)$ and $v(x)$ are the transient and steady state parts of the solution, respectively. State the boundary value problems that $v(x)$ and $w(x,t)$, respectively, satisfy.

Solution.

$$\begin{aligned} &v''(x) = -s(x) \\ &v(0) = T_0, \quad v(a) = T_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t} \\ &w(0,t) = 0, \quad w(a,t) = 0 \\ &w(x,0) = f(x) - v(x) \end{aligned}$$

□

b) Suppose $k = 1$ and $s(x) = 6x$. Find $v(x)$.

Solution. We have $v''(x) = -6x$ so $v'(x) = -3x^2 + A$ and $v(x) = -x^3 + Ax + B$. From $v(0) = T_0$ we get $B = T_0$. From $v(a) = T_1$ we get $-a^3 + A \cdot a + T_0 = T_1$, which gives $A = \frac{a^3+T_1-T_0}{a} = a^2 + \frac{T_1-T_0}{a}$. Thus

$$v(x) = -x^3 + \left(a^2 + \frac{T_1 - T_0}{a}\right) x + T_0.$$

□

Problem 4: (24 points) Find the solution of the heat problem

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$
$$u(0,t) = 0, \quad u(2,t) = \pi, \quad t > 0;$$
$$u(x,0) = \frac{\pi x}{2} - 3\sin(\pi x) + 5\sin(2\pi x), \quad 0 \le x \le 2.$$

Solution. We first need to find the steady-state solution $v(x)$. Note that $v''(x) = 0$ and $v(0) = 0, v(2) = \pi$. This gives $v(x) = \frac{\pi x}{2}$. We then need to solve the following homogeneous problem

$$\frac{\partial^2 w}{\partial x^2} = 4\frac{\partial w}{\partial t}, \qquad 0 < x < 2, \quad t > 0;$$
$$w(0,t) = 0, \quad w(2,t) = 0, \quad t > 0;$$
$$w(x,0) = -3\sin(\pi x) + 5\sin(2\pi x), \quad 0 \le x \le 2.$$

However, we know that the solution to this problem is given by

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 kt} \sin(\lambda_n x)$$

where $\lambda_n = \frac{n\pi}{a}$ and $k = \frac{1}{4}$, $a = 2$ in this problem. Therefore

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

The coefficients c_n can be determined from $w(x,0) = -3\sin(\pi x) + 5\sin(2\pi x)$, so $c_2 = -3$, $c_4 = 5$ and $c_n = 0$ for all other values of n. It follows that

$$w(x,t) = -3e^{-\frac{\pi^2}{4}t} \sin(\pi x) + 5e^{-\pi^2 t} \sin(2\pi x).$$

The solution to the initial problem is $u(x,t) = w(x,t) + v(x)$, so

$$u(x,t) = -3e^{-\frac{\pi^2}{4}t} \sin(\pi x) + 5e^{-\pi^2 t} \sin(2\pi x) + \frac{\pi x}{2}.$$

□

Some useful formulas

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$