# MAT 341 – Applied Real Analysis
Fall 2015

## Midterm 2 – November 5, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator. You **are** allowed to bring a note card to the exam (8.5 x 5.5in - front and back), but no other notes are allowed.

**Please show your work!** To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

| PROBLEM | SCORE |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| TOTAL | |

**Problem 1:** (12 points) The *telegraph equation* governs the flow of voltage, or current, in a transmission line and has the form:

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \qquad 0 < x < 100, \quad t > 0.$$

The coefficients $c$, $k$, $a$ are constants related to electrical parameters in the line. Assuming that $F(x,t) = 0$ and $u(x,t) = \phi(x)T(t)$, carry out a separation of variables and find the eigenvalue problem for $\phi$. Take the boundary conditions to be

$$\frac{\partial u}{\partial x}(0,t) = 0 \;\text{ and }\; u(100,t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by $T(t)$.

**Problem 2:** (20 points) Solve the heat problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < 2, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad \text{where} \quad f(x) = \begin{cases} T_0 & \text{if} \quad 0 < x < 1 \\ T_1 & \text{if} \quad 1 \leq x < 2 \end{cases}$$

**Problem 3:**

a) (12 points) Find the eigenvalues $\lambda_n$ and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 1$$

$$\phi(0) = 0, \qquad \phi'(1) - \phi(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

*(Problem 3 continued)*

b) (5 points) Consider the function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 0.5 \\ 1 - x & \text{if} \quad 0.5 \leq x < 1. \end{cases}$$

Suppose $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of the function $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ from part $a$). Write down a formula for the coefficients $c_n$. You are **not** asked to compute the coefficients.

c) (7 points) To what value does the series converge at $x = 0.5$? What about at $x = 0$ and $x = 0.3$?

**Problem 4:** (22 points) Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t > 0$$

$$u(x,t) \ \text{ bounded as } x \to \infty$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \quad \text{where} \quad f(x) = \begin{cases} \pi - x & \text{if} \ \ 0 < x < \pi \\ 0 & \text{if} \quad \pi \le x \end{cases}$$

**Problem 5:** (22 points) If an elastic string is *free* at one end, the boundary condition to be satisfied there is that $\dfrac{\partial u}{\partial x} = 0$. On the other hand, if it is *fixed* at one end, the boundary condition to be satisfied there is that $u = 0$. Find the displacement $u(x,t)$ in an elastic string of length $a = 1$, fixed at $x = 0$ and free at $x = a$, set in motion with no initial velocity from the initial position $u(x,0) = \sin\left(\dfrac{3\pi x}{2}\right)$.

a) State the boundary value problem that $u(x,t)$ satisfies. Include the initial conditions.

b) Find $u(x,t)$.

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$