MAT 341 – Applied Real Analysis
Fall 2015

Final – December 11, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Problem 1: (18 points) Check TRUE or FALSE, no other explanation is necessary.

a) A periodic, continuous function can have two different Fourier series, but they both converge to $f(x)$.

TRUE | FALSE

b) Suppose the Fourier series of a function $f(x)$ converges uniformly for $0 < x < 1$. Then $f(x)$ cannot have jump discontinuities on the interval $(0, 1)$.

TRUE | FALSE

c) The general solution to the wave equation obtained by separation of variables is the same as the solution obtained by D'Alembert's method.

TRUE | FALSE

d) Suppose $\phi_n(x)$ and $\phi_m(x)$ are eigenvalues of a regular Sturm-Liouville problem on the interval $0 < x < 1$. Then $\int_0^1 \phi_n(x)\phi_m(x)\,dx = 0$ whenever $m \neq n$.

TRUE | FALSE

e) The functions $v_1(r,\theta) = r^{-n}\cos(n\theta)$ and $v_2(r,\theta) = r^n\cos(n\theta)$ are both solutions to the potential equation $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

TRUE | FALSE

f) If $w(x,y,t)$ is the solution of a two-dimensional wave problem with homogeneous boundary conditions, then $\lim_{t\to\infty} w(x,y,t) = 0$.

TRUE | FALSE

g) Suppose the solution of a certain two-dimensional heat problem on the square $0 < x < 1$, $0 < y < 1$ is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(m\pi x)\cos(n\pi y)\exp(-(m^2+n^2)\pi^2 kt).$$

If $u(x,0,0) = f(x)$, then a_{mn} are the coefficients of the Fourier sine series of $f(x)$.

TRUE | FALSE

h) The bounded solutions to the equation $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda^2 r\phi = 0$ are given by $\phi(r) = AJ_0(\lambda r)$, for some constant A.

TRUE | FALSE

i) Suppose $u(x,y) = \int_0^\infty B(\lambda)\frac{\sinh((a-x)\lambda)}{\sinh(\lambda a)}\sin(\lambda y)\,d\lambda$ is solution to a potential equation in the strip $0 < x < a$, $0 < y$. Then $B(\lambda) = \frac{2}{\pi}\int_0^\infty u(0,y)\sin(\lambda y)\,dy$.

TRUE | FALSE

Problem 2: (14 points) Consider the function

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } \ 0 < x < 1, \\ 0 & \text{if } \ 1 \leq x < 2. \end{cases}$$

Find the Fourier cosine series for f. Does the Fourier cosine series converge uniformly? Explain.

Problem 3: (12 points) The eigenvalues and eigenfunctions to the following problem

$$(e^x \phi')' + e^x \lambda^2 \phi = 0, \quad 0 < x < 2$$
$$\phi(0) = 0 \quad \phi(2) = 0$$

are $\lambda_n = \dfrac{\sqrt{1 + n^2\pi^2}}{2}$ and $\phi_n(x) = \exp\left(-\frac{x}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$.

a) Find the coefficients for the expansion of the function $f(x) = \exp\left(-\frac{x}{2}\right)$, $0 < x < 2$, in terms of the eigenfunctions ϕ_n.

b) To what values does the series converge at $x = 1$ and $x = 2$? Explain.

Problem 4: (12 points) Consider the two-dimensional heat problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0$$

with boundary conditions

$$u(0, y, t) = \sin(5y), \qquad u(a, y, t) = 0, \qquad 0 < y < b, \quad t > 0$$

$$u(x, 0, t) = 0, \qquad u(x, b, t) = \cos(5x), \qquad 0 < x < a, \quad t > 0$$

and initial condition:

$$u(x, y, 0) = xy, \qquad 0 < x < a, \quad 0 < y < b.$$

a) State the initial value – boundary value problem satisfied by the steady-state solution $v(x, y)$. What is the PDE that $v(x, y)$ satisfies? You are **NOT** asked to solve it.

b) State the initial value – boundary value problem satisfied by the transient solution $w(x, y, t)$. You are **NOT** asked to solve it.

Problem 5: (16 points) Find the solution $u(x, y)$ of Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in the rectangle $0 < x < 2$, $0 < y < 3$, that satisfies the boundary conditions

$$u(0, y) = 0, \qquad u(2, y) = 0, \qquad 0 < y < 3,$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, \qquad u(x, 3) = \sin\left(\frac{\pi x}{2}\right) - 17 \sin\left(\frac{5\pi x}{2}\right), \qquad 0 < x < 2.$$

Problem 6: (16 points) Consider the potential equation on a half disk:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \qquad 0 < \theta < \pi, \qquad 0 < r < 2$$

$$\frac{\partial v}{\partial \theta}(r, 0) = 0, \qquad \frac{\partial v}{\partial \theta}(r, \pi) = 0, \qquad 0 < r < 2$$

a) Set $v(r, \theta) = R(r)\Theta(\theta)$ to separate the variables and write down the associated eigenvalue problem for Θ. Write down a differential equation that is verified by $R(r)$.

b) Solve the eigenvalue problem for Θ and find the eigenfunctions $\Theta_n(\theta)$.

(Problem 6 continued)

c) Suppose the function $v(r,\theta)$ is bounded as $r \to 0^+$. Find the fundamental solutions $v_n(r,\theta) = R_n(r)\Theta_n(\theta)$.

d) Suppose the function $v(r,\theta)$ is bounded as $r \to 0^+$. Find the general solution to this PDE if the initial condition is

$$v(2,\theta) = 1 + \cos(2015\,\theta), \qquad 0 < \theta < \pi.$$

Problem 7: (12 points) Solve the following initial value – boundary value problem: :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0$$
$$u(x,0) = 0, \qquad 0 < x < 1,$$

knowing that $\frac{\partial u}{\partial x}(0,t) = \sin\left(\frac{341\pi ct}{2}\right)$, $t > 0$. Can the solution be written as

$$u(x,t) = \phi(x+ct) - \phi(x-ct),$$

for some function ϕ?

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$