MAT 341 – Applied Real Analysis
Spring 2017

Midterm 2 – April 11, 2017

Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator. You **are** allowed to bring a note card to the exam (8.5 x 5.5in - front and back), but no other notes are allowed.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1: (18 points) Find the Fourier integral representation of

$$f(x) = \begin{cases} \pi x & \text{if } \; 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of the Fourier integral at $x = 0$? At $x = 1$?

SOLUTION. We first compute the Fourier integral coefficients:

$$\begin{aligned} A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx = \frac{1}{\pi}\int_0^1 \pi x\cos(\lambda x)\,dx \\ &= \int_0^1 x\cos(\lambda x)\,dx = \left(\frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}\right)\Bigg|_0^1 \\ &= \frac{\sin(\lambda)}{\lambda} + \frac{\cos(\lambda)}{\lambda^2} - \frac{1}{\lambda^2} \end{aligned}$$

and

$$\begin{aligned} B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx = \frac{1}{\pi}\int_0^1 \pi x\sin(\lambda x)\,dx \\ &= \int_0^1 x\sin(\lambda x)\,dx = \left(\frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}\right)\Bigg|_0^1 \\ &= \frac{\sin(\lambda)}{\lambda^2} - \frac{\cos(\lambda)}{\lambda} \end{aligned}$$

The Fourier integral representation is

$$\int_0^{\infty} A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\,d\lambda = \frac{f(x-) + f(x+)}{2}.$$

At $x = 0$ the integral equals 0. At $x = 1$ the integral equals $\frac{\pi}{2}$. $\square$

Problem 2: (16 points) Consider the heat problem in a semi-infinite rod:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\pi}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \qquad \text{where} \quad f(x) = \begin{cases} \pi x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad 1 \leq x \end{cases}$$

b) Let $u(x,t) = \phi(x)T(t)$. Write down an ODE for ϕ together with the boundary and boundedness conditions.

SOLUTION. The eigenvalue problem for ϕ is

$$\phi'' + \lambda^2\phi = 0, \qquad 0 < x < \pi$$

$$\phi'(0) = 0$$

$$\phi(x) \text{ bounded as } x \to \infty$$

□

c) Find the general solution $u(x,t)$.

SOLUTION.

SOLUTION. The solution is given by

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)e^{-\pi\lambda^2 t}\,d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx = \frac{2}{\pi}\int_0^1 \pi x\cos(\lambda x)\,dx \\ &= 2\int_0^1 x\cos(\lambda x)\,dx = 2\frac{\lambda\sin(\lambda) + \cos(\lambda) - 1}{\lambda^2}. \end{aligned}$$

Therefore the solution is

$$u(x,t) = 2\int_0^\infty \frac{\lambda\sin(\lambda) + \cos(\lambda) - 1}{\lambda^2}\cos(\lambda x)e^{-\pi\lambda^2 t}\,d\lambda.$$

Note that the value for $A(\lambda)$ was already computed in Problem 1. □

Problem 3: (22 points) Consider the conduction of heat in a rod with insulated lateral surface whose left end is held at constant temperature and whose right end is exposed to convective heat transfer. Suppose the PDE satisfied by the temperature in the rod is:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0$$
$$u(0,t) = 0, \qquad t > 0$$
$$2u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0$$
$$u(x,0) = x, \quad 0 < x < 1.$$

a) Let $u(x,t) = \phi(x)T(t)$. Write down the eigenvalue problem for ϕ (that is, and ODE satisfied by ϕ and the boundary conditions).

Solution. The eigenvalue problem for ϕ is

$$\phi'' + \lambda^2\phi = 0, \qquad 0 < x < 1$$
$$\phi(0) = 0$$
$$2\phi(1) + \phi'(1) = 0$$

□

b) Solve the eigenvalue problem for ϕ and determine the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$.

Solution. This is a convection problem. We know that $\lambda = 0$ is not an eigenvalue. If $\lambda^2 > 0$ then $\phi(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x)$. From $\phi(0) = 0$ we get $C_1 = 0$. From the second condition we get $2\phi(1) + \phi'(1) = 2C_2\sin(\lambda) - C_2\lambda\cos(\lambda) = 0$. This yields

$$\sin(\lambda) = -\frac{1}{2}\lambda\cos(\lambda) \quad\Rightarrow\quad \tan(\lambda) = -\frac{\lambda}{2}.$$

The eigenvalues are λ_n, $n = 1, 2, \ldots$, where λ_n is the n-th root of this equation. The eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$. □

(Problem 3 continued)

c) Find the general solution $u(x,t)$.

Solution. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) e^{-\lambda_n^2 t}$$

where

$$c_n = \frac{\displaystyle\int_0^1 x\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx} = \frac{\displaystyle\int_0^1 x\sin(\lambda_n x)\,dx}{\displaystyle\int_0^1 \sin^2(\lambda_n x)\,dx}$$

To finish the computation we evaluate

$$\int_0^1 x\sin(\lambda_n x)\,dx = \left.\frac{\sin(\lambda_n x)}{\lambda_n^2}\right|_0^1 - \left.\frac{x\cos(\lambda_n x)}{\lambda_n}\right|_0^1 = \frac{\sin(\lambda_n)}{\lambda_n^2} - \frac{\cos(\lambda_n)}{\lambda_n}$$

and

$$\int_0^1 \sin^2(\lambda_n x)\,dx = \int_0^1 \frac{1-\cos(2\lambda_n x)}{2}\,dx = \frac{1}{2} - \frac{\sin(2\lambda_n)}{4\lambda_n},$$

where λ_n is the n-th root of the equation $\tan(\lambda) = -\frac{\lambda}{2}$. $\square$

Problem 4: (20 points) Find the eigenvalues and the corresponding eigenfunctions of the problem:

$$\phi'' + \lambda^2 \phi = 0, \qquad 0 < x < \pi$$

$$\phi(0) - \phi(\pi) = 0, \qquad \phi'(0) = 0$$

Is this a regular Sturm-Liouville problem?

SOLUTION. If $\lambda = 0$, then $\phi(x) = Ax + B$. From $\phi'(0) = 0$ we find $A = 0$. The condition $\phi(0) - \phi(\pi) = B - B = 0$ gives no information about B. Hence $\lambda = 0$ is an eigenvalue and $\phi_0(x) = 1$ is an eigenfunction.

If $\lambda^2 > 0$, then $\phi(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$. From $\phi'(0) = C_2\lambda = 0$ we find $C_2 = 0$. Thus $\phi(x) = C_1 \cos(\lambda x)$. The first boundary condition $\phi(0) - \phi(\pi) = C_1(1 - \cos(\lambda\pi)) = 0$ yields $\cos(\lambda\pi) = 1$, so $\lambda\pi = 2n\pi$, for $n = 1, 2, \ldots$.

The eigenvalues are $\lambda_n = 2n$, for $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $\phi_n(x) = \cos(2nx)$.

This is not a regular Sturm-Liouville problem. □

Problem 5: (24 points) Consider the following vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(a,t) = 0, \qquad t > 0$$

$$u(x,0) = 0, \qquad 0 < x < a$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < a, \qquad \text{where} \quad g(x) = \begin{cases} 0 & \text{if} \quad 0 < x < \frac{a}{2} \\ 2c & \text{if} \quad \frac{a}{2} \le x < a. \end{cases}$$

a) Find $u(x,t)$ using separation of variables.

Solution. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)\cos(\lambda_n ct) + b_n \sin(\lambda_n x)\sin(\lambda_n ct),$$

where $\lambda_n = \frac{n\pi}{a}$, $n = 1, 2, \ldots$. Since $u(x,0) = 0$, we get $a_n = 0$. The other initial condition gives

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{a} \sin\left(\frac{n\pi x}{a}\right) = g(x).$$

Therefore

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{n\pi c}\int_{\frac{a}{2}}^a 2c\sin\left(\frac{n\pi x}{a}\right)dx \\ &= -\frac{4}{n\pi}\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Bigg|_{\frac{a}{2}}^{a} = \frac{4a}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) \\ &= \frac{4a(-1)^{n+1}}{n^2\pi^2} \end{aligned}$$

The solution to the given PDE is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4a(-1)^{n+1}}{n^2\pi^2}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right).$$

$\square$

b) Find $u(x,t)$ using D'Alembert's solution to the wave equation.

SOLUTION. We first compute

$$G(x) = \frac{1}{c}\int_0^a g(y)\,dy = \begin{cases} 0 & \text{if} \quad 0 < x < \dfrac{a}{2} \\ 2x - a & \text{if} \quad \dfrac{a}{2} \le x < a. \end{cases}$$

Note that for $\frac{a}{2} \le x < a$ we have

$$\frac{1}{c}\int_0^a g(y)\,dy = \int_{a/2}^x 2\,dy = 2x - a.$$

The even extension G_e of G has the formula

$$G_e(x) = \begin{cases} 0 & \text{if} \quad -\dfrac{a}{2} < x < \dfrac{a}{2} \\ 2x - a & \text{if} \quad \dfrac{a}{2} \le x < a \\ -2x - a & \text{if} \quad -a < x \le -\dfrac{a}{2}. \end{cases}$$

Let $\tilde{G}_e$ be the periodic extension of G_e of period $2a$. The solution to the PDE is

$$u(x,t) = \frac{1}{2}\left(\tilde{G}_e(x+ct) - \tilde{G}_e(x-ct)\right).$$

□

c) Using the solution from part b) compute $u\left(a, \dfrac{a}{c}\right)$.

SOLUTION. Using part b) we find

$$u\left(a, \frac{a}{c}\right) = \frac{1}{2}\left(\tilde{G}_e(2a) - \tilde{G}_e(0)\right) = 0,$$

since $\tilde{G}_e$ is periodic of period $2a$. Note that part c) can be solved independently of part b). □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$