MAT 341 Applied Real Analysis

Final Exam, Fall 2019

Name:________________________________

This is a closed book, closed notes test. No consultations with others. Calculators are not allowed.

Please turn off and take off the desk cell phones and other electronic gadgets. Only the exam and pens/pencils should be on your desk.

Please explain all your answers and show all work. Answers without explanation will receive little credit.

The problems are not in the order of difficulty. You may want to look through the exam and do the easier questions first.

If short on time, focus on questions you understand best. Complete detailed answers for fewer questions will typically result in higher grades than incomplete answers to more questions.

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1	2	3	4	5	6	Total

Reference Page

Fourier series representation of a periodic function $f(x)$ of period $2a$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{a}\right) + b_n \sin\left(\frac{\pi n x}{a}\right),$$

where

$$a_0 = \frac{1}{2a}\int_{-a}^{a} f(x)\,dx, \quad a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\left(\frac{\pi n x}{a}\right)\,dx, \quad b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{\pi n x}{a}\right)\,dx.$$

Fourier integral representation of $f(x)$ satisfying appropriate conditions:

$$f(x) \sim \int_0^{\infty} \left(A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\right) d\lambda, \quad -\infty < x < \infty,$$

where $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx$, $B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx$.

For the vibrating string problem

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t), \quad 0 < x < a, \quad t > 0$$

with fixed ends $u(0,t) = u(a,t) = 0$ and initial data $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, D'Alembert solution is given as a sum of two waves $u(x,t) = \psi(x+ct) + \phi(x-ct)$, where

$$\psi(x+ct) = \frac{1}{2}(\tilde{f}_{odd}(x+ct) + \tilde{G}_{even}(x+ct)), \qquad \psi(x+ct) = \frac{1}{2}(\tilde{f}_{odd}(x-ct) - \tilde{G}_{even}(x-ct)),$$

with $G(x) = \frac{1}{c}\int_0^x g(y)\,dy$.

Potential equation in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

1. Consider the function

$$f(x) = \begin{cases} -2, & \text{for } 0 < x < \frac{1}{2}, \\ 1 & \text{for } x = \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

(a) To find the *cosine* Fourier series of the function $f(x)$ on the interval $(0, 1)$, what extension of this function do you need to consider? Sketch the graph of this extension on $(-\infty, +\infty)$.

(b) Find the *cosine* Fourier series of the function $f(x)$ on the interval $(0, 1)$.

(c) Does this Fourier series converge at points $x = 0$? $x = \frac{1}{2}$? $x = 4$? If converges, what does this Fourier series converge to? Explain your answers.

2. Find eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\phi'' + k\phi = 0, \quad \phi(0) = 0, \quad \phi(2) = 0.$$

You need to consider possible cases for signs of eigenvalues (positive/negative/zero); please address the cases carefully and show all work.
You may use part (a) in parts (b) and (c) below, as well as in one of the other questions; please do not repeat this work!

(b) Find all product solutions for the problem

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t), \quad 0 < x < 2, \quad t > 0,$$
$$u(0,t) = u(2,t) = 0, \quad t > 0.$$

Use the separation of variables method. Using the product solutions you found, write a formula for the general solution.

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(c) Consider the potential problem in the half-infinite strip $0 < x < 2$, $y > 0$:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\
&u(0, y) = 0, \quad u(2, y) = 0, \quad y > 0, \\
&u(x, y) \text{ bounded as } y \to +\infty.
\end{aligned}
$$

Using the separation of variables method, find product solutions, and then use them to write a formula for the general solution.

3. Consider the boundary value problem:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 2, \qquad 0 < y < 2,\\
&u(x,0) = 0, && 0 < x < 2,\\
&u(x,2) = 0, && 0 < x < 2,\\
&u(0,y) = 0, && 0 < y < 2,\\
&u(2,y) = \sin \pi y, && 0 < y < 2.
\end{aligned}
$$

(a) Is this a Dirichlet problem or a Neumann problem?

(b) Solve this problem completely.

(c) Consider the boundary value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 2, \qquad 0 < y < 2,\\
&u(x,0) = 0, && 0 < x < 2,\\
&u(x,2) = \sin \pi x, && 0 < x < 2,\\
&u(0,y) = 0, && 0 < y < 2,\\
&u(2,y) = \sin \pi y, && 0 < y < 2.
\end{aligned}$$

Explain how you would solve this problem if you already know the solution to the problem in part (a). You may actually write down the solution (with appropriate explanations), or just carefully explain the procedure.

4. The following initial value-boundary value problem describes the motion of a vibrating string.

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}(x,t), && 0 < x < 4, \quad t > 0,\\
&u(0,t) = u(4,t) = 0, && t > 0,\\
&u(x,0) = f(x), && 0 < x < 4,\\
&\frac{\partial u}{\partial t}(x,0) = 0, && 0 < x < 4.
\end{aligned}
$$

In this problem, the function $f(x)$ is given by

$$
f(x) = \begin{cases} x, & \text{for } 0 \le x < 1,\\ 1, & \text{for } 1 \le x \le 3,\\ 4 - x, & \text{for } 3 < x \le 4. \end{cases}
$$

(a) Give the solution $u(x,t)$ as superposition of two waves. Express these waves using appropriate periodic extension of the initial data. State clearly what extension you are using. (You do not have to derive this solution from the original equation.)

(b) Sketch the graph of the extension you need in part (a).

(c) Find the position of the midpoint of the string at times $t = \frac{1}{2}$, $t = 10$. Explain your answers.

(d) Sketch the graph of $u(x, t)$ for $t = \frac{1}{2}$, $0 < x < 4$. Show all work to explain how you found this graph. Is your graph consistent with the midpoint position from part (b)?

(e) What is the first moment $t > 0$ when the string returns to its original position? What is the second such moment? Explain your answers.

5. (a) Find the steady state solution for the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}(x,t) + \cos x = \frac{\partial u}{\partial t}(x,t), \qquad 0 < x < \pi, \qquad t > 0\\
&u(0,t) = 1, \qquad t > 0\\
&\frac{\partial u}{\partial x}(\pi,t) = \pi, \quad t > 0\\
&u(x,0) = T_0, \quad 0 < x < 1, \text{ where } T_0 \text{ is a constant.}
\end{aligned}$$

(b) State the initial value - boundary value problem satisfied by the transient solution. Please show all work. (You do not have to solve the resulting problem.)

6. Consider the following heat problem in a semi-infinite rod:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t), \qquad 0 < x < +\infty, \qquad t > 0 \\
&u(0,t) = 0, \qquad t > 0 \\
&u(x,t) \text{ bounded as } x \to +\infty, \\
&u(x,0) = f(x), \qquad x > 0.
\end{aligned}
$$

(a) Find all product solutions for this problem and write an expression for the general solution.
(For full credit, you will need to consider different cases for signs of eigenvalues. You will still receive most points if you solve the question correctly but skip this analysis.)

(b) Suppose that in the problem as above, initial heat distribution is given by the function

$$f(x) = \begin{cases} 0, & \text{for } 0 < x < 1, \\ 1, & \text{for } 1 \le x \le 3, \\ 0, & \text{for } x > 3. \end{cases}$$

Solve this boundary value-initial value problem completely.