Review

- What is curl of $\mathbf{F} = \langle M, N \rangle$, i.e., in the 2-D case?
- What does "irrotational" mean?
- What is noteworthy about the vector field

$$\mathbf{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle?$$

- Let C be the unit circle traversed once counterclockwise. What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?
- Is there any difference in meaning between the expressions $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C M\,dx + N\,dy$?

Green's theorem

- This theorem puts curl into a broader perspective
- It applies specifically to the plane. In later sections, we cover Stoke's theorem and the Divergence theorem, which are analogous theorems in $\mathbb{R}^3$

Green's theorem

Let R be a simply connected region with piecewise smooth boundary, oriented counterclockwise. Let $\mathbf{F} = \langle M, N \rangle$, where M, N have continuous first partial derivatives. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA.$$

Putting the picture together

- Interpretation of curl: line integral over a "small loop"
- Interpretation: motion of a paddle wheel
- Combining loops
- Notation: $\oint F \cdot d\mathbf{r}$ or $\circlearrowleft\!\!\!\!\!\!\int F \cdot d\mathbf{r}$,

  $\circlearrowright\!\!\!\!\!\!\int F \cdot d\mathbf{r}$

  (also called counterclockwise/clockwise <u>circulation</u> of **F** around C)

Using Green's theorem

- Green's theorem is especially useful for evaluating a complicated line integral by replacing it with a simpler double integral
- Green's theorem can also be useful in the other direction: replacing a double integral with a line integral
- Finding the area of a region with Green's theorem:

$$\text{Area}(R) = \frac{1}{2} \int_C x\,dy - y\,dx,$$

where R is the region bounded by the simple closed curve C.

Divergence form of Green's theorem

- Green's theorem can be reinterpreted in terms of divergence

Green's theorem, version II

Let C be a simple closed curve and $\mathbf{N}$ the outward pointing normal vector, with R the region enclosed by C. Let $\mathbf{F}$ be a vector field. Then

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$