Review

- Green's theorem problem: Let $\mathbf{F}(x, y) = \langle e^{y^2}, e^{x^2} \rangle$. Find
$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$
for the curve given.
- $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is often called the circulation of $\mathbf{F}$ over C. What does this mean from a physical point of view?
- Why is curl also called the circulation density?

Green's theorem

Let R be a simply connected region with piecewise smooth boundary, oriented counterclockwise. Let $\mathbf{F} = \langle M, N \rangle$, where M, N have continuous first partial derivatives. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA.$$

Divergence form of Green's theorem

- Green's theorem can be reinterpreted in terms of divergence

Green's theorem, version II

Let C be a simple closed curve and $\mathbf{N}$ the outward pointing normal vector, with R the region enclosed by C. Let $\mathbf{F}$ be a vector field. Then

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

- The integral on the left is called a <u>flux</u> integral

We will encounter two generalizations of Green's theorem:

- Stoke's theorem: a similar theorem for surfaces in $\mathbb{R}^3$ and their boundary curves (for the **curl form** of Green's theorem)
- The divergence theorem: a similar theorem for closed regions in $\mathbb{R}^3$ and their boundary surfaces (for the **divergence form** of Green's theorem)

Ways of describing surfaces

So far in this course, we've used two main ways to describe surfaces in $\mathbb{R}^3$

- The level set of a function $F(x, y, z)$ is typically a surface (e.g. equations of planes, spheres, conic sections)
- The graph $z = f(x, y)$ of a function (e.g., upper and lower half spheres)

Parametric surfaces

- A third way, and the focus of this lecture, is by a parametrization: $(x, y, z) = \mathbf{r}(u, v)$, where $\mathbf{r}$ is a vector-valued function of two variables
- Depending on the problem, we might use another pair of variables like (θ, ϕ) or (r, θ) instead of (u, v).

- $\mathbf{r}_u$ and $\mathbf{r}_v$ are both tangent vectors to the surface at each point. Together, they determine the tangent plane to the surface
- The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is the normal vector to the surface (if it is nonzero)
- The surface area of a parametrized surface is given by the formula

$$SA = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$$