REMARKS ON "QUASICONFORMAL PARAMETRIZATION OF METRIC SURFACES WITH SMALL DILATATION"

I wrote this paper as a graduate student. Now, with a couple more years of experience, I see there are a few things worth saying that might help the reader.

- The main argument in this paper is based on considering an ellipse that minimizes the Banach–Mazur distance to a given convex body. At the time of writing, I found little information on the topic. Since then, I've learned that this is a topic that has been somewhat studied. The standard name is *distance ellipse* (or *distance ellipsoid* in higher dimensions). See the MathOverflow post https://mathoverflow.net/questions/306199/ellipsoid-minimizing-banach-mazur-distance-to-convex-body for more details.
- In the proof of Lemma 2.1, I take for granted that a distance ellipse has two linearly independent contact points with the respective convex body. This fact is not entirely obvious. It turns out to be true, and it was proved in D. R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika, 26(1) (1979), 18-29.
- It would have been helpful to point this out explicitly: the mapping I produce has the property that it minimizes the mean dilatation $\sqrt{K_O(f)K_I(f)}$ among all quasiconformal parametrizations of the surface. This is useful when understanding this result in relation to other work on "energy-minimizing" parametrizations of metric surfaces, where a variety of different energies are considered. See for example A. Lytchak, S. Wenger, *Canonical parameterizations of metric disks*, Duke Math. J. 169 (2020), no. 4, 761–797, where the parametrizations produced minimize the outer dilatation $K_O(f)$ among all quasiconformal parametrizations of the surface; note that this is the same as in K. Rajala's original result.
- T. Ikonen, in his paper *Uniformization Of metric surfaces using isothermal coordinates*, https://arxiv.org/abs/1909.09113, introduces a more conceptual point of view to the topic. He calls a minimizer of mean dilatation an *isothermal* parametrization of that surface. This name is of course based on the classical isothermal parametrization of Riemannian 2-manifolds, which is being generalized. In his paper, T. Ikonen generalizes K. Rajala's result, including the contribution in this paper, to arbitrary metric surfaces (that is, of arbitrary topological type).
- Finally, a standard example of where my parametrization differs from the one in K. Rajala's paper, or equivalently where the John ellipse and distance ellipse are different. Write a point $z \in \mathbb{R}^2$ in coordinates as $z = (x, y)$. Define a norm by $N(z) = \|z\|_2$ if $0 \leq x, y$ and $N(z) = \max\{|x|, |y|\}$ if $x \leq 0 \leq y$. The unit ball is "half circle/half square".