MAT 341 – Applied Real Analysis
FALL 2015

Midterm 1 – October 1, 2015

SOLUTIONS

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
TOTAL	

Problem 1: (25 points) Consider the function

$$f(x) = \begin{cases} -x & \text{if} \quad -2 \le x < 0 \\ x & \text{if} \quad 0 \le x < 2, \end{cases} \qquad f(x+4) = f(x).$$

Find the Fourier series for f. Determine whether the series converges uniformly or not. To what value does the Fourier series converge at $x = 2015$?

Solution. Notice that f is even, so we can use the half-formulas when computing the Fourier coefficients. The Fourier series is just a cosine series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right).$$

We have $a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = 1$ and $a_n = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)\,dx = 4\frac{(-1)^n - 1}{n^2\pi^2}$ (using the formula at the end of the booklet). The Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} 4\frac{(-1)^n - 1}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right).$$

The function is continuous, with piecewise continuous derivative, so the Fourier series converges uniformly everywhere. This can be seen also from the coefficients as

$$\sum_{n=1}^{\infty} |a_n| = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{|(-1)^n - 1|}{n^2} \le \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges. At $x = 2015$, the Fourier series converges to $f(2015) = f(2016-1) = f(-1) = 1$. We have used the fact that f is periodic of period 4. $\square$

Problem 2: (25 points) Suppose that the Fourier series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$.

a) What is the Fourier series of $1 - 2f(x)$?

Solution.

$$1 - 2f(x) = 1 - 2\sum_{n=1}^{\infty} e^{-341n} \cos(n\pi x)$$

$\square$

b) What is the Fourier series of $F(x) = \int_0^x f(y)\,dy$?

Solution.

$$F(x) = \int_0^x \sum_{n=1}^{\infty} e^{-341n} \cos(n\pi y)\; dy = \sum_{n=1}^{\infty} \frac{e^{-341n}}{n\pi} \sin(n\pi x)$$

$\square$

c) Find the Fourier series of $f''(x)$ if it exists. Otherwise, explain why it does not exist.

Solution.

$$f''(x) = -\sum_{n=1}^{\infty} n^2\pi^2 e^{-341n} \cos(n\pi x).$$

This series converges uniformly because $\sum_{n=1}^{\infty} |n^2 a_n| = \pi^2 \sum_{n=1}^{\infty} \frac{n^2}{e^{341n}} < \infty$ (which converges by the integral test). Notice also that the denominator is a polynomial, while the nominator is an exponential, hence the series converges. $\square$

d) What is the period of f? Can f have jump discontinuities or is it a continuous function?

Solution. The Fourier series is periodic of period 2, hence f is periodic of period 2. The function is continuous (by part c) we already know that f is twice differentiable, hence f is continuous). $\square$

Problem 3: (25 points) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - S\frac{\partial u}{\partial x} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \qquad t > 0$$

with boundary conditions

$$u(0,t) = T_0, \quad u(2,t) = 0, \qquad t > 0$$

and initial condition $u(x,0) = f(x)$, $0 \le x \le 2$. (S and T_0 are positive constants.)

a) Find the steady-state solution $v(x)$. What is the ODE that $v(x)$ satisfies?

SOLUTION. The steady-state solution verifies the equation $v''(x) - Sv'(x) = 0$, with boundary conditions $v(0) = T_0$ and $v(2) = 0$. The characteristic equation is $r^2 - Sr = 0$ and has roots $r = S$ and $r = 0$. The solution is $v(x) = C_1 + C_2 e^{Sx}$. We find the coefficients from the boundary conditions. We have $C_1 + C_2 = T_0$ and $C_1 + C_2 e^{2S} = 0$. Hence $C_1 = \frac{T_0 e^{2S}}{e^{2S}-1}$ and $C_2 = -\frac{T_0}{e^{2S}-1}$ and

$$v(x) = \frac{T_0 e^{2S}}{e^{2S} - 1} - \frac{T_0 e^{Sx}}{e^{2S} - 1}.$$

□

b) State the initial value–boundary value problem satisfied by the transient solution $w(x,t)$. You are NOT asked to solve this problem.

SOLUTION. By definition $w(x,t) = u(x,t) - v(x)$. Using the equations for u from the hypothesis and for v from part a) we find

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < 2, \qquad t > 0;\\
&w(0,t) = 0, \quad w(2,t) = 0, \qquad t > 0;\\
&w(x,0) = f(x) - v(x), \quad 0 \le x \le 2.
\end{aligned}$$

where $v(x)$ is the steady-state solution from part a). □

Problem 4: (25 points) Solve the heat problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad t > 0;\\
&u(0,t) = 0, \;\; u(1,t) = \beta, \quad t > 0;\\
&u(x,0) = \beta x + \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.
\end{aligned}$$

Solution. We first find the steady-state solution $v(x) = \beta x$. As shown in the lecture, the transient solution $w(x,t)$ verifies the PDE

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x^2} = \frac{1}{4}\frac{\partial w}{\partial t}, \qquad 0 < x < 1, \quad t > 0;\\
&w(0,t) = 0, \;\; w(1,t) = 0, \quad t > 0;\\
&w(x,0) = \sin\left(\frac{\pi x}{2}\right), \quad 0 \le x \le 1.
\end{aligned}$$

and the solution of this PDE is given by

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-4n^2\pi^2 t},$$

where

$$c_n = 2\int_0^1 \sin(n\pi x)\sin\left(\frac{\pi x}{2}\right)\, dx.$$

Note that these are not orthogonal functions! These functions have different periods: $\sin(n\pi x)$ has period 2, while $\sin\left(\frac{\pi x}{2}\right)$ has period 4. We compute the integral, using the formulas at the end of the booklet and find $c_n = \frac{(-1)^n 8n}{\pi(1-4n^2)}$. The solution to the given PDE is

$$u(x,t) = \beta x + \sum_{n=1}^{\infty} \frac{(-1)^n 8n}{\pi(1-4n^2)} \sin(n\pi x) e^{-4n^2\pi^2 t}.$$

$\square$

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\sin^2(a) = \frac{1-\cos(2a)}{2} \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$