Review

- Surface area of a graph $z = f(x, y)$ over $R \subset \mathbb{R}^2$:

$$SA = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

- More generally, surface area of a parametric surface $(x, y, z) = \mathbf{r}(u, v)$ defined on R:

$$SA = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

- How is the first formula a special case of the second one?

Surface integrals (the first type)

- We can integrate over surfaces with respect to surface area. This is done simply by adding a function f to the previous integral formula.
- Formally, we can derive this definition as a limit of Riemann sums similarly to other integrals in the past. We'll skip these details in class.

Let S be a surface parametrized by $\mathbf{r}(u, v)$ on the region D and let $f(x, y, z)$ be a continuous function on S. Then we define the integral of f over S by

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

Surface integrals (the first type)

- If the surface is a graph $z = f(x, y)$, then the previous formula becomes

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, z) \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

- If f represents the density of the surface, then the surface integral can be interpreted as the total mass of the surface.

Flux integrals: introduction

- The second type of surface integrals are called “flux integrals”.
- The difference is that we integrate a vector field, instead of a scalar function. This is similar to the two types of line integrals.
- Physically, the flux integral measures the amount of flow of a fluid across a surface
- Mathematically, we integrate the dot product of the vector field with the unit normal vector at each point of surface

Orientation of a surface

- There is a certain condition needed to define the flux integral:
- An orientation of a surface is a consistent (i.e., continuous) choice of unit normal vector at each point. (Recall that at each point there are *two* choices of unit normal vector.)
- You can think of an orientation as a consistent choice of “inside” and “outside”.
- A surface is orientable if it has an orientation.
- The main mathematical significance of the “Möbius strip” is as an example of a non-orientable surface.

Orientation of a surface

- The orientation of a parametric surface is inherited from the parametrization $\mathbf{r}(u, v)$
- The unit normal vectors belonging to the orientation are given by $\mathbf{N} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$
- Reversing the order of u and v reverses the orientation

Flux integrals

Let $\mathbf{F}(x, y, z) = \langle M, N, P \rangle$ be a vector field with continuous first partial derivatives, and let S be a surface oriented by a unit normal vector $\mathbf{N}$. The flux or flux integral of $\mathbf{F}$ across S is

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS.$$