Practice Exam 1 Key

1. $F(x,y,z) = x^2 + 2y^2 - z^2 = 21$

$\nabla F(x,y,z) = \langle 2x, 4y, -2z \rangle$

$\nabla F(2,3,1) = \langle 4, 12, -2 \rangle$ $\quad 4(x-2) + 12(y-3) - 2(z-1) = 0$

$\vec{n} = \dfrac{\langle 4, 12, -2 \rangle}{\sqrt{16+144+4}} = \left\langle \dfrac{2}{\sqrt{41}}, \dfrac{6}{\sqrt{41}}, \dfrac{-1}{\sqrt{41}} \right\rangle$

2. a. $\nabla F(x,y,z) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}, 1 \right\rangle$

$\nabla F(1,0,2) = \langle 2, 0, 1 \rangle$

$\nabla F(1,0,2) \cdot \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle = \frac{12}{7} + \frac{2}{7} = \boxed{2}$

b. $\vec{u} = \dfrac{\langle 2, 0, 1 \rangle}{\sqrt{4+1}} = \langle 2/\sqrt{5}, 0, 1/\sqrt{5} \rangle$

3. $F_x(x,y) = x^2 - y^2 - 1 = 0$ $\qquad F_y(x,y) = -2xy = 0$

$\Longrightarrow x = 0$ or $y = 0$

$x = 0 \Longrightarrow -y^2 = 1 \longrightarrow$ no solutions

$y = 0 \Longrightarrow x^2 - 1 = 0 \Longrightarrow x = \pm 1$

$F_{xx} = 2x \quad F_{xy} = -2y \quad F_{yy} = -2x$

$d = \begin{vmatrix} 2x & -2y \\ -2y & -2x \end{vmatrix} = -4x^2 - 4y^2 \qquad d(\pm 1, 0) = -4 < 0$

two saddle points: $(1,0), (-1,0)$

4. Lagrange multipliers: $\nabla f = \lambda \nabla g$

$\begin{cases} 2 = \lambda(2x) \\ 3 = \lambda(2y) \\ 4 = \lambda(2z) \\ x^2+y^2+z^2 = 1 \end{cases}$

$x = 1/\lambda, \; y = 3/2\lambda, \; z = 2/\lambda$

$\Longrightarrow 1/\lambda^2 + 9/4\lambda^2 + 4/\lambda^2 = 1$

$\lambda^2 = 1 + 9/4 + 4 = 29/4 \qquad \lambda = \pm\dfrac{\sqrt{29}}{2}$

$(x,y,z) = (\pm 2/\sqrt{29}, \pm 3/\sqrt{29}, \pm 4/\sqrt{29})$

max: $f(2/\sqrt{29}, 3/\sqrt{29}, 4/\sqrt{29}) = (4+9+16)/\sqrt{29} = 29/\sqrt{29} = \sqrt{29}$

min: $f(-2/\sqrt{29}, -3/\sqrt{29}, -4/\sqrt{29}) = -29/\sqrt{29} = -\sqrt{29}$

5.a. $\int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} (\rho\cos(\phi))\,\rho^2\sin(\phi)\,d\phi\,d\rho\,d\theta = 2\pi\int_0^1\int_0^{\pi/2}\rho^3\cos(\phi)\sin(\phi)\,d\phi\,d\rho$

$= 2\pi\int_0^1 \rho^3 \underbrace{\left.\frac{\sin^2(\phi)}{2}\right|_0^{\pi/2}}_{1/2-0} d\rho = \pi\int_0^1 \rho^3 d\rho = \left.\frac{\pi}{4}\rho^4\right|_0^1 = \frac{\pi}{4}$

b. $\int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} (\rho\cos(\phi))^2\rho^2\sin(\phi)\,d\phi\,d\rho\,d\theta = 2\pi\left(\int_0^1\rho^4 d\rho\right)\left(\int_0^{\pi/2}\cos^2(\phi)\sin(\phi)\,d\phi\right)$

$= 2\pi\left(\left.\frac{1}{5}\rho^5\right|_0^1\right)\left.\left(\frac{-1}{3}\cos^3(\phi)\right)\right|_0^{\pi/2} = 2\pi\left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = \frac{2\pi}{15}$

$\bar{z} = \frac{2\pi/15}{\pi/4} = 8/15$ $\qquad (0, 0, 8/15)$

$\bar{x} = \bar{y} = 0$ by symmetry

6. $z = 16 - x^2 - y^2 = f(x,y)$

$f_x = -2x$

$f_y = -2y$

$SA = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2}\,dA = \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\; r\,dr\,d\theta = 2\pi\cdot\frac{1}{8}\frac{2}{3}\left.(1+4r^2)^{3/2}\right|_0^4$

$= \frac{\pi}{6}\left(65^{3/2} - 1\right)$

7.a. $\vec{r}\,'(t) = \left\langle 1, \frac{e^t - e^{-t}}{2}\right\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + \frac{e^{2t} - 2 + e^{-2t}}{4}} = \sqrt{\frac{1}{4}(e^{2t} + 2 + e^{-2t})} = \sqrt{\frac{(e^t+e^{-t})^2}{4}} = \frac{e^t + e^{-t}}{2}$

$\vec{T}(t) = \left\langle \frac{2}{e^t+e^{-t}}, \frac{e^t - e^{-t}}{e^t+e^{-t}}\right\rangle$

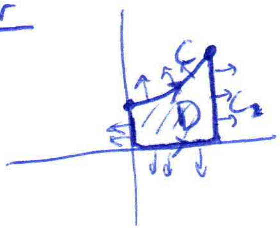
$\vec{N}(t) = \left\langle -\frac{e^t-e^{-t}}{e^t+e^{-t}}, \frac{2}{e^t+e^{-t}}\right\rangle$

b. $\vec{F}\cdot\vec{N} = -\frac{e^t-e^{-t}}{e^t+e^{-t}} + \frac{e^t+e^{-t}}{2}\cdot\frac{2}{e^t+e^{-t}} = 1-\frac{e^t-e^{-t}}{e^t+e^{-t}}$

$$\int_C \vec{F}\cdot\vec{N}\,ds = \int_0^1 \left(1-\frac{e^t-e^{-t}}{e^t+e^{-t}}\right)\left(\frac{e^t+e^{-t}}{2}\right)dt = \int_0^1 \frac{1}{2}\left(e^t+e^{-t}+e^t-e^{-t}\right)dt$$

$$= \int_0^1 \frac{1}{2}(2e^t)\,dt = \int_0^1 e^t\,dt = \boxed{e-1}$$

Or



$$\iint_D \operatorname{div}\vec{F}\,dA = \int_{C_2}\vec{F}\cdot\vec{N}\,ds - \int_C \vec{F}\cdot\vec{N}\,ds$$

$\operatorname{div}\vec{F} = 0+1 = 1$

$$\iint_D \operatorname{div}\vec{F}\,dA = \int_0^1\int_0^{\frac{e^x+e^{-x}}{2}} 1\,dy\,dx = \int_0^1 \frac{e^x+e^{-x}}{2}\,dx = \left.\frac{e^x-e^{-x}}{2}\right|_0^1 = \frac{e-e^{-1}}{2}$$

$$\int_{C_2}\vec{F}\cdot\vec{N}\,ds = \underbrace{-1}_{\text{left edge}} + \underbrace{0}_{\text{bottom edge}} + \underbrace{\frac{e+e^{-1}}{2}}_{\text{right edge}} = \frac{e+e^{-1}}{2}-1$$

$$\frac{e+e^{-1}}{2}-1-\frac{e-e^{-1}}{2} = \boxed{e-1}$$

8. a. $f(x,y) = \int 2xe^{x^2+y^2}\,dx$

$\Longrightarrow f(x,y) = e^{x^2+y^2} + g(y)$

$f_y(x,y) = 2ye^{x^2+y^2} + g'(y) = 2ye^{x^2+y^2} + e^y \Longrightarrow g'(y) = e^y$

$f(x,y) = e^{x^2+y^2} + e^y$

A potential function exists, so $\vec{F}(x,y)$ is conservative.

Alternatively, we can compute $\operatorname{curl}\vec{F} = 4xye^{x^2+y^2} - 4xye^{x^2+y^2} = 0$.

b. $\int_C \vec{F}\cdot d\vec{r} = f(2,3)-f(0,0) = e^{4+9}+e^3-e-e = \boxed{e^{13}+e^3-2e}$

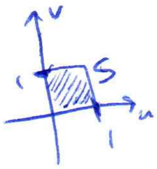
9. $\vec{F} = \langle \arctan x + ye^{xy},\ xe^{xy} + \sin y + x \rangle$

$\operatorname{curl} \vec{F} = xye^{xy} + 1 - xye^{xy} = 1$

$\iint_D 1 \, dA = 3 \cdot 4 = \boxed{12}$

10. $u = x + y$, $v = x - y$ $\Rightarrow$ $2x = u + v$, $x = \frac{u+v}{2}$; $2y = u - v$, $y = \frac{u-v}{2}$

$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/4 - 1/4 = -1/2$



$\iint_S e^{4\left(\frac{u+v}{2}\right) + 2\left(\frac{u-v}{2}\right)} \left| -1/2 \right| dA = \iint_S \frac{1}{2} e^{3u+v} \, dA$

$= \frac{1}{2} \int_0^1 \int_0^1 e^{3u} e^{v} \, du \, dv = \frac{1}{2} \left(e^{v} \Big|_0^1 \right) \left(\frac{1}{3} e^{3u} \Big|_0^1 \right) = \boxed{\frac{1}{6}(e-1)(e^3-1)}$