MAT 341 APPLIED REAL ANALYSIS
LECTURE 01 FINAL EXAM
DECEMBER 20, 2018

Name:____

I.D.:____

The test is a closed book exam, sheets from Midterms 1 and 2 and an extra sheet (letter size, both sides) with formulas can be consulted. No other sources or notes can be used; cell phones, tablets and calculators are not allowed. Answer each question in the space provided and on the reverse side of the sheets. Show your work: no credit will be given for the unjustified answers.

1	2	3	4	5	6	7	8	EC	SUM
20 pts	30 pts	20 pts	20 pts	15 pts	30 pts	25 pts	20 pts	25 pts	180 pts

1. (a) (10 points) Find the Fourier series of the function $f(x) = x$ on the interval $-1 < x < 1$.

(b) (5 points) Find the Fourier series of the function $f(x) = x^2$ on the interval $-1 < x < 1$ by integrating the Fourier series in part (a).

(c) (5 points) Use part (b) to evaluate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

(*Hint*: Find the mean value of $f(x) = x^2$ on $-1 < x < 1$.)

2. Find the eigenvalues and the eigenfunctions and state the orthogonality relation for the eigenfunctions of the following singular Sturm-Liouville eigenvalue problems.

(a) (15 points)

$$(x\varphi')' + \lambda x\varphi = 0, \quad 0 < x < a,$$
$$\varphi(0) \quad \text{is bounded}, \quad \varphi(a) = 0.$$

(b) (15 points)

$$[(1 - x^2)\varphi']' + \lambda\varphi = 0, \quad -1 < x < 1,$$
$$\varphi(-1) \quad \text{is bounded}, \qquad \varphi(1) \quad \text{is bounded}.$$

3 (a) (5 points) Find the steady-state solution of the problem

$$\begin{aligned} u_t &= K u_{zz}, & 0 < z < 1, && t > 0, \\ u(0,t) &= 1, & u(1,t) = 2, && t > 0, \\ u(z,0) &= 1, & 0 < z < 1. && \end{aligned}$$

(b) (15 points) Solve the initial value–boundary value problem in part (a).

4. (a) (10 points) Find the solution $u(\rho, \varphi)$ of Laplace's equation in the disk $\rho < R$, satisfying the boundary condition $u(R, \varphi) = 1 + \cos 2\varphi + 3 \sin 3\varphi$, $-\pi < \varphi < \pi$.

(b) (10 points) Find the bounded solution $u(\rho, \varphi)$ of Laplace's equation in the exterior disk $\rho > R$, satisfying the boundary condition $u(R, \varphi) = 3+4 \cos 4\varphi+5 \sin 7\varphi$, $-\pi < \varphi < \pi$.

5. (a) (10 points) Use the d'Alembert solution of the wave equation to solve the problem

$$\begin{aligned} y_{tt} &= c^2 y_{ss}, && 0 < s < a, && 0 < t, \\ y(0,t) &= 0, && y(a,t) = 0, && 0 < t, \\ y(s,0) &= 0, && 0 < s < a, && \\ y_t(s,0) &= \sin\left(\frac{\pi s}{a}\right), && 0 < s < a. && \end{aligned}$$

(b) (5 points) Sketch the solution at times $t = a/2c$ and $t = a/c$.

6. Consider the vibrating membrane problem: two-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad x^2 + y^2 < a^2$$

with the boundary condition $u(x, y, t) = 0$ if $x^2 + y^2 = a^2$ and initial conditions

$$u(\rho, \varphi, 0) = u_1(\rho, \varphi) \quad \text{and} \quad u_t(\rho, \varphi, 0) = u_2(\rho, \varphi),$$

where $0 < \rho < a$ and $-\pi < \varphi < \pi$.

(a) (15 points) Find the solution $u(\rho, \varphi, t)$ of the vibrating membrane problem in the case where $u_1(\rho, \varphi) = 1$ and $u_2(\rho, \varphi) = 0$.

(b) (15 points) Find the solution $u(\rho, \varphi, t)$ of the vibrating membrane problem in the case where $u_1(\rho, \varphi) = 0$ and

$$u_2(\rho, \varphi) = J_2\left(\frac{\alpha_{21}\rho}{a}\right) \sin 2\varphi + 3J_7\left(\frac{\alpha_{71}\rho}{a}\right) \sin 7\varphi.$$

(Here α_{mn} is the n-th zero of the Bessel function $J_m(x)$ of order m).

7. Consider the initial-value problem for the heat equation

$$u_t = K(u_{xx} + u_{yy})$$

in the square $0 < x < 1$, $0 < y < 1$ with the non-homogeneous boundary conditions

$$u(0, y, t) = 1, \; u(1, y, t) = 2, \; u_y(x, 0, t) = 0, \; u_y(x, 1, t) = 0.$$

(a) (10 points) Find the steady-state solution.

(b) (15 points) Solve the initial-value problem for the heat equation with the non-homogeneous boundary conditions given above and the initial condition

$$u(x, y, 0) = 1.$$

8. (20 points) Find the solution of three-dimensional Laplace's equation in the ball of radius a, satisfying the boundary condition

$$u(a, \theta, \varphi) = \cos\theta + 3\cos 2\theta.$$

Extra Credit (25 points)

(a) (10 points) Find the eigenvalues and the eigenfunctions of the following Sturm-Liouville eigenvalue problem

$$(x\varphi')' + \lambda x\varphi = 0, \quad 0 < x < a,$$
$$\varphi(0) \quad \text{is bounded}, \quad \varphi'(a) = 0.$$

(b) (15 points) Using the generating function for the Legendre polynomials,

$$\frac{1}{\sqrt{1 - 2st + t^2}} = \sum_{n=0}^{\infty} P_n(s)t^n, \quad -1 < t < 1,$$

prove the recurrence relation

$$(n + 1)P_{n+1}(s) = (2n + 1)sP_n(s) - nP_{n-1}(s),$$

where $P_{-1}(s) = 0$.
(*Hint*: Differentiate the generating function with respect to t and rearrange the terms.)